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CONVERGENCE OF AN ITERATIVE SCHEME FOR REICH TYPE NON-SELF NONEXPANSIVE MAPPINGS

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Abstract. This paper aims to introduce and study a new class of mappings called a pair of Reich type non-self nonexpansive mappings. In the case that H is a real Hilbert space a sequence converges strongly to a common best proximity point of a pair of Reich type non-self nonexpansive mappings. Our scheme does not involve computation of a closed and convex set C_n and use metric projection from H onto C_n for each $n \geq 1$. Our results improve and generalize many of the results in the literature.

1. Introduction

Let A and B be two nonempty subsets of a real Hilbert space H. A mapping $T : A \to A$ is called nonexpansive if for all $u, v \in A$ we have $||Tu-Tv|| \leq ||u-v||$. A point $u \in A$ is said to be a fixed point of T if $T(u) = u$. The set of fixed points of T will be denoted by $F(T)$. In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable mapping. If the mapping T is non-self mapping, that is, $T : A \to B$ with $A \cap B = \emptyset$, then the equation $T(u) = u$ does not have a solution. In this situation, it is natural to determine an approximate solution u such that the error $\varphi(u) = d(u, Tu)$ is minimum. Such point u is a global minimization of the function φ and it is called the set of best proximity point of T if $d(u, Tu) = d(A, B)$, where $d(A, B) = \inf\{||u - z|| : u \in A, z \in B\}$. The set of best proximity points of T in A is denoted by $Best_A(T)$.

It is well-known that the existence of best proximity points for some nonlinear mappings can be applied to solve problems in equilibrium problems, economics and others (see, for example [19,20,28,41]). In 2017, Pirbavafa and Vaezpour [28] studied that existence of equilibrium pair in free abstract of economies can be guaranteed by best proximity point theory. In recent years, the concept of best proximity point attracted the attention of many mathematicians, see for instance [6–8]. For the existence of best proximity points of mappings, we refer [3, 11, 25, 38] and references therein.

For a given pair of nonempty subsets (A, B) of a Hilbert space H, its proximal pair is the pair (A_0, B_0) defined by

$$A_0 := \{u \in A : ||u - v|| = d(A, B), \text{for some } v \in B\},$$
$$B_0 := \{v \in B : ||u - v|| = d(A, B), \text{for some } u \in A\}.$$

Proximal pair may be empty. The pair (A_0, B_0) is nonempty weakly compact convex whenever A and B are nonempty weakly compact and convex (see, [36]). We remark that if (A, B) is a pair of nonempty subsets of a Hilbert space H such that $d(A, B) >$

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0, then $A_0 \subset \partial A$ and $B_0 \subset \partial B$, where ∂A denotes the boundary of A (see, [36]). We note that the best proximity point reduces to a fixed point of T if T is a self mapping.

Let $T : A \to A$ be a mapping with the following property:

$$||T(u) - T(v)|| \leq \frac{1}{2}\big(||Tu - u|| + ||Tv - v||\big) \text{ for all } u, v \in A. \tag{1.1}$$

A mapping satisfying (1.1) is known as *Kannan nonexpansive mapping* and need not be continuous (see, [16,17]). We note that the class of nonexpansive mappings and the class of mappings satisfying (1.1) are independent (see, [26]). In 1980, Greguš [12] considered these two classes of mappings and studied the following class of mappings:

$$||Tu - Tv|| \leq a||u - v|| + b||Tu - u|| + c||Tv - v|| \text{ for all } u, v \in A, \tag{1.2}$$

where $a, b, c \geq 0$ such that $a + b + c = 1$. If $a + b + c < 1$, then the mapping T satisfying (1.2) is known as Reich contraction (see, [25,33–35]). It can be easily seen that if the mapping satisfies (1.2), then it satisfies the following condition:

$$||Tx - Ty|| \leq a||x - y|| + \alpha(||Tx - x|| + ||Ty - y||) \text{ for all } x, y \in A, \tag{1.3}$$

where $a, \alpha \geq 0$ such that $a + 2\alpha = 1$ and $\alpha = \frac{b+c}{2}$. If $\alpha \in [0, 1)$, then $a = 1 - 2\alpha$ and (1.3) becomes

$$||Tu - Tv|| \leq \alpha||Tu - u|| + \alpha||Tv - v|| + (1 - 2\alpha)||u - v|| \text{ for all } u, v \in A. \tag{1.4}$$

These class of mappings was initially studied in 2019, by Pandey *et al.* [26]. If $\alpha = \frac{1}{2}$, then (1.4) becomes the class of mappings satisfying (1.1) and if $\alpha = 0$, then (1.4) reduces to nonexpansive mapping.

In 2020, Pant *et al.* [27] extended the mapping satisfying (1.4) to non-self cases as follows:

Definition 1.1. A mapping $T : A \to B$ is called *Reich type non-self nonexpansive* if there exists an $\alpha \in [0, 1)$ such that for all $u, v \in A$

$$||Tu - Tv|| \leq \alpha||u - P_A(Tu)|| + \alpha||v - P_A(Tv)|| + (1 - 2\alpha)||u - v||. \tag{1.5}$$

We observe that, every non-self nonexpansive mapping is Reich type non-self nonexpansive mapping but the converse need not be true (see, [26]).

Definition 1.2. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. Then the sequence $\{u_n\}$ in A is said to be *approximate best proximity point sequence* for T if

$$\lim_{n\to\infty} d(u_n, Tu_n) = d(A, B).$$

Definition 1.3. Let H be a real Hilbert space and (A, B) be a pair of nonempty subsets of H. A mapping $T : A \to B$ is said to satisfy *proximal point property* if for every sequence $\{u_n\}$ in A such that $u_n \rightharpoonup x \in A$ and $\{u_n\}$ is an approximate best proximity point sequence for T, we have $||u - Tu|| = d(A, B)$ or $u \in Best_A(T)$.

Several authors (see, e.g., [1, 2, 4, 9–11, 13, 15, 20, 21, 23, 24, 31, 32, 37, 42]) studied convergence results of fixed points and common fixed points using some well known iterative processes. Thus, it is natural to consider the problem of best proximity and common best proximity points of non-self mappings. In line with this, a number of best proximity point results have been obtained by many mathematicians [11, 15, 20–22] and the references therein.

More recently, Pant *et al.* [27] studied the method of approximation of best proximity points of a Reich type non-self nonexpansive mapping T using the following Krasnosel'skií -Mann type algorithm:

$$\begin{aligned} &u_1 \in A_0, \\ &u_{n+1} = P_A(\beta_n P_B u_n + (1-\beta_n)Tu_n), \end{aligned} \tag{1.6}$$

where $\beta_n \in [a, b] \subset (0, 1)$. They proved that the sequence $\{u_n\}$ weakly converges to a best proximity point of T in A under mild assumptions on T. In addition, they employed hybrid algorithm to obtain strong convergence theorem of best proximity point for a Reich type non-self nonexpansive mapping T, in Hilbert spaces. In fact, they proved the following theorem.

Theorem P ([27])**.** *Let H be a Hilbert space and (A, B) be a pair of nonempty closed convex subsets of H. Let $T : A \to B$ be a Reich type non-self nonexpansive mapping such that $T(A_0) \subset B_0$ and satisfy the proximal property. Let $\beta_n \in [0, \beta]$ for each $n \in \mathbb{N}$, $\beta \in (0, 1)$, $Best_A(T) \neq \emptyset$, $x \in H$ and $C_1 = A_0$. Given $u_1 = P_{C_1}(x)$, define a sequence $\{u_n\}$ as follows:*

$$\begin{cases} v_n = \beta_n u_n + (1-\beta_n)P_A(T(u_n)), \\ C_{n+1} := \{w \in C_n : ||v_n - w|| \le ||u_n - w_n||\}, \\ u_{n+1} = P_{C_{n+1}}(x). \end{cases} \tag{1.7}$$

Then, the sequence $\{u_n\}$ converges strongly to $y = P_{Best_A(T)}(x)$.

We remark that the computation of u_{n+1} in Algorithms (1.7) is not simple in applications because of the involvement of computations of C_n for each $n \ge 1$ and the metric projection on C_n.

It is our purpose in this paper to introduce a pair of Reich type non-self nonexpansive mappings and study an *Ishikawa type iterative process* that converges strongly to a common best proximity point of a pair of Reich type nonexpansive non-self mappings. As a consequence, we obtain the *Mann type iteration scheme* for approximating the best proximity point of Reich type nonexpansive non-self mapping. Moreover, the assumption that T satisfies the proximity property is not required. Our schemes do not involve computation of C_n to obtain u_{n+1} for each $n \ge 1$. Our theorems extend and unify most of the results that have been proved for this important class of nonlinear mappings.

2. Preliminaries

This section contains some basic definitions and results that will be used in our subsequent analysis. Let $A \subset H$ be a nonempty, closed, and convex subset of H.

For any $x \in H$, the projection mapping $P_A : H \to A$ is defined by

$$\| P_A x - x \| = \inf_{y \in A} \| x - y \| .$$

It is also known that P_A satisfies

$$\| P_A x - P_A y \|^2 \leq \langle P_A x - P_A y, x - y \rangle, \text{ for all } x, y \in H. \tag{2.1}$$

In particular, P_A is nonexpansive.

We shall need the following definitions.

Definition 2.1 ([29,30])**.** Let (X, d) be a metric space and (A, B) a pair of nonempty subsets of X such that $A_0 \neq \emptyset$. Then, the pair (A, B) is said to satisfy the P-property if,

$$d(u_1, v_1) = d(A, B) \text{ and } d(u_2, v_2) = d(A, B) \text{ implies } d(u_1, u_2) = d(v_1, v_2),$$

where $u_1, u_2 \in A_0$ and $v_1, v_2 \in B_0$.

It is shown in [30] that the pair (A, B) satisfies the P-property if (A, B) is a pair of nonempty, closed and convex subsets of a Hilbert space H.

The following lemma will be used in our convergence analysis.

Lemma 2.2 ([39])**.** *Let H be a Hilbert space and (A, B) be a pair of nonempty subsets of H such that B is closed and convex. Then, $||u - P_B(u)|| = d(A, B)$ for all $u \in A_0$.*

Lemma 2.3 ([39])**.** *Let H be a Hilbert space and (A, B) be a pair of nonempty subsets of H such that A is closed and convex. Let $T : A \to B$ be a mapping such that $T(A_0) \subset B_0$. Then, $F(P_A \circ T|_{A_0}) = Best_A(T)$.*

Lemma 2.4 ([40])**.** *Let H be a Hilbert space and (A, B) be a pair of nonempty subsets of H such that B is closed and convex. Let $T : A \to B$ be a mapping such that $T(A_0) \subset B_0$. Then, $P_B(u) = T(u)$ for all $u \in Best_A(T)$.*

3. Main results

In this section, we introduce a pair of Reich type non-self nonexpansive mappings and prove a strong convergence theorem for finding a common element of the set of solutions for the pair of mappings.

Definition 3.1. Let A and B be subsets of a Hilbert space H and let T and S be mappings from A into B. The pair T and S is said to be Reich type non-self nonexpansive mappings if

$$||Tu - Sv|| \leq \alpha||u - P_A(Tu)|| + \alpha||v - P_A(Sv)|| + (1 - 2\alpha)||u - v||, \tag{3.1}$$

for all $u, v \in A$, $\alpha \in [0, 1)$.

Example of a pair of Reich type non-self nonexpansive mappings is given below.

Example 3.2 ([27])**.** Let $A = [0, 4]$ and $B = [5, 6]$ be subsets of $\mathbb{R}$ endowed with the usual norm. Define a mapping $T, S : A \to B$ by:

$$Tx = \begin{cases} 5 + x, & \text{if } x \in [0, 1]; \\ 5, & \text{otherwise,} \end{cases} \tag{3.2}$$

and

$$Sx = \begin{cases} 6, & \text{if } x \in [0,1]; \\ 5, & \text{otherwise.} \end{cases} \tag{3.3}$$

We note that $P_A(Sx) = 4 = P_A(Tx)$ for all $x \in [0,4]$. Moreover, we consider the following cases to show that T and S form a pair of Reich type non-self nonexpansive mappings for $\alpha \in [\frac{1}{4}, \frac{1}{2}]$.

a) If $x \in [0,1]$ and $y \in (1,4]$, then we have

$$||Tx - Sy|| = |x| \quad \leq \quad \alpha||x - P_ATx|| + \alpha||y - P_ASy|| + (1-2\alpha)|x-y|.$$

b) If $x \in [0,1]$ and $y \in [0,1]$, then one can show that

$$||Tx - Sy|| = 1 - x \quad \leq \quad \alpha||x - P_ATx|| + \alpha||y - P_ASy|| + (1-2\alpha)|x-y|.$$

c) If $y \in [0,1]$ and $x \in (1,4]$, then we obtain

$$||Tx - Sy|| = 1 \quad \leq \quad \alpha||x - P_ATx|| + \alpha||y - P_ASy|| + (1-2\alpha)|x-y|.$$

d) If $y \in (1,4]$ and $y \in (1,4]$, then we get

$$||Tx - Sy|| = 0 \quad \leq \quad \alpha||x - P_ATx|| + \alpha||y - P_ASy|| + (1-2\alpha)|x-y|.$$

Therefore, we conclude that T and S form a pair of Reich type non-self nonexpansive mappings with constant $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ and the common best proximity point $x = 4$.

By observing the construction of Ishikawa's iteration [14], we construct the following *Ishikawa's type scheme* for a pair of Reich type non-self nonexpansive mappings $T, S : A \to B$, where A and B are closed and convex subsets of a Hilbert space H. Assume that $T(A_0) \subset B_0$ and $S(A_0) \subset B_0$. Let $x_0 \in A_0$. Since $Tx_0 \in B_0$ there exists $u_0 \in A_0$ such that $||u_0 - Tx_0|| = d(A,B)$. Define $y_0 = (1-\delta_0)x_0 + \delta_0 u_0 \in A_0$. This yields that $Sy_0 \in B_0$ and there exists $v_0 \in A_0$ such that $||v_0 - Sy_0|| = d(A,B)$. Next, we define $x_1 = (1-\eta_0)x_0 + \eta_0 v_0$. Thus, by continuing this process, we derive that

$$\begin{cases} y_n = (1-\delta_n)x_n + \delta_n u_n, \\ x_{n+1} = (1-\eta_n)x_n + \eta_n v_n, \end{cases} \tag{3.4}$$

where $u_n \in A_0$ such that $||u_n - Tx_n|| = d(A,B)$ and $v_n \in A_0$ such that $||v_n - Sy_n|| = d(A,B)$, for $\eta_n, \delta_n \in [0,1], \forall n \in \mathbb{N}$.

We now prove our main theorem for a common best proximity point of a pair of Reich type non-self nonexpansive mappings.

Theorem 3.3. *Let H be a real Hilbert space and let A, B be closed and convex subsets of H. Assume that $T, S : A \to B$ are a pair of Reich type non-self nonexpansive mappings with constant $\alpha \in [0,1)$. Suppose the common best proximity point set is nonempty. Then, the sequence $\{x_n\}$ generated by (3.4), where $\alpha \in [0,1)$, $0 < \eta \leq \eta_n < 1$ and $0 \leq \delta_n \leq \delta < 1$, converges strongly to a common best proximity point of T and S.*

Proof. Now, we divide the proof into five steps.

Step 1. We prove that the sequence $\{x_n\}$ is bounded. Let p be a common best proximity point of T and S. So, $||p - Tp|| = d(A,B)$, $||p - Sp|| = d(A,B)$. Thus, by P-property, we have $Tp = Sp$. Moreover, we note that $p = P_ATp$ and since T and

S form a pair of Reich type non-self nonexpansive mappings, for $w \in A$, we obtain that

$$\begin{aligned} ||Sw - Sp|| &= ||Sw - Tp|| \\ &\leq \alpha\big[||w - P_ASw|| + ||p - P_ATp||\big] + (1-2\alpha)||w-p|| \\ &\leq \alpha\big[||p - P_ASw|| + ||p - w||\big] + (1-2\alpha)||w-p|| \\ &\leq \alpha\big[||P_ASw - P_ASp|| + ||p - w||\big] + (1-2\alpha)||w-p|| \\ &\leq \alpha\big[||Sw - Sp|| + ||p - w||\big] + (1-2\alpha)||w-p||. \end{aligned}$$

Therefore,

$$||Sw - Sp|| \leq ||w - p||. \tag{3.5}$$

Similarly, $||Tw - Tp|| \leq ||w - p||$. Now, from (3.4) and P-property we get

$$\begin{aligned} ||x_{n+1} - p|| &= ||(1-\eta_n)x_n + \eta_n v_n - p|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n||v_n - p|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n||Sy_n - Sp|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n||y_n - p|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n||(1-\delta_n)x_n + \delta_n u_n - p|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n(1-\delta_n)||x_n - p|| + \eta_n\delta_n||u_n - p|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n(1-\delta_n)||x_n - p|| + \eta_n\delta_n||Tx_n - Tp|| \\ &\leq (1-\eta_n)||x_n - p|| + \eta_n(1-\delta_n)||x_n - p|| + \eta_n\delta_n||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned}$$

Therefore, by induction we derive that $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty}||x_n - v_n|| = 0$. Suppose that $\lim_{n\to\infty}||x_n - v_n|| \neq 0$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a real number $\epsilon_0 > 0$, such that $||x_{n_k} - v_{n_k}|| \geq \epsilon_0$, for every $k \geq 1$. Moreover, for $p \in Best_A(T)$ and the P-property, we obtain

$$\begin{aligned} ||x_{n_k} - v_{n_k}|| &\leq ||x_{n_k} - p|| + ||p - v_{n_k}|| \\ &= ||x_{n_k} - p|| + ||Sp - Sy_{n_k}|| \\ &\leq ||x_{n_k} - p|| + ||p - y_{n_k}|| \\ &= ||x_{n_k} - p|| + ||(1-\delta_{n_k})(x_{n_k} - p) + \delta_{n_k}(u_{n_k} - p)|| \\ &\leq ||x_{n_k} - p|| + (1-\delta_{n_k})||x_{n_k} - p|| + \delta_{n_k}||u_{n_k} - p|| \\ &\leq ||x_{n_k} - p|| + (1-\delta_{n_k})||x_{n_k} - p|| + \delta_{n_k}||Tx_{n_k} - Tp|| \\ &\leq ||x_{n_k} - p|| + (1-\delta_{n_k})||x_{n_k} - p|| + \delta_{n_k}||x_{n_k} - p|| \\ &= 2||x_{n_k} - p||. \end{aligned}$$

This implies that

$$||x_{n_k} - p|| \;\geq\; \frac{1}{2}||x_{n_k} - v_{n_k}|| \geq \frac{\epsilon_0}{2} = \epsilon_1 > 0. \tag{3.7}$$

In addition, by P-property and (3.5), we have

$$||v_n - p|| \;=\; ||Sy_n - Sp|| \leq ||y_n - p||$$

$$\begin{aligned}
&= ||(1-\delta_n)(x_n-p)+\delta_n(u_n-p)|| \\
&\leq (1-\delta_n)||x_n-p||+\delta_n||u_n-p|| \\
&\leq (1-\delta_n)||x_n-p||+\delta_n||Tx_n-Tp|| \\
&\leq (1-\delta_n)||x_n-p||+\delta_n||x_n-p|| \\
&\leq ||x_n-p||.
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists $L > 0$ such that $||x_n - x^*|| \leq L$ and hence

$$\left\|\frac{x_{n_k}-p}{||x_{n_k}-p||}-\frac{v_{n_k}-p}{||x_{n_k}-p||}\right\| = \left\|\frac{x_{n_k}-v_{n_k}}{||x_{n_k}-p||}\right\| \geq \frac{\epsilon_0}{L} > 0, \tag{3.8}$$

and

$$\left\|\frac{x_{n_k}-p}{||x_{n_k}-p||}\right\| = 1, \left\|\frac{v_{n_k}-p}{||x_{n_k}-p||}\right\| \leq 1. \tag{3.9}$$

Thus, the fact that H is uniformly convex implies that there exists $\gamma > 0$ such that

$$\left\|\frac{x_{n_k}-p}{||x_{n_k}-p||}+\frac{v_{n_k}-p}{||x_{n_k}-p||}\right\| \leq 2-\gamma. \tag{3.10}$$

Now, from (3.4) and the inequality in (3.10), we have

$$\begin{aligned}
||x_{n_k+1}-p|| &= ||(1-\eta_{n_k})x_{n_k}+\eta_{n_k}v_{n_k}-p|| \\
&\leq (1-2\eta_{n_k})||x_{n_k}-p|| + ||\eta_{n_k}(x_{n_k}-p)+\eta_{n_k}(v_{n_k}-p)|| \\
&= (1-2\eta_{n_k})||x_{n_k}-p|| + \eta_{n_k}||x_{n_k}-p||\left\|\frac{x_{n_k}-p}{||x_{n_k}-p||}+\frac{v_{n_k}-p}{||x_{n_k}-p||}\right\| \\
&\leq (1-2\eta_{n_k})||x_{n_k}-p|| + (2-\gamma)\eta_{n_k}||x_{n_k}-p|| \\
&= (1-\gamma\eta_{n_k})||x_{n_k}-p|| \\
&\leq ||x_{n_k}-p|| - \gamma\eta_{n_k}\epsilon_1 \\
&\leq ||x_{n_k}-p|| - \gamma\eta\epsilon_1,
\end{aligned}$$

and this implies that $||x_{n_k+1}-p|| \leq ||x_{n_k}-p||$. Furthermore, we derive that

$$\begin{aligned}
||x_{n_k+1}-p|| &\leq ||x_{n_k}-p|| - \gamma\eta\epsilon_1 \\
&\leq ||x_{n_k-1}-p|| - \gamma\eta\epsilon_1 \\
&\leq \dots \\
&\leq ||x_{n_{k-1}}-p|| - \gamma\eta\epsilon_1,
\end{aligned}$$

and

$$\begin{aligned}
||x_{n_k}-p|| &\leq ||x_{n_{k-1}}-p|| - \gamma\eta\epsilon_1 \\
&\leq ||x_{n_{k-2}}-p|| - 2\gamma\eta\epsilon_1 \\
&\leq \dots \\
&\leq ||x_{n_1}-p|| - (k-1)\gamma\eta\epsilon_1. \qquad (3.11)
\end{aligned}$$

Letting $k \to \infty$, to both sides of (3.11) we obtain that $\lim_{k\to\infty} ||x_{n_k}-p|| \leq 0$, which is a contradiction to (3.7). Therefore, $\lim_{k\to\infty} ||x_n - v_n|| = 0$.

Step 3. We show that $\lim_{n\to\infty}||x_n - u_n|| = 0$. Note that, since $||u_n - Tx_n|| = d(A,B) = ||v_n - Sy_n||$ we have $||u_n - P_A Tx_n|| = 0$ and $||v_n - P_A Sy_n|| = 0$. Moreover, by P-property we get $||u_n - v_n|| = ||Tx_n - Sy_n||$ and hence

$$
\begin{aligned}
||x_n - u_n|| &\leq ||x_n - v_n|| + ||v_n - u_n|| \\
&= ||x_n - v_n|| + ||Tx_n - Sy_n|| \\
&\leq ||x_n - v_n|| + \alpha[||x_n - P_A Tx_n|| + ||y_n - P_A Sy_n||] \\
&\quad +(1-2\alpha)||x_n - y_n|| \\
&\leq ||x_n - v_n|| + \alpha[||x_n - u_n|| + ||u_n - P_A Tx_n|| + ||y_n - v_n|| \\
&\quad +||v_n - P_A Sy_n||\big] + (1-2\alpha)||x_n - y_n|| \\
&\leq ||x_n - v_n|| + \alpha[||x_n - u_n|| + ||y_n - v_n||\big] + (1-2\alpha)||x_n - y_n||, \\
&\leq ||x_n - v_n|| + \alpha||x_n - u_n|| + (1-2\alpha)||x_n - y_n||, \\
&\quad +\alpha[||(1-\delta_n)x_n + \delta_n u_n - v_n||] \\
&\leq ||x_n - v_n|| + \alpha||x_n - u_n|| + (1-2\alpha)||x_n - y_n|| \\
&\quad +\alpha(1-\delta_n)||x_n - v_n|| + \alpha\delta_n||u_n - v_n|| \\
&\leq ||x_n - v_n|| + \alpha||x_n - u_n|| + (1-2\alpha)||x_n - y_n|| \\
&\quad +\alpha(1-\delta_n)||x_n - v_n|| + \alpha\delta_n[||u_n - x_n|| + ||x_n - v_n||] \\
&\leq (1+\alpha)||x_n - v_n|| + \big[(\alpha(1+\delta_n) + (1-2\alpha)\delta_n\big]|x_n - u_n||,
\end{aligned}
$$

which implies that

$$
||x_n - u_n|| \leq \frac{1+\alpha}{D}||x_n - v_n|| \to 0, \tag{3.12}
$$

as $n \to \infty$, where $D = (1 - \alpha(1-\delta_n) - \delta_n > 0$, and hence

$$
||y_n - u_n|| = (1-\delta_n)||x_n - u_n|| \to 0 \text{ as } n \to \infty. \tag{3.13}
$$

Step 4: We show that if the sequence $\{x_n\}$ converges to x, then x is the common best proximity point of T and S.

The fact that $||u_n - Tx_n|| = d(A,B) = ||v_n - Sy_n||$ and P-property imply that $||u_n - v_n|| = ||Tx_n - Sy_n||$. Thus, we obtain

$$
\begin{aligned}
||u_n - v_n|| &= ||Tx_n - Sy_n|| \\
&\leq \alpha\big[|x_n - P_A Tx_n|| + ||y_n - P_A Sy_n||\big] + (1-2\alpha)||x_n - y_n| \\
&\leq \alpha\big[||x_n - u_n|| + ||u_n - P_A Tx_n||\big] + (1-2\alpha)||x_n - y_n|| \\
&\quad +\alpha\big[||y_n - v_n|| + ||v_n - P_A Sy_n||\big] \\
&\leq \alpha\big[||x_n - u_n|| + ||y_n - v_n||\big] + (1-2\alpha)||x_n - y_n|| \\
&\leq \alpha\big[||x_n - v_n|| + ||v_n - u_n|| + (1-\delta_n)||x_n - v_n|| + \delta_n||u_n - v_n||\big] \\
&\quad +(1-2\alpha)\delta_n[||u_n - v_n|| + ||x_n - v_n||] \\
&\leq (2\alpha + \delta_n(1-3\alpha))||x_n - v_n|| + (\alpha + \delta_n(1-\alpha))||v_n - u_n||,
\end{aligned}
$$

which implies that

$$
||u_n - v_n|| \leq \frac{2\alpha + \delta_n(1-3\alpha)}{1 - (\alpha + \delta_n(1-\alpha))}||x_n - v_n|| \to 0, \text{ as } n \to \infty. \tag{3.14}
$$

Furthermore, from (3.4), (3.12) and (3.14) we get $||x_n - y_n|| = \delta_n||u_n - x_n|| \to 0$ and $||y_n - v_n|| \leq (1-\delta_n)||x_n - v_n|| + \delta_n||v_n - u_n|| \to 0$ as $n \to \infty$. By assumption $x_n \to x$ implies that $v_n \to x$ and $u_n \to x$ and hence from (3.12) we get $y_n \to x$. Thus, we get

$$\begin{aligned}
||u_n - Sx|| - d(A,B) &\leq ||u_n - Tx_n|| + ||Tx_n - Sx|| - d(A,B)\\
&= ||Tx_n - Sx||\\
&\leq \alpha||x_n - P_A(Tx_n))|| + \alpha||x - P_A(Sx)|| + (1-2\alpha)||x_n - x||\\
&\leq \alpha\Big[||x_n - u_n|| + ||u_n - P_A(Tx_n)||\Big]\\
&\quad + \alpha\Big[||x - Sx|| + ||Sx - P_A(Sx)||\Big] + (1-2\alpha)||x_n - x||,
\end{aligned} \tag{3.15}$$

and taking the limit both sides as $n \to \infty$ we obtain that $(1-\alpha)\big[||x - Sx|| - d(A,B)\big] \leq 0$ and hence the fact that $\alpha < 1$ implies that $||x - Sx|| = d(A,B)$. Similarly, we obtain that $||x - Tx|| = d(A,B)$.

Step 5. Next, we show that $\{u_n\}$ is Cauchy. Note that

$$||u_n - u_{n+m}|| \quad \leq \quad ||u_n - v_{n+m}|| + ||v_{n+m} - u_{n+m}||. \tag{3.16}$$

From $||u_n - Tx_n|| = d(A,B) = ||v_{n+m} - Sy_{n+m}||$ and P-property, we have $||u_n - v_{n+m}|| = ||Tx_n - Sy_{n+m}||$. Therefore,

$$\begin{aligned}
||u_n - u_{n+m}|| &\leq ||Tx_n - Sy_{n+m}|| + ||u_{n+m} - v_{n+m}||\\
&\leq \alpha||x_n - P_ATx_n|| + \alpha||y_{n+m} - P_ASy_{n+m}|| + (1-2\alpha)||x_n - y_{n+m}||\\
&\quad + ||u_{n+m} - v_{n+m}||\\
&\leq \alpha\big[||x_n - u_n|| + ||u_n - P_ATx_n||\big]\\
&\quad + \alpha\big[||y_{n+m} - v_{n+m}|| + ||v_{n+m} - P_ASy_{n+m}||\big]\\
&\quad + (1-2\alpha)\big[||x_n - u_n|| + ||u_n - u_{n+m}|| + ||u_{n+m} - y_{n+m}||\big]\\
&\quad + ||u_{n+m} - v_{n+m}||\\
&\leq \alpha\big[||x_n - u_n|| + ||y_{n+m} - v_{n+m}||\big]\\
&\quad + (1-2\alpha)\big[||x_n - u_n|| + ||u_n - u_{n+m}|| + ||u_{n+m} - y_{n+m}||\big]\\
&\quad + ||u_{n+m} - v_{n+m}||.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
||u_n - u_{n+m}|| \quad &\leq \quad \frac{(1-\alpha)}{2\alpha}||x_n - u_n|| + \frac{1}{2}||y_{n+m} - v_{n+m}||\\
&\quad + \frac{(1-2\alpha)}{2\alpha}||u_{n+m} - y_{n+m}|| + \frac{1}{2\alpha}||u_{n+m} - v_{n+m}||.
\end{aligned} \tag{3.17}$$

Therefore,

$$||u_n - u_{n+m}|| \to 0 \text{ as } n \to m, n \to \infty.,$$

This shows that $\{u_n\}$ is Cauchy. Thus, there exists $q \in A_0$ such that $\lim_{n\to\infty} u_n = q$ and hence $\lim_{n\to\infty} x_n = q$. Therefore, by Step 4, q is the common best proximity point of T and S. This completes the proof. $\square$

If in Theorem 3.3, we assume that $T = S$, we obtain the following corollary.

Corollary 3.4. *Let H be a real Hilbert space and A, B be closed and convex subsets of H. Assume that $T : A \to B$ is a Reich type non-self nonexpansive mappings with constant $\alpha \in [0, 1)$. Suppose best proximity point set $Best_A(T) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by (3.4) with $T = S$, where $\alpha \in [0, 1)$, $0 < \eta \leq \eta_n < 1$ and $0 \leq \delta_n \leq \delta < 1$, converges strongly to a common best proximity point of T.*

If in Theorem 3.3, we assume that $T = S$ and $\delta_n = 0$, then we obtain the following scheme called Mann's type iteration scheme:

$$x_{n+1} = (1 - \eta_n)x_n + \eta_n v_n, \tag{3.18}$$

where $v_n \in A_0$ such that $||v_n - Tx_n|| = d(A, B)$, and $\eta_n \in [0, 1], \forall n \in \mathbb{N}$.

Next, we establish a convergence of Mann's type scheme (3.5) for a best proximity point of a single Reich type non-self nonexpansive mapping.

Theorem 3.5. *Let A, B be closed and convex subsets of a real Hilbert space H. Assume that $T : A \to B$ is Reich type non-self nonexpansive mapping with constant $\alpha \in [0, 1)$. Suppose best proximity point set $Best_A(T) \neq \emptyset$. For arbitrary $v_0 \in A_0$, the sequence generated by (3.18), where $0 < \eta \leq \eta_n < 1$, converges strongly to the best proximity point of T.*

Proof. Taking $T = S$ and $\delta_n = 0$, in the proof of Theorem 3.3 we obtain the required assertion. $\square$

Corollary 3.6. *Let A, B be closed and convex subsets of a real Hilbert space H. Assume that $T : A \to B$ is Kannan nonexpansive mapping. Suppose best proximity point set $Best_A(T) \neq \emptyset$. For arbitrary $v_0 \in A_0$, the sequence generated by (3.18), where $0 < \eta \leq \eta_n < 1$, converges strongly to the best proximity point of T.*

Proof. The proof follows from Theorem 3.5 with $\alpha = \frac{1}{2}$. $\square$

Corollary 3.7. *Let A, B be closed and convex subsets of a real Hilbert space H. Assume that $T : A \to B$ is non-self nonexpansive mapping. Suppose best proximity point set $Best_A(T) \neq \emptyset$. For arbitrary $v_0 \in A_0$, the sequence generated by (3.18), where $0 < \eta \leq \eta_n < 1$, converges strongly to the best proximity point of T.*

Proof. The proof follows from Theorem 3.5 with $\alpha = 0$. $\square$

Remark 3.8. In this paper, we have proposed an algorithm for approximating a common best proximity point of a pair of Reich type non-self nonexpansive mappings in real Hilbert spaces. The main result in this paper extends the results of Pant *et al.* [27] to a common best proximity point of a pair of Reich type non-self nonexpansive mappings. Our scheme does not involve computation of C_n to obtain u_{n+1} for each $n \geq 1$. Our scheme does not involve computation of C_n to obtain u_{n+1} for each $n \geq 1$.

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