Journal of Nonlinear and Convex Analysis
Volume 25, Number 11, 2024, 2793–2811

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

TWO-STEP INERTIAL ALGORITHMS FOR SOLVING SPLIT FEASIBILITY AND FIXED POINT PROBLEMS WITH LINE SEARCH STEPS

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Dedicated to the Memories of Professor K. Goebel and Professor W. A. Kirk

Abstract. In this paper, we consider general inertial algorithms for finding a common solution of split feasibility problem and fixed point problem of a κ-strictly pseudocontractive mapping in real Hilbert spaces. We construct two inertial steps to accelerate the convergence of the algorithms with Armijo-like step size rule. We also prove the weak and strong convergence theorems of these algorithms under some mild conditions without prior knowledge of the Lipschitz constant of the cost mappings. In addition, we provide numerical experiments to illustrate the efficiency and advantage of the proposed method compared with other recent methods in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, respectively, and let C and Q be nonempty closed convex subsets of H. The split feasibility problem (SFP) was first proposed by Censor and Elfving [9] in 1994. Since then, it has attracted much attention due to its applications in image reconstruction, intensity modulated radiation therapy, etc, see [7,10,11]. The SFP is formulated as follows: find a point $x^* \in H$ with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

where $A : H \to H$ is a bounded linear operator. The solution set of SFP (1.1) is denoted by S. It is found that the CQ algorithm proposed in [6] is the most effective algorithm to solve SFP (1.1), the iterative process of which is defined as

$$x_{n+1} = P_C(x_n - \lambda A^*(I - P_Q)Ax_n), \quad n \geq 0, \tag{1.2}$$

where the step size λ is chosen in the open interval $\left(0, \frac{2}{\|A\|^2}\right)$, P_C and P_Q are metric projections onto C and Q, respectively, A^* is the adjoint operator of A. If P_C and P_Q have closed form expressions, such as C and Q are half spaces or closed balls, then the algorithm is easier to implement. If there is no such situation, the implementation of the algorithm needs a lot of complicated calculation processes, although the performance of the algorithm can be proved in theory. Furthermore,

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2020 *Mathematics Subject Classification.* 47J25, 47J20, 49J40, 47H09.

Key words and phrases. Split feasibility problem, fixed point, inertial, line search.

This work was partially supported by the Natural Science Foundation of China (Nos. 12471130; 11771126).

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the choice of step size depends on the operator norm, which is not always an easy task.

In what follows, we define the convex objective function f by

$$f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2, \quad x \in H, \tag{1.3}$$

where I is the identity operator. Then f is differentiable and has a Lipschitz gradient given by

$$\nabla f(x) = A^*(I - P_Q)Ax, \quad x \in H. \tag{1.4}$$

It turns out that the CQ algorithm (1.2) can be written in the form of a gradient projection algorithm

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \quad n \geq 0,$$

where $\lambda_n \in (0, \frac{2}{L})$ and $L = \|A\|^2$ is the Lipschitz constant of ∇f.

Qu and Xiu [29] introduced Armijo-line searches in Euclidean spaces to solve SFP (1.1) by modifying the relaxed CQ algorithm. Thereafter, Yang [19] extended it to Hilbert spaces as follows

$$\begin{cases} y_n = P_{\widehat{C}_n}(x_n - \lambda_n \nabla f_n(x_n)), \\ x_{n+1} = P_{\widehat{C}_n}(x_n - \lambda_n \nabla f_n(y_n)), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\lambda_n = \gamma l^{m_n}$, with $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$, and m_n being the smallest nonnegative integer such that

$$\lambda_n \|\nabla f_n(x_n) - \nabla f_n(y_n)\| \leq \mu \|x_n - y_n\|. \tag{1.6}$$

Here $\widehat{C}_n$ and $\widehat{Q}_n$ are defined by $\widehat{C}_n = \{x \in H \mid c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\}$, with $\xi_n \in \partial c(x_n)$, and $\widehat{Q}_n = \{y \in H \mid q(Ax_n) + \langle \zeta_n, y - Ax_n \rangle \leq 0\}$, with $\zeta_n \in \partial q(Ax_n)$. Moreover, $f_n(x) = \frac{1}{2}\|(I - P_{\widehat{Q}_n})Ax\|^2$ so that $\nabla f_n(x) = A^*(I - P_{\widehat{Q}_n})Ax$ for $x \in H$. They proved that $\{x_n\}$ weakly converges to a solution of (SFP). Many authors constructed variable step sizes without knowing the prior knowledge of operator norm, see [21, 22, 26, 30, 38–40, 42].

To improve the performance of the algorithm, Alvarez and Attouch [2] introduced the inertial technique, which is also widely used as an accelerating method to solve monotone inclusion problems, see [4,5,23–25,27,35,43]. Dang et al. [12] proposed an inertial relaxed CQ algorithm. The iterative scheme is as follows: for any $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = P_{\overline{C}_n}(w_n - \lambda_n \nabla f_n(w_n)), \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $0 \leq \theta_n < \theta < 1$, $\overline{C}_n = \{x \in H \mid c(w_n) + \langle \xi_n, x - w_n \rangle \leq 0\}$, $\xi_n \in \partial c(w_n)$, $\overline{Q}_n = \{y \in H \mid q(Aw_n) + \langle \zeta_n, y - Aw_n \rangle \leq 0\}$, $\zeta_n \in \partial q(Aw_n)$ and, $\nabla f_n(x) = A^*(I - P_{\overline{Q}_n})Ax$ for $x \in H$. A relaxed projection algorithm with a line search process was also constructed to solve (SFP), see [34]. The convergence result was also obtained under some suitable assumptions.

Let $T : H \to H$ be a nonexpansive mapping. The fixed point problem (FPP) is expressed as finding a point $x \in H$ such that

$$Tx = x. \tag{1.8}$$

The set of fixed points of T is denoted by $F(T)$. It is known that Mann iterative algorithm is more efficient among many iterative algorithms for solving fixed point problems involving nonexpansive mappings in the form of

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \quad n \geq 0, \tag{1.9}$$

where $\{\lambda_n\}$ is a sequence of nonnegative real numbers in $[0,1]$. If $F(T) \neq \emptyset$, then it is known that the sequence $\{x_n\}$ generated by (1.9) converges weakly to a fixed point of T under the divergence condition $\sum_{n=1}^{\infty} \lambda_n(1-\lambda_n) = \infty$.

Inertial-type algorithms were originated from the heavy ball method (an implicit discretization) of the second-order time dynamical system [1, 28], which has attracted much attention, due to its acceleration of the speed of the convergence of the original algorithms [15, 17].

Tan and Cho [36] proposed an inertial Mann-like algorithm for fixed points of nonexpansive mappings in Hilbert spaces. Their algorithm reads as follows

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (1-\alpha_n)w_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad \forall n \geq 1, \end{cases} \tag{1.10}$$

where $T : H \to H$ is a nonexpansive mapping such that $F(T) \neq \emptyset$. It is also proved that the iterative sequence $\{x_n\}$ generated by (1.10) converges to a fixed point of T in norm under some appropriate assumptions. We notice that inertial techniques were used to construct a number of iterative algorithms [16, 18, 33, 37].

