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STRONG CONVERGENCE THEOREMS FOR FIXED POINTS AND ATTRACTIVE POINTS OF BERINDE NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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This work is dedicated to the memories of Professor Kazimierz Goebel and Professor William Art Kirk.

Abstract. In this paper, we establish strong convergence theorems for fixed points and attractive points of Berinde nonexpansive mappings in complete CAT(0) spaces, utilizing the Mann iterative process. Given the non-necessity of convexity in the set of fixed points for Berinde nonexpansive mappings, and inspired by the concept of attractive points introduced by Takahashi and Takeuchi [21], we employ the convexity property to prove strong convergence theorems that deal with a metric projection mappings onto sets of attractive points in CAT(0) spaces.

1. Introduction

Fixed point theorems are crucial in both theoretical and applied mathematics because they provide a strong foundation for analyzing and solving an extensive number of mathematical problems. Let $T : C \to H$ be a mapping, where C is a nonempty subset of a Hilbert space H. The set of fixed points of T is denoted by $F(T)$, that is,

$$F(T) = \{z \in C : z = Tz\}.$$

The set of attractive points of T is denoted by $A(T)$, that is,

$$A(T) = \{z \in H : \|z - Ty\| \leq \|z - y\|, \quad \forall y \in C\}.$$

A mapping $T : C \to C$ is said to be quasi-nonexpansive if $\|Ty - z\| \leq \|y - z\|$ for all $y \in C$ and for all $z \in F(T)$. For such mappings, it is known that the sets of fixed points are convex. Another technique for solving the fixed point problem is employing the concept of attractive points, which firstly introduced by Takahashi and Takeuchi [21]. Using the properties of attractive points, the authors successfully proved a nonlinear ergodic theorem in Hilbert spaces without assuming convexity.

Moreover, Takahashi and Takeuchi [21] proved the existence of fixed points using the existence of attractive points under some conditions as follows:

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Lemma 1.1 ([21]). *Let C be a nonempty closed convex subset of H and let T be a mapping from C into itself. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$. In particular, if $z \in A(T)$, then $P_C z \in F(T)$ where P_C is the metric projection from H onto C.*

For general mappings defined on Hilbert spaces, a set of fixed points does not necessarily have to be convex, but a set of attractive points is always convex (see [21]).

Lemma 1.2 ([21]). *Let C be a nonempty subset of H and let T be a mapping from C into H. Then $A(T)$ is a closed convex subset of H.*

The concept and some fundamental properties of attractive points were expanded in CAT(0) spaces by [1, 7, 12, 16]. The set of all attractive points of a mapping $T : C \to X$, where X is a metric space and C is a nonempty subset of X, defined as

$$A(T) = \{z \in X : d(z, Ty) \leq d(z, y), \quad \forall y \in C\}. \tag{1.1}$$

In 2013, Kunwai et al. [16] proved some results of $A(T)$ in CAT(0) spaces, analogous to Lemma 1.1 and Lemma 1.2.

Let (X, d) be a complete metric space and C be a nonempty subset of X. A mapping $T : C \to C$ is called a contraction if there exists $c \in (0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in C$. A new type of contraction mapping was introduced by Berinde [2], called weak contraction or (δ, L)-contraction. It is defined by the existence of constants $\delta \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in C$,

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

It is very interesting to study fixed point problems of these mappings because the weak contractions unify a large class of contractive-type mappings.

Chumpungam [6] generalized the concept of weak contractions to what are termed Berinde nonexpansive mappings, extending them to Banach spaces. That is, for all $x, y \in C$,

$$d(Tx, Ty) \leq d(x, y) + Ld(y, Tx). \tag{1.2}$$

By setting $L = 0$, we observe that every nonexpansive mapping is a $(1, 0)$-contraction mapping.

Meanwhile, the researchers focused on fixed point problems for nonlinear mappings, most of which are quasi-nonexpansive (for examples see [1, 7, 11, 12, 16, 17, 21, 23–25]).

The following example demonstrates a Berinde nonexpansive mapping that is neither quasi-nonexpansive nor nonexpansive. Moreover, observe that $F(T)$ is not a convex set.