Ceng et al. [8] introduced and analyzed an extragradient method for finding a common element of the solution set S and the fixed point set $F(T)$ of a nonexpansive mapping T in Hilbert spaces. The algorithm is formulated as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)TP_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \quad \forall n \geq 0, \end{cases} \tag{1.11}$$

where $T : C \to C$ is a nonexpansive mapping with $F(T) \cap S \neq \emptyset$ and $f_{\alpha_n}(x) = f(x) + (\alpha_n/2)\|x\|^2$ so that $\nabla f_{\alpha_n} = \nabla f + \alpha_n I = A^*(I - P_Q)A + \alpha_n I$. This algorithm combined the extragradient algorithm with the regularization algorithm. They proved that the sequence $\{x_n\}$ generated by (1.11) converges weakly to an element of $F(T) \cap S$ under mild conditions.

Dong et al. [14] introduced a general inertial Mann algorithm and proved the weak convergence of proposed algorithm under some conditions. The scheme is given by:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda_n)y_n + \lambda_n T z_n \end{cases} \tag{1.12}$$

for all $n \geq 1$, where $T : H \to H$ is a nonexpansive mapping. It is easy to show that the general inertial Mann algorithm includes other algorithms as special cases. They [14] proved that the sequence $\{x_n\}$ generated by (1.12) converges weakly to a point of $F(T)$.

In this paper, motivated and inspired by the above-mentioned work, we provide weakly or strongly convergent algorithms for solving split feasibility problem and fixed point problem of a κ-strictly pseudocontractive mapping. In Section 2, we recall some basic definitions and existing lemmas to be used in our proofs. In Section 3, the weak and strong convergence of the proposed algorithms are analyzed. We present in Section 4 some numerical experiments to compare our methods with other methods in the existing literature. A conclusion is also included in Section 5.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Given a sequence $\{x_n\}$ in H and a point $x \in H$. We use the standard notation: $x_n \to x$ means that $\{x_n\}$ converges in norm to x, and respectively, $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Also given a nonempty closed convex subset C of H. The normal cone to C at a point $z \in H$ is defined as

$$N_C z := \begin{cases} \{x \in H : \ \langle x, y - z \rangle \leq 0 \quad \forall y \in C\}, & \text{if } z \in C, \\ \emptyset, & \text{if } z \notin C. \end{cases}$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$, one has $\langle x - y, f - g \rangle \geq 0$. A monotone mapping T is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $(x, f) \in G(T)$, i.e., $f \in Tx$.

Lemma 2.1. (i) *For any $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

(ii) *For $x, y \in H$, $t \in \mathbb{R}$, we have*

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

Definition 2.2. Let C be a nonempty closed convex subset of H. For every $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \min\{\|x - y\| \mid y \in C\}.$$

The operator P_C is called the metric projection from H onto C.

Definition 2.3. Given a mapping $T : H \to H$ is a mapping.

(1) T is said to be L-Lipschitz if there exists a nonnegative constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H.$$

If $L = 1$, then T is said to be nonexpansive. If $L < 1$, then T is said to be contractive. It is known that P_C is nonexpansive.

(2) T is said to be firmly nonexpansive if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2 \quad \forall x, y \in H.$$

(3) T is called co-coercive (or inverse strongly monotone, ISM for short) on H with a modulus $\alpha > 0$ if

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2 \quad \forall x, y \in H.$$

In this case, we say that T is α-co-coercive, or α-ISM. We know that $I - P_C$ is 1-co-coercive (i.e., firmly nonexpansive).

(4) T is said to be κ-strictly pseudocontractive if there exists $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in H.$$

Every nonexpansive mapping is clearly 0-strictly pseudocontractive.

Lemma 2.4 ([20])**.** *Let C be a nonempty closed and convex subset of a real Hilbert space H and P_C be the metric projection from H onto C. Then, for all $x, y \in H$ and $z \in C$, we have*

(i) $\langle x - P_C x, z - P_C x\rangle \leq 0$;
(ii) $\|P_C x - P_C y\| \leq \|x - y\|$;
(iii) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$;
(iv) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|(I - P_C)x\|^2$.

Lemma 2.5 ([3])**.** *If $T : H \to H$ is a κ-strictly pseudocontractive mapping for some $0 \leq \kappa < 1$, then T satisfies the following properties:*

(1) *T is Lipschitz continuous with Lipschitz constant $L = (1 + \kappa)/(1 - \kappa)$.*
(2) *$F(T)$ is closed and convex.*
(3) *$I - T$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup \bar{x}$ and $(I - T)x_n \to 0$, then $\bar{x} \in F(T)$.*

Lemma 2.6 ([8])**.** *Let a point $x^* \in C$ be given. Then the following statements are equivalent.*

(a) *x^* solves SFP (1.1).*
(b) *x^* solves the fixed point equation (for each $\lambda > 0$):*

$$x^* = P_C(x^* - \lambda\nabla f(x^*)) = P_C(x^* - \lambda A^*(I - P_Q)Ax^*).$$

(c) *x^* solves the variational inequality (VI) with respect to the gradient of f, that is,*

$$\langle \nabla f(x^*), x - x^*\rangle \geq 0, \quad x \in C. \tag{2.1}$$

We will use $VI(C, \nabla f)$ to denote the solution set of VI (2.1).

Lemma 2.7 ([1])**.** *Let $\{\psi_n\}, \{\delta_n\}$ and $\{\alpha_n\}$ be the sequences in $[0, +\infty)$ such that*

$$\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$$

for each $n \geq 1$. Suppose $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(i) *$\sum_{n=1}^{\infty} \left[\psi_n - \psi_{n-1}\right]_+ < +\infty$, where $[t]_+ = \max\{t, 0\}$;*
(ii) *there exists $\psi^* \in [0, +\infty)$ such that $\lim_{n\to+\infty} \psi_n = \psi^*$.*

Lemma 2.8 ([44])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be a κ-strict pseudocontraction with a fixed point. Define $S : C \to H$ by $Sx = \alpha x + (1-\alpha)Tx$ for each $x \in C$. Then, for $\alpha \in [\kappa, 1)$, S is nonexpansive and $F(S) = F(T)$.*

Lemma 2.9 ([3])**.** *Let K be a nonempty subset of H and $\{x_n\}$ be a sequence in H. Suppose that the following two conditions are satisfied:*

(i) *for each $x \in K$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(ii) *every sequential weak cluster point of $\{x_n\}$ lies in K.*

Then the sequence $\{x_n\}$ converges weakly to a point in K.

Lemma 2.10 ([32])**.** *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n \delta_n, \ \ n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$. Suppose the conditions below are satisfied:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *whenever $\{n_k\}$ is a subsequence of positive integers such that* $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \ge 0$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Algorithms and convergence analysis

In this section we introduce two two-step inertial iterative algorithms for finding a point in the set $S \cap F(T)$, that is, a common solution to SFP (1.1) and FPP (1.8). We shall prove the weak convergence of the first algorithm, and the strong convergence of the second algorithm.

3.1. **Algorithm 1.** Let x_0, x_1 be arbitrarily chosen in H. Given constants $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \frac{1}{3})$. Our algorithm generates a sequence $\{x_n\}$ by the following iteration process:

$$\begin{cases} u_n = x_n + a_n(x_n - x_{n-1}), \\ w_n = x_n + b_n(x_n - x_{n-1}), \\ y_n = P_C\big(w_n - \lambda_n \nabla f(w_n)\big), \\ z_n = P_C\big(w_n - \lambda_n \nabla f(y_n)\big), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T_n z_n, \end{cases} \tag{3.1}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\begin{aligned} &\lambda_n \max\Big\{\|\nabla f(z_n) - \nabla f(y_n)\|, \|\nabla f(y_n) - \nabla f(w_n)\|\Big\} \\ &\le \mu\Big(\|z_n - y_n\| + \|y_n - w_n\|\Big). \end{aligned} \tag{3.2}$$

Moreover, $T_n = \gamma_n I + (1-\gamma_n)T$, $\{\gamma_n\} \subset (0, 1)$ fulfils the condition $\liminf_{n\to\infty}(\gamma_n - \kappa) > 0$, and $T : C \to H$ is a κ-strictly pseudocontraction for some $\kappa \in [0, 1)$.

To analyze the convergence of Algorithm 1, we assume the following two conditions:

C1) $\{a_n\} \subset [0,a]$ and $\{b_n\} \subset [0,b]$ are nondecreasing with $a_1 = b_1 = 0$, $b_n - a_n \leq b_{n+1} - a_{n+1} \leq 0$ and $a, b \in [0,1)$;

(C2) β, σ, $\delta > 0$, $\{\beta_n\}$ is nonincreasing, and

$$\delta > \frac{a\xi(1+\xi) + a\sigma}{1-a^2}, \quad 0 < \beta \leq \beta_n \leq \frac{\delta - a[\xi(1+\xi) + a\delta + \sigma]}{\delta[1 + \xi(1+\xi) + a\delta + \sigma]}, \tag{3.3}$$

where $\xi = \max\{a, b\}$.

Lemma 3.1. *Let $\gamma > 0$, $l \in (0,1)$ and $\mu \in (0, \frac{1}{3})$. Then the line search rule* (3.2) *is well defined and*

$$\frac{\mu l}{L} < \lambda_n \leq \gamma,$$

where $L = \|A\|^2$.

Proof. From (1.3) and (1.4), we have $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ and $\nabla f(x) = A^*(I - P_Q)Ax$. Noting the fact that $I - P_Q$ is (firmly) nonexpansive, we obtain that ∇f is L-Lipschtiz and moreover, $\frac{1}{L}$-co-coercive. So, we get

$$\|\nabla f(z_n) - \nabla f(y_n)\| \leq L\|z_n - y_n\|, \quad \|\nabla f(y_n) - \nabla f(w_n)\| \leq L\|y_n - w_n\|.$$

It turns out that

$$\max\Big\{\|\nabla f(z_n) - \nabla f(y_n)\|, \|\nabla f(y_n) - \nabla f(w_n)\|\Big\} \leq L\big(\|z_n - y_n\| + \|y_n - w_n\|\big).$$

By definition, $\lambda_n = \gamma l^{m_n}$ and since $l \in (0,1)$, we trivially see that $\lambda_n \leq \gamma$. However, since λ_n is the largest value that satisfies (3.2), we get

$$\begin{aligned}
&\frac{\lambda_n}{l}\max\Big\{\Big\|\nabla f\Big(P_C\Big(w_n - \frac{\lambda_n}{l}\nabla f(y_n)\Big)\Big) - \nabla f\Big(P_C\Big(w_n - \frac{\lambda_n}{l}\nabla f(w_n)\Big)\Big)\Big\|, \\
&\qquad \Big\|\nabla f\Big(P_C\Big(w_n - \frac{\lambda_n}{l}\nabla f(w_n)\Big)\Big) - \nabla f(w_n)\Big\|\Big\} \\
&> \mu\Big(\Big\|P_C\Big(w_n - \frac{\lambda_n}{l}\nabla f(y_n)\Big) - P_C\Big(w_n - \frac{\lambda_n}{l}\nabla f(w_n)\Big)\Big\| \\
&\quad + \Big\|P_C\Big(w_n - \frac{\lambda_n}{l}\nabla f(w_n)\Big) - w_n\Big\|\Big).
\end{aligned}$$

Consequently, we have $\frac{\lambda_n}{l} \cdot L > \mu$, that is, $\lambda_n > \frac{\mu l}{L}$. The proof is complete. $\square$

Theorem 3.2. *Assume that $\{\lambda_n\}$ satisfies the line search condition (3.2). Assume also $S \cap F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the Algorithm (3.1) converges weakly to a point of $S \cap F(T)$.*

Proof. Let $p \in S \cap F(T)$; thus $p \in C$, $Ap \in Q$, and $Tp = p$. Using Lemma 2.4, we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(w_n - \lambda_n \nabla f(y_n)) - p\|^2 \\
&\leq \|w_n - \lambda_n\nabla f(y_n) - p\|^2 - \|w_n - \lambda_n \nabla f(y_n) - z_n\|^2 \\
&= \|w_n - p\|^2 - \|w_n - z_n\|^2 + 2\lambda_n\langle \nabla f(y_n), p - z_n\rangle \\
&= \|w_n - p\|^2 - \|w_n - z_n\|^2 + 2\lambda_n\langle \nabla f(y_n), p - y_n\rangle + 2\lambda_n\langle \nabla f(y_n), y_n - z_n\rangle. \qquad (3.4)
\end{aligned}$$

According to the fact that $I - P_C$ is firmly nonexpansve and $\nabla f(z) = 0$, we derive

$$\langle \nabla f(y_n), y_n - p\rangle = \langle \nabla f(y_n) - \nabla f(p), y_n - p\rangle$$

$$= \langle (I-P_Q)Ay_n - (I-P_Q)Ap, Ay_n - Ap\rangle$$

$$\geq \|(I-P_Q)Ay_n\|^2. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\begin{aligned}\|z_n-p\|^2 &\leq \|w_n-p\|^2 - \|w_n-z_n\|^2 - 2\lambda_n\|(I-P_Q)Ay_n\|^2 + 2\lambda_n\langle \nabla f(y_n), y_n - z_n\rangle \\ &= \|w_n-p\|^2 - [\|w_n-y_n\|^2 + \|y_n-z_n\|^2 + 2\langle w_n-y_n, y_n-z_n\rangle] \\ &\quad - 2\lambda_n\|(I-P_Q)Ay_n\|^2 + 2\lambda_n\langle\nabla f(y_n), y_n-z_n\rangle \\ &= \|w_n-p\|^2 - \|w_n-y_n\|^2 - \|y_n-z_n\|^2 - 2\lambda_n\|(I-P_Q)Ay_n\|^2 \\ &\quad + 2\langle w_n - \lambda_n\nabla f(y_n) - y_n, z_n-y_n\rangle \\ &= \|w_n-p\|^2 - \|w_n-y_n\|^2 - \|y_n-z_n\|^2 - 2\lambda_n\|(I-P_Q)Ay_n\|^2 \\ &\quad + 2\langle w_n-\lambda_n\nabla f(w_n) - y_n, z_n-y_n\rangle + 2\lambda_n\langle\nabla f(w_n) - \nabla f(y_n), z_n-y_n\rangle.\end{aligned} \tag{3.6}$$

As $y_n = P_C\big(w_n - \lambda_n\nabla f(w_n)\big)$ and $z_n \in C$, it follows from Lemma 2.4 that

$$\langle w_n - \lambda_n\nabla f(w_n) - y_n, z_n - y_n\rangle \leq 0. \tag{3.7}$$

Moreover, by virtue of (3.2),

$$\begin{aligned}2\lambda_n\langle\nabla f(w_n)-\nabla f(y_n), z_n-y_n\rangle &\leq 2\lambda_n\|\nabla f(w_n)-\nabla f(y_n)\|\cdot\|z_n-y_n\| \\ &\leq 2\mu(\|z_n-y_n\|+\|y_n-w_n\|)\cdot\|z_n-y_n\| \\ &\leq 3\mu\|z_n-y_n\|^2 + \mu\|y_n-w_n\|^2.\end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) and keeping $\mu \in (0, \frac{1}{3})$ in mind, we get

$$\begin{aligned}\|z_n-p\|^2 &\leq \|w_n-p\|^2 - \|w_n-y_n\|^2 - \|y_n-z_n\|^2 - 2\lambda_n\|(I-P_Q)Ay_n\|^2 \\ &\quad + 3\mu\|z_n-y_n\|^2 + \mu\|y_n-w_n\|^2 \\ &= \|w_n-p\|^2 - (1-\mu)\|y_n-w_n\|^2 - (1-3\mu)\|z_n-y_n\|^2 \\ &\quad - 2\lambda_n\|(I-P_Q)Ay_n\|^2 \\ &\leq \|w_n-p\|^2.\end{aligned} \tag{3.9}$$

Recalling that T_n is nonexpansive and $F(T_n) = F(T)$, we immediately have, for $p \in F(T)$,

$$\|T_nz_n - p\|^2 \leq \|z_n-p\|^2. \tag{3.10}$$

From (3.1), we get

$$\begin{aligned}\|w_n-p\|^2 &= \|(1+b_n)(x_n-p) - b_n(x_{n-1}-p)\|^2 \\ &= (1+b_n)\|x_n-p\|^2 - b_n\|x_{n-1}-p\|^2 + b_n(1+b_n)\|x_n-x_{n-1}\|^2\end{aligned}$$

and

$$\begin{aligned}\|u_n-p\|^2 &= \|(1+a_n)(x_n-p) - a_n(x_{n-1}-p)\|^2 \\ &= (1+a_n)\|x_n-p\|^2 - a_n\|x_{n-1}-p\|^2 + a_n(1+a_n)\|x_n-x_{n-1}\|^2.\end{aligned}$$

Then, from (3.9) and (3.10),

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\beta_n)(u_n-p) + \beta_n(T_nz_n-p)\|^2 \\ &= (1-\beta_n)\|u_n-p\|^2 + \beta_n\|T_nz_n-p\|^2 - \beta_n(1-\beta_n)\|T_nz_n-u_n\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|u_n-p\|^2+\beta_n\|w_n-p\|^2-\beta_n(1-\beta_n)\|T_nz_n-u_n\|^2\\
&=(1-\beta_n)[(1+a_n)\|x_n-p\|^2-a_n\|x_{n-1}-p\|^2\\
&\quad+a_n(1+a_n)\|x_n-x_{n-1}\|^2]\\
&\quad+\beta_n[(1+b_n)\|x_n-p\|^2-b_n\|x_{n-1}-p\|^2\\
&\quad+b_n(1+b_n)\|x_n-x_{n-1}\|^2]-\beta_n(1-\beta_n)\|T_nz_n-u_n\|^2\\
&=[(1-\beta_n)(1+a_n)+\beta_n(1+b_n)]\|x_n-p\|^2\\
&\quad-\Big[(1-\beta_n)a_n+\beta_nb_n\Big]\|x_{n-1}-p\|^2\\
&\quad+[(1-\beta_n)a_n(1+a_n)+\beta_nb_n(1+b_n)]\|x_n-x_{n-1}\|^2\\
&\quad-\beta_n(1-\beta_n)\|T_nz_n-u_n\|^2. \qquad (3.11)
\end{aligned}
$$

Let $\theta_n=a_n(1-\beta_n)+b_n\beta_n$. Then from (C1), (C2) and $\beta_n\in(0,1)$, it follows that $\theta_n\subset[0,\xi]$ is nondecreasing with $\theta_1=0$. Then (3.11) is reduced to

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \;\leq\;& (1+\theta_n)\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2-\beta_n(1-\beta_n)\|T_nz_n-u_n\|^2\\
&+\Big[(1-\beta_n)a_n(1+a_n)+\beta_nb_n(1+b_n)\Big]\|x_n-x_{n-1}\|^2. \qquad (3.12)
\end{aligned}
$$

On the other hand, by (3.1), we get

$$
\begin{aligned}
\|T_nz_n-u_n\|^2&=\Big\|\frac{1}{\beta_n}(x_{n+1}-x_n)+\frac{a_n}{\beta_n}(x_{n-1}-x_n)\Big\|^2\\
&=\frac{1}{\beta_n^2}\|x_{n+1}-x_n\|^2+\frac{a_n^2}{\beta_n^2}\|x_{n-1}-x_n\|^2+2\frac{a_n}{\beta_n^2}\langle x_{n+1}-x_n,x_{n-1}-x_n\rangle\\
&\geq\frac{1}{\beta_n^2}\|x_{n+1}-x_n\|^2+\frac{a_n^2}{\beta_n^2}\|x_{n-1}-x_n\|^2\\
&\quad+\frac{a_n}{\beta_n^2}\left(-\rho_n\|x_{n+1}-x_n\|^2-\frac{1}{\rho_n}\|x_{n-1}-x_n\|^2\right), \qquad (3.13)
\end{aligned}
$$

where $\rho_n=\frac{1}{a_n+\delta\beta_n}$. Substituting (3.13) into (3.12) yields

$$
\begin{aligned}
\|x_{n+1}-p\|^2&-(1+\theta_n)\|x_n-p\|^2+\theta_n\|x_{n-1}-p\|^2\\
&\leq\frac{(1-\beta_n)(a_n\rho_n-1)}{\beta_n}\|x_{n+1}-x_n\|^2+\mu_n\|x_n-x_{n-1}\|^2, \qquad (3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_n&=(1-\beta_n)a_n(1+a_n)+\beta_nb_n(1+b_n)\\
&\quad+a_n(1-\beta_n)\frac{1-a_n\rho_n}{\rho_n\beta_n}\geq 0. \qquad (3.15)
\end{aligned}
$$

By the definition of ρ_n, we get $\delta=\frac{1-a_n\rho_n}{\rho_n\beta_n}$. From (3.15),

$$
\begin{aligned}
\mu_n&=(1-\beta_n)a_n(1+a_n)+\beta_nb_n(1+b_n)+a_n(1-\beta_n)\delta\\
&\leq\xi(1+\xi)+a\delta. \qquad (3.16)
\end{aligned}
$$

Let $\phi_n = \|x_n - p\|^2$ and $\psi_n = \phi_n - \theta_n\phi_{n-1} + \mu_n\|x_n - x_{n-1}\|^2$ for all $n \in \mathbb{N}$. Since $\{\theta_n\}$ is nondecreasing and $\phi_n \geq 0$, from (3.14), we get

$$\begin{aligned}\psi_{n+1} - \psi_n &\leq \phi_{n+1} - (1+\theta_n)\phi_n + \theta_n\phi_{n-1} + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ &\leq \left[\frac{(1-\beta_n)(a_n\rho_n - 1)}{\beta_n} + \mu_{n+1}\right]\|x_{n+1} - x_n\|^2. \end{aligned} \tag{3.17}$$

Next, we show that

$$\frac{(1-\beta_n)(a_n\rho_n - 1)}{\beta_n} + \mu_{n+1} \leq -\sigma. \tag{3.18}$$

Since $\rho_n = \frac{1}{a_n + \delta\beta_n}$, then,

$$\begin{aligned}&\frac{(1-\beta_n)(a_n\rho_n - 1)}{\beta_n} + \mu_{n+1} \leq -\sigma \\ &\Leftrightarrow \beta_n(\mu_{n+1} + \sigma) + (1-\beta_n)(a_n\rho_n - 1) \leq 0 \\ &\Leftrightarrow \beta_n(\mu_{n+1} + \sigma) - \frac{\delta\beta_n(1-\beta_n)}{a_n + \delta\beta_n} \leq 0 \\ &\Leftrightarrow (a_n + \delta\beta_n)(\mu_{n+1} + \sigma) + \delta\beta_n \leq \delta.\end{aligned}$$

From (3.3) and (3.16), we have

$$(a_n + \delta\beta_n)(\mu_{n+1} + \sigma) + \delta\beta_n \leq (a + \delta\beta_n)\Big[\xi(1+\xi) + a\delta + \sigma\Big] + \delta\beta_n \leq \delta.$$

Hence, (3.18) is verified. From (3.17) and (3.18)

$$\psi_{n+1} - \psi_n \leq -\sigma\|x_{n+1} - x_n\|^2, \tag{3.19}$$

which implies that $\{\psi_n\}$ is nonincreasing. Furthermore,

$$-\xi\phi_{n-1} \leq \phi_n - \xi\phi_{n-1} \leq \psi_n \leq \psi_1. \tag{3.20}$$

Since $\psi_1 = \phi_1 \geq 0$ ($\theta_1 = a_1 = b_1 = 0$), we get

$$\phi_n \leq \xi\phi_{n-1} + \psi_1 \leq \cdots \leq \xi^n\phi_0 + \psi_1\sum_{k=1}^{n-1}\xi^k \leq \xi^n\phi_0 + \frac{\psi_1}{1-\xi}. \tag{3.21}$$

From (3.19), (3.20) and (3.21), we have

$$\sigma\sum_{k=1}^{n}\|x_{k+1} - x_k\|^2 \leq \psi_1 - \psi_{n+1} \leq \psi_1 + \xi\phi_n \leq \xi^{n+1}\phi_0 + \frac{\psi_1}{1-\xi},$$

which implies that

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 < +\infty. \tag{3.22}$$

From (3.14), (3.22) and Lemma 2.7, we conclude that $\lim_{n\to\infty}\|x_n - p\|$ exists. We also have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (3.1),

$$\begin{aligned}\|u_n - x_{n+1}\| &\leq \|x_n - x_{n+1}\| + a_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - x_{n+1}\| + a\|x_n - x_{n-1}\| \\ &\to 0 \ (n \to \infty).\end{aligned} \tag{3.23}$$

Similarly,

$$\|w_n - x_{n+1}\| \to 0 \ (n \to \infty). \tag{3.24}$$

On the other hand, from (3.9), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(u_n - p) + \beta_n(T_n z_n - p)\|^2 \\
&= (1-\beta_n)\|u_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|T_n z_n - u_n\|^2 \\
&\le (1+\theta_n)\|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 - \beta_n(1-\beta_n)\|T_n z_n - u_n\|^2 \\
&\quad + \Big[(1-\beta_n)a_n(1+a_n) + \beta_n b_n(1+b_n)\Big]\|x_n - x_{n-1}\|^2 \\
&\quad - \beta_n\Big[(1-\mu)\|w_n - y_n\|^2 + (1-3\mu)\|z_n - y_n\|^2 + 2\lambda_n\|(I-P_Q)Ay_n\|^2 \\
&\quad + (1-\gamma_n)(\gamma_n - \kappa)\|Tz_n - z_n\|^2\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\beta_n(1-\beta_n)\|T_n z_n - u_n\|^2 + \beta_n\Big[(1-\mu)\|w_n - y_n\|^2 + (1-3\mu)\|z_n - y_n\|^2 \\
&\quad + 2\lambda_n\|(I-P_Q)Ay_n\|^2 + (1-\gamma_n)(\gamma_n-\kappa)\|Tz_n - z_n\|^2\Big] \\
&\qquad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \theta_n\Big(\|x_n - p\|^2 - \|x_{n-1} - p\|^2\Big) \\
&\qquad\quad + \Big[(1-\beta_n)a_n(1+a_n) + \beta_n b_n(1+b_n)\Big]\|x_n - x_{n-1}\|^2 \to 0 \quad (n \to \infty).
\end{aligned}$$

So, as $n \to \infty$, we get

$$\begin{aligned}
&\|T_n z_n - u_n\| \to 0, \quad \|w_n - y_n\| \to 0, \quad \|z_n - y_n\| \to 0, \\
&\|(I-P_Q)Ay_n\| \to 0, \quad \|Tz_n - z_n\| \to 0.
\end{aligned} \tag{3.25}$$

Let x be a sequential weak cluster point of $\{x_n\}$. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$; thus, $w_{n_k} \rightharpoonup x$, $y_{n_k} \rightharpoonup x$, $z_{n_k} \rightharpoonup x$ as $k \to \infty$. Now, we show that $x \in F(T)$. Since $z_{n_k} \rightharpoonup x$, $\|Tz_{n_k} - z_{n_k}\| \to 0$ $(n \to \infty)$, using Lemma 2.5 we obtain $x \in F(T)$.

Next, we show that $x \in S$. Let

$$T'v := \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T' is maximal monotone and $0 \in T'v$ if and only if $v \in VI(C, \nabla f)$ (see [31] for more details). Let $G(T')$ be the graph of T' and $(v, w) \in G(T')$. Then $w \in T'v = \nabla f(v) + N_C v$ for $v \in C$, and $w - \nabla f(v) \in N_C v$. By the definition of $N_C v$, we get

$$\langle v - u, w - \nabla f(v)\rangle \ge 0, \quad \forall u \in C.$$

Since $z_n = P_C(w_n - \lambda_n \nabla f(y_n))$ and $v \in C$, we have

$$\langle w_n - \lambda_n\nabla f(y_n) - z_n, v - z_n\rangle \le 0 \implies \left\langle v - z_n, \frac{z_n - w_n}{\lambda_n} + \nabla f(y_n)\right\rangle \ge 0.$$

From $w - \nabla f(v) \in N_C v$ and $z_{n_k} \in C$, we get

$$\langle v - z_{n_k}, w\rangle \ge \langle v - z_{n_k}, \nabla f(v)\rangle$$

$$
\begin{aligned}
&\geq \langle v - z_{n_k}, \nabla f(v)\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - w_{n_k}}{\lambda_{n_k}} + \nabla f(y_{n_k})\right\rangle \\
&= \langle v - z_{n_k}, \nabla f(v)\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - w_{n_k}}{\lambda_{n_k}}\right\rangle - \langle v - z_{n_k}, \nabla f(y_{n_k})\rangle \\
&= \langle v - z_{n_k}, \nabla f(v) - \nabla f(z_{n_k})\rangle \\
&\quad + \langle v - z_{n_k}, \nabla f(z_{n_k}) - \nabla f(y_{n_k})\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - w_{n_k}}{\lambda_{n_k}}\right\rangle.
\end{aligned}
$$

Hence, from (3.25) we have $\langle v - x, w\rangle \geq 0$. Since T' is maximal monotone, then $0 \in T'x$, and hence $x \in VI(C, \nabla f)$. Thus it is clear that $x \in S$ from Lemma 2.6, that is, $x \in S \cap F(T)$. From Lemma 2.9, it follows that $\{x_n\}$ converges weakly to a point in $S \cap F(T)$. This completes the proof. □

3.2. **Algorithm 2.** In order to introduce an algorithm that converges in norm, we employ a viscosity approximation technique. Let x_0, x_1 be arbitrary in H, $\varphi : H \to H$ be a contraction mapping with constant $\tau \in [0, 1)$. Assume that $a, b \in [0, 1)$ and $\{\epsilon_n\}, \{\eta_n\}$ are positive sequences such that $\sum_{n=1}^{\infty} \epsilon_n < \infty, \sum_{n=1}^{\infty} \eta_n < \infty$. Given constants $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \frac{1}{3})$. Choose a_n and b_n such that $0 < a_n < \overline{a_n}$ and $0 < b_n < \overline{b_n}$, respectively, where

$$
\overline{a_n} = \begin{cases} \min\left\{a, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1} \\ a, & x_n = x_{n-1} \end{cases}
$$

$$
\overline{b_n} = \begin{cases} \min\left\{b, \dfrac{\eta_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1} \\ b, & x_n = x_{n-1} \end{cases}
$$

Compute:

$$
\begin{cases}
u_n = x_n + a_n(x_n - x_{n-1}), \\
w_n = x_n + b_n(x_n - x_{n-1}), \\
y_n = P_C\big(w_n - \lambda_n \nabla f(w_n)\big), \\
z_n = P_C\big(w_n - \lambda_n \nabla f(y_n)\big), \\
x_{n+1} = \beta_n \varphi(u_n) + (1 - \beta_n) T_n z_n,
\end{cases}
\tag{3.26}
$$

where λ_n is as defined in (3.2), $T_n = \gamma_n I + (1 - \gamma_n)T$, $\gamma_n \in [\kappa, 1)$, and $T : C \to H$ is a κ-strict pseudocontraction with $\kappa \in [0, 1)$. Suppose the following conditions are satisfied:

(C3) $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$ and $\lim_{n\to\infty} \frac{\eta_n}{\beta_n} = 0$;

(C4) $\{a_n\} \subset (0, 1)$, $\{b_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$.

Theorem 3.3. *Assume that $\{\lambda_n\}$ satisfies the line search condition (3.2), the conditions (C3)-(C4) hold, $\liminf_{n\to\infty}(\gamma_n - \kappa) > 0$, and $S \cap F(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the Algorithm 2 converges strongly to $p \in S \cap F(T)$, where $p = P_{S\cap F(T)}\varphi(p)$.*

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Let $p \in S \cap F(T)$. Since φ is a contraction mapping, from (3.9), (3.10) and (3.26), it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n\big(\varphi(u_n) - p\big) + (1-\beta_n)\big(T_n z_n - p\big)\| \\
&\le \beta_n(\|\varphi(u_n) - \varphi(p)\| + \|\varphi(p) - p\|) + (1-\beta_n)\|z_n - p\| \\
&\le \beta_n\tau\|u_n - p\| + \beta_n\|\varphi(p) - p\| + (1-\beta_n)\|z_n - p\| \\
&\le \beta_n\tau(\|x_n - p\| + a_n\|x_n - x_{n-1}\|) + \beta_n\|\varphi(p) - p\| \\
&\quad + (1-\beta_n)(\|x_n - p\| + b_n\|x_n - x_{n-1}\|) \\
&\le \beta_n\tau\|x_n - p\| + a_n\|x_n - x_{n-1}\| + \beta_n\|\varphi(p) - p\| \\
&\quad + (1-\beta_n)\|x_n - p\| + b_n\|x_n - x_{n-1}\| \\
&= [1 - \beta_n(1-\tau)]\|x_n - p\| \\
&\quad + \beta_n\Big(\|\varphi(p) - p\| + \frac{a_n}{\beta_n}\|x_n - x_{n-1}\| + \frac{b_n}{\beta_n}\|x_n - x_{n-1}\|\Big). \qquad (3.27)
\end{aligned}$$

According to $(C3)$ and the definition of a_n, b_n, we get

$$\frac{a_n}{\beta_n}\|x_n - x_{n-1}\| \le \frac{\epsilon_n}{\beta_n} \to 0 \ (n \to \infty), \quad \frac{b_n}{\beta_n}\|x_n - x_{n-1}\| \le \frac{\eta_n}{\beta_n} \to 0 \ (n \to \infty).$$

Let constants $M_1, M_2 > 0$ satisfy $\frac{a_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1$, $\frac{b_n}{\beta_n}\|x_n - x_{n-1}\| \le M_2$. From (3.27), we find that

$$\begin{aligned}
\|x_{n+1} - p\| &\le [1 - \beta_n(1-\tau)]\|x_n - p\| + \beta_n(1-\tau)\frac{\|\varphi(p) - p\| + M_1 + M_2}{1-\tau} \\
&\le \max\left\{\|x_n - p\|, \ \frac{\|\varphi(p) - p\| + M_1 + M_2}{1-\tau}\right\} \\
&\le \cdots \\
&\le \max\left\{\|x_0 - p\|, \ \frac{\|\varphi(p) - p\| + M_1 + M_2}{1-\tau}\right\}.
\end{aligned}$$

This sufficiently implies the boundedness of $\{x_n\}$, and so are $\{w_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{\varphi(u_n)\}$.

Next, we show that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le [1 - \beta_n(1-\tau)]\|x_n - p\|^2 + \beta_n(1-\tau) \cdot \frac{1}{1-\tau} \\
&\cdot \left[\frac{a_n}{\beta_n}\|x_n - x_{n-1}\| \cdot M_3 + \frac{b_n}{\beta_n}\|x_n - x_{n-1}\| \cdot M_4 + 2\langle \varphi(p) - p, x_{n+1} - p\rangle\right] \qquad (3.28)
\end{aligned}$$

where $M_3, M_4 > 0$ are constants (which will be made clear later on). As a matter of fact, using Lemma 2.1 and (3.10), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n(\varphi(u_n) - p) + (1-\beta_n)(T_n z_n - p)\|^2 \\
&= \beta_n\|\varphi(u_n) - \varphi(p) + \varphi(p) - p\|^2 \\
&\quad + (1-\beta_n)\|T_n z_n - p\|^2 - \beta_n(1-\beta_n)\|T_n z_n - \varphi(u_n)\|^2 \\
&\le \beta_n\big(\|\varphi(u_n) - \varphi(p)\|^2 + 2\langle \varphi(p) - p, \varphi(u_n) - p\rangle\big) \\
&\quad + (1-\beta_n)\big(\|z_n - p\|^2 - (1-\gamma_n)(\gamma_n - \kappa)\|Tz_n - z_n\|^2\big)
\end{aligned}$$

$$
\begin{aligned}
&- \beta_n(1-\beta_n)\|T_n z_n - \varphi(u_n)\|^2 \\
&\leq \beta_n\big(\tau^2\|u_n - p\|^2 + 2\|\varphi(p) - p\| \cdot \|\varphi(u_n) - p\|\big) + (1-\beta_n)\|z_n - p\|^2 \\
&- (1-\beta_n)(1-\gamma_n)(\gamma_n - \kappa)\|Tz_n - z_n\|^2 \\
&- \beta_n(1-\beta_n)\|T_n z_n - \varphi(u_n)\|^2. \qquad (3.29)
\end{aligned}
$$

Set $M_5 := (2\|\varphi(p) - p\|)\sup_{n\geq 1}\|\varphi(u_n) - p\|$. From the definitions of u_n and w_n, we get

$$
\begin{aligned}
\|u_n - p\|^2 &\leq \big(\|x_n - p\| + \beta_n M_1\big)^2 \\
&= \|x_n - p\|^2 + 2\|x_n - p\| \cdot \beta_n M_1 + \beta_n^2 M_1^2 \\
&\leq \|x_n - p\|^2 + \beta_n M_6,
\end{aligned}
$$

where $M_6 = \sup_{n\geq 1}\big(2\|x_n - p\| \cdot M_1 + \beta_n M_1^2\big)$. Similarly,

$$
\|w_n - p\|^2 \leq \|x_n - p\|^2 + \beta_n M_7,
$$

where $M_7 = \sup_{n\geq 1}(2\|x_n - p\| \cdot M_2 + \beta_n M_2^2)$. From (3.9) and (3.29), it follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n(\|x_n - p\|^2 + \beta_n M_6 + M_5) + (1-\beta_n)(\|x_n - p\|^2 + \beta_n M_7) \\
&\quad - (1-\beta_n)(1-\mu)\|y_n - w_n\|^2 - (1-\beta_n)(1-3\mu)\|z_n - y_n\|^2 \\
&\quad - (1-\beta_n)2\lambda_n\|(I - P_Q)Ay_n\|^2 \\
&\quad - (1-\beta_n)(1-\gamma_n)(\gamma_n - \kappa)\|Tz_n - z_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|T_n z_n - \varphi(u_n)\|^2 \\
&\leq \|x_n - p\|^2 + \beta_n M_8 - (1-\beta_n)(1-\mu)\|y_n - w_n\|^2 \\
&\quad - (1-\beta_n)(1-3\mu)\|z_n - y_n\|^2 - 2(1-\beta_n)\lambda_n\|(I - P_Q)Ay_n\|^2 \\
&\quad - (1-\beta_n)(1-\gamma_n)(\gamma_n - \kappa)\|Tz_n - z_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|T_n z_n - \varphi(u_n)\|^2, \qquad (3.30)
\end{aligned}
$$

where $M_8 = \sup_{n\geq 1}(\beta_n M_6 + M_5 + (1-\beta_n)M_7)$. On the other hand,

$$
\begin{aligned}
\|u_n - p\|^2 &\leq \big(\|x_n - p\| + a_n\|x_n - x_{n-1}\|\big)^2 \\
&\leq \|x_n - p\|^2 + a_n\|x_n - x_{n-1}\| \cdot M_3,
\end{aligned}
$$

where $M_3 = \sup_{n\geq 1}(2\|x_n - p\| + a_n\|x_n - x_{n-1}\|) > 0$. Similarly,

$$
\|w_n - p\|^2 \leq \|x_n - p\|^2 + b_n\|x_n - x_{n-1}\| \cdot M_4,
$$

where $M_4 = \sup_{n\geq 1}(2\|x_n - p\| + b_n\|x_n - x_{n-1}\|) > 0$. Using Lemma 2.1, (3.9) and (3.10), we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n\big(\varphi(u_n) - \varphi(p)\big) + (1-\beta_n)(T_n z_n - p) + \beta_n\big(\varphi(p) - p\big)\|^2 \\
&\leq \|\beta_n\big(\varphi(u_n) - \varphi(p)\big) + (1-\beta_n)(T_n z_n - p)\|^2 + 2\beta_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq \beta_n\|\varphi(u_n) - \varphi(p)\|^2 + (1-\beta_n)\|T_n z_n - p\|^2 + 2\beta_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq \beta_n\tau\|u_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 + 2\beta_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq \beta_n\tau\|x_n - p\|^2 + a_n\|x_n - x_{n-1}\| \cdot M_3 + (1-\beta_n)\|x_n - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + b_n\|x_n - x_{n-1}\| \cdot M_4 + 2\beta_n\langle \varphi(p) - p, x_{n+1} - p\rangle \\
&= (1-(1-\tau)\beta_n)\,\|x_n - p\|^2 + M_3 a_n\|x_n - x_{n-1}\| + M_4 b_n\|x_n - x_{n-1}\| \\
&\quad + 2\beta_n\langle \varphi(p) - p, x_{n+1} - p\rangle.
\end{aligned}
$$

This is obviously equivalent to (3.28).

Now let p be the unique fixed point of the contraction $P_{S\cap F(T)}\varphi$; thus $p = P_{S\cap F(T)}\varphi(p)$, or the unique solution to the variational inequality:

$$\langle \varphi(p) - p, q - p\rangle \le 0, \quad q \in S \cap F(T). \tag{3.31}$$

Setting $s_n = \|x_n - p\|^2$, $\gamma_n = (1-\tau)\beta_n$, and

$$\delta_n = \frac{1}{1-\tau}\left\{\frac{a_n M_3}{\beta_n}\|x_n - x_{n-1}\| + \frac{b_n M_4}{\beta_n}\|x_n - x_{n-1}\| + 2\langle \varphi(p) - p, x_{n+1} - p\rangle\right\},$$

we then rewrite (3.28) in the form

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n\delta_n. \tag{3.32}$$

In order to use Lemma 2.10 to prove that $s_n \to 0$, we take a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \ge 0$. In order to verify that $\limsup_{k\to\infty}\delta_{n_k} \le 0$, we observe that

$$\frac{a_n}{\beta_n}\|x_n - x_{n-1}\| \le \frac{\varepsilon_n}{\beta_n} \to 0 \ (n\to\infty), \quad \frac{b_n}{\beta_n}\|x_n - x_{n-1}\| \le \frac{\eta_n}{\beta_n} \to 0 \ (n\to\infty).$$

It turns out that

$$\limsup_{k\to\infty}\delta_{n_k} = \frac{2}{1-\tau}\cdot\limsup_{k\to\infty}\langle \varphi(p) - p, x_{n_k+1} - p\rangle. \tag{3.33}$$

In the meanwhile, an easy observation of (3.30) is $s_{n_k+1} - s_{n_k} \le M_8\beta_{n_k} \to 0$. Hence, we must have $\lim_{k\to\infty}(s_{n_k+1} - s_{n_k}) = 0$. Then again from (3.30) we obtain

(i) $\|y_{n_k} - w_{n_k}\| \to 0$,
(ii) $\|z_{n_k} - y_{n_k}\| \to 0$,
(iii) $\|(I - P_Q)Ay_{n_k}\| \to 0$,
(iv) $\|Tz_{n_k} - z_{n_k}\| \to 0$,
(v) $\|T_{n_k}z_{n_k} - \varphi(u_{n_k})\| \to 0$.

With no loss of generality, we may assume $x_{n_k} \rightharpoonup x^*$. We shall show that $x^* \in S \cap F(T)$. Since $T_n z_n = \gamma_n z_n + (1-\gamma_n)Tz_n$ for all n, we derive that

$$
\begin{aligned}
\|T_{n_k}z_{n_k} - z_{n_k}\| &= (1-\gamma_{n_k})\|Tz_{n_k} - z_{n_k}\| \le \|Tz_{n_k} - z_{n_k}\| \to 0, \\
\|x_{n_k+1} - T_{n_k}z_{n_k}\| &= \beta_{n_k}\|\varphi(u_{n_k}) - T_{n_k}z_{n_k}\| \to 0, \\
\|x_{n_k+1} - x_{n_k}\| &\le \|x_{n_k+1} - T_{n_k}z_{n_k}\| + \|T_{n_k}z_{n_k} - z_{n_k}\| \\
&\quad + \|z_{n_k} - y_{n_k}\| + \|y_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0.
\end{aligned}
$$

Consequently, it follows that $x_{n_k+1} \rightharpoonup x^*$, $w_{n_k} \rightharpoonup x^*$, $y_{n_k} \rightharpoonup x^*$ (thus $Ay_{n_k} \rightharpoonup Ax^*$), and $z_{n_k} \rightharpoonup x^*$. Furthermore, by (iii) and (iv) together with the demiclosedness principle of nonexpansive mappings, we arrive at $(I - P_Q)Ax^* = 0$ and $Tx^* = x^*$. This yields that $x^* \in S \cap F(T)$ and (3.34) is then reduced to

$$\limsup_{k\to\infty}\delta_{n_k} = \frac{2}{1-\tau}\langle \varphi(p) - p, x^* - p\rangle \le 0 \tag{3.34}$$

due to VI (3.31) with $q = x^*$.

Therefore, Lemma 2.10 is applicable to (3.32) to get $s_n \to 0$ as $n \to \infty$; namely, $x_n \to p$ in norm. The proof is complete. $\square$

4. Numerical experiments

In this section, we provide some numerical experiments in signal recovery to compare our algorithm with those of Suantai [34] and Gibali [19]. Our numerical experiments have been performed in Windows 10 using MATLAB R2016b. Let $H = \mathbb{R}^2$, $C = \{x \in H \mid \|x\| \leq 1\}$, $Q = \{x \in H \mid x = 0\}$ and the matrix A be randomly generated by a standardized normal distribution. Suppose that $f : H \to \mathbb{R}$ is defined by

$$f(x) = \frac{1}{2}\|x\|^2, \quad x \in H.$$

We define T by $Tx := x - \frac{1}{2}\sin x$ for all $x \in C$. (Note that $\sin x$ is defined componentwise for $x \in H$.) Then T is a 0.8-strictly pseudocontractive mapping. It is readily seen that $(0,0)^\top$ is the unique common solution of problems (1.1) and (1.8). The nearest point projection onto C is

$$P_C(x) = \begin{cases} x, & \text{if } x \in C, \\ \dfrac{x}{\|x\|}, & \text{otherwise.} \end{cases}$$

To show the efficiency of our algorithm, we compare it with the algorithms proposed in Suantai [34] and Gibali [19]. For the sake of convenience, we denote Algorithm 1 by Algo I, the algorithm in [34] by Algo II, and the algorithm in [19] by Algo III, respectively. Furthermore, we use $\|x_n\| < 10^{-5}$ as a stopping criterion.

It is easy to see that our proposed algorithm Algo I converges faster than both Algo II and Algo III, which indicates that our algorithm can indeed accelerate the convergence of some existing algorithms.

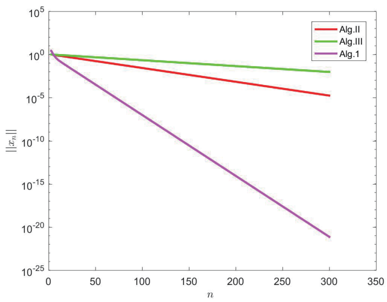

FIGURE 1. The number of iterations for $a_n = 0.5$, $b_n = 0.5$, $\beta_n = 0.5$

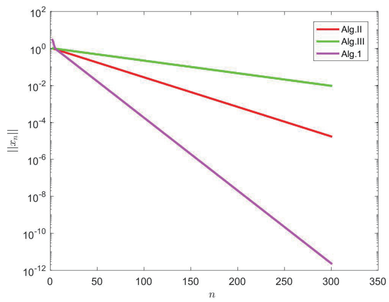

FIGURE 2. The number of iterations for $a_n = 0.4$, $b_n = 0.0$, $\beta_n = 0.5$

5. Conclusion

In this paper, we proposed and studied the convergence of the general inertial algorithms with Armijo type step sizes in real Hilbert spaces to solve the split feasibility and fixed point problems for κ-strictly pseudocontractive mappings. In addition, we also established the weak and strong convergence theorems of these algorithms under mild conditions. Finally, we presented some numerical experiments on our methods in comparison with other existing methods. The results showed that our methods improve and extend the corresponding results in Suantai [34] and Gibali [19] to a certain extent.

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Manuscript received October 9, 2023
revised May 11, 2024

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