Example 1.3 ([6]). Let $X = \mathbb{R}, C = [0, 1]$ and define a mapping $T : C \to C$ by

$$Tx = \begin{cases} x^2, & \text{if } x \in [0, \frac{1}{2}), \\ 1, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then T is $(1,4)$-contraction but is not nonexpansive. Since $T(0)=0$ and $T(1)=1$, $F(T)=\{0,1\}$. By choosing $(\frac{1}{3},1)\in C\times F(T)$, we have

$$\left|T\Big(\frac{1}{3}\Big)-1\right|=\frac{8}{9}>\frac{2}{3}=\left|\frac{1}{3}-1\right|.$$

Thus, T is not quasi-nonexpansive.

Research on the problems of fixed points for Berinde nonexpansive mappings continues to inspire many researchers, as evidenced by [5, 6, 15] and references therein.

In this paper, we utilize the Mann iterative process [18] and the concept of attractive points, as defined in (1.1), to prove strong convergence theorems for fixed points and attractive points of Berinde nonexpansive mappings under some conditions in complete CAT(0) spaces. Moreover, since the set of fixed points of Berinde nonexpansive mappings may be nonconvex, we apply the convexity of the set of attractive points to prove strong convergence theorems that deal with a metric projection mapping onto the set of attractive points.

2. Preliminaries

Throughout this paper, we use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{N}$ the set of positive integers. Let (X,d) be a metric space. A geodesic path (or shortly a geodesic) joining u to v in X is a map c from a closed interval $[0,l]\subseteq\mathbb{R}$ to X, such that $c(0)=u, c(l)=v$ and $d(c(s),c(t))=|s-t|$ for all $s,t\in[0,l]$. The image of c is called a geodesic segment joining u and v when it is unique and denoted by $[u,v]$. We denote the unique point $w\in[u,v]$, such that $d(u,w)=\lambda d(u,v)$ and $d(v,w)=(1-\lambda)d(u,v)$ by $(1-\lambda)u\oplus\lambda v$, where $0\leq\lambda\leq 1$.

The metric space (X,d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining a and b for each $a,b\in X$.

A geodesic triangle $\triangle(u_1,u_2,u_3)$ in a geodesic space (X,d) consists of three points u_1,u_2,u_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of the points (the edges of $\triangle$). A comparison triangle for $\triangle(u_1,u_2,u_3)$ in (X,d) is a triangle $\bar{\triangle}(u_1,u_2,u_3):=\triangle(\bar{u}_1,\bar{u}_2,\bar{u}_3)$ in the Euclidean plane $\mathbb{R}^2$, such that $d_{\mathbb{R}^2}(\bar{u}_m,\bar{u}_n)=d(u_m,u_n)$ for all $m,n\in\{1,2,3\}$.

A geodesic space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and $\triangle$ be a comparison triangle in $\mathbb{R}^2$. Subsequently, the triangle is said to satisfy the CAT(0) inequality if

$$d(u,v)\leq d_{\mathbb{R}^2}(\bar{u},\bar{v}),$$

for all $u,v\in\triangle$ and all comparison points $\bar{u},\bar{v}\in\bar{\triangle}$.

For any points p,q,r in a CAT(0) space and let s be the midpoint of the segment $[q,r]$, then the CAT(0) inequality implies the so-called (CN) inequality, i.e.,

$$d^2(p,s)\leq\frac{1}{2}d^2(p,q)+\frac{1}{2}d^2(p,r)-\frac{1}{4}d^2(q,r).$$

It is worth noting that a uniquely geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality, see [4] for more information.

Now, some elementary facts about CAT(0) spaces are mentioned as follows.

Lemma 2.1 ([4])**.** *Let (X,d) be a CAT(0) space, $p,q,s,t \in X$ and $\lambda \in [0,1]$. Then*

(1) $d\big(\lambda p \oplus (1-\lambda)s, \lambda q \oplus (1-\lambda)t\big) \leq \lambda d(p,q) + (1-\lambda)d(s,t)$,

(2) $d\big(\lambda p \oplus (1-\lambda)s, q\big) \leq \lambda d(p,q) + (1-\lambda)d(s,q)$.

Lemma 2.2 ([9])**.** *Let (X,d) be a CAT(0) space and $p,q,s \in X$. Then*

$$d^2\big(\lambda p \oplus (1-\lambda)s, q\big) \leq \lambda d^2(p,q) + (1-\lambda)d^2(s,q) - \lambda(1-\lambda)d^2(p,s),$$

for all $\lambda \in [0,1]$.

The concept of quasilinearization, in metric space X, introduced by Berg and Nikolaev [3]. Note that, for $(a,u) \in X \times X$, we call $\overrightarrow{au}$ a vector in $X \times X$. Quasilinearization is a map $\langle *, * \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined as

$$2\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = d^2(p,s) + d^2(q,r) - d^2(p,r) - d^2(q,s), \tag{2.1}$$

for all $p,q,r,s \in X$. It can be observed that $\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = \langle \overrightarrow{rs}, \overrightarrow{pq} \rangle$, $\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = -\langle \overrightarrow{qp}, \overrightarrow{rs} \rangle$ and $\langle \overrightarrow{pu}, \overrightarrow{rs} \rangle + \langle \overrightarrow{uq}, \overrightarrow{rs} \rangle = \langle \overrightarrow{pq}, \overrightarrow{rs} \rangle$, for all $p,q,r,s,u \in X$.

We recall the concept of a metric projection in CAT(0) spaces. Let C be a nonempty closed convex subset of a complete CAT(0) space (X,d). Then for any $u \in X$, we know that there exists a unique nearest point $z \in C$, such that

$$d(u,z) = \inf_{v \in C} d(u,v).$$

In this case, z is called the unique nearest point of u in C. The mapping P_C is called metric projection from X onto a nonempty closed convex subset C defined by $P_C x = z$. In 2013, Dehghan and Rooin [8] used the concept of quasilinearization to characterize a metric projection in CAT(0) spaces.

Lemma 2.3 ([8])**.** *Let C be a nonempty convex subset of a complete CAT(0) space (X,d), $s \in X$ and $t \in C$. Subsequently,*

$$t = P_C s \quad \textit{if and only if} \quad \langle \overrightarrow{ts}, \overrightarrow{wt} \rangle \geq 0,$$

for all $w \in C$.

In 2008, Kirk and Panyanak [14] introduced a geometric condition on CAT(0) spaces called the (Q_4) *condition.* A CAT(0) space (X,d) is said to satisfy the (Q_4) *condition* if for all $u,v,s,t \in X$,

$$d(s,u) < d(u,t) \text{ and } d(s,v) < d(v,t) \text{ imply } d(s,m) < d(m,t), \quad \text{for all } m \in [u,v].$$

In 2013, Kakavandi [13] modified the (Q_4) *condition* as: A CAT(0) space (X,d) is said to satisfy the $(\overline{Q}_4)$ *condition* if for all $u,v,s,t \in X$,

$$d(s,u) \leq d(u,t) \text{ and } d(s,v) \leq d(v,t) \text{ imply } d(s,m) \leq d(m,t), \quad \text{for all } m \in [u,v].$$

It is mentioned in [13] that $(\overline{Q}_4)$ *condition* implies (Q_4) *condition.* There are some CAT(0) spaces that do not satisfy the $(\overline{Q}_4)$ condition. However, Hilbert spaces, $\mathbb{R}$-trees, and every CAT(0) space of constant curvature satisfy the $(\overline{Q}_4)$ *condition* (see [10]).

In 2013, Kunwai et al. [16] proved that the existence of attractive points can guarantee the existence of fixed points in complete CAT(0) space as follows:

Lemma 2.4 ([16]). *Let (X, d) be a complete CAT(0) space. Let $T : C \to C$ be a mapping defined on a nonempty closed convex subset C of X. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$. In particular, if $z \in A(T)$, then $P_C z \in F(T)$ where P_C is the metric projection from X onto C.*

The following results is also obtained by Kunwai et al. [16].

Lemma 2.5 ([16]). *Let (X, d) be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $\{u_n\}$ be a bounded sequence in X. If $d(u_{n+1}, u) \leq d(u_n, u)$ for any $u \in C$, then $\{P_C u_n\}$ converges strongly to some $u_0 \in C$, where P_C is the metric projection from X onto C.*

While nonexpansive mappings guarantee the convexity of the set of fixed points in Hilbert spaces [20], this property is not necessarily guaranteed for Berinde nonexpansive mappings, as illustrated in Example 1.3. However, it is shown in [16] that the set of all attractive points for mappings in complete CAT(0) spaces satisfying the $(\overline{Q}_4)$ *condition* is convex and closed as follows :

Lemma 2.6 ([16]). *Let (X, d) be a complete CAT(0) space satisfying the $(\overline{Q}_4)$ condition. Let $T : C \to X$ be a mapping defined on a nonempty subset C of X. Then $A(T)$ is a closed convex subset of X.*

3. Strong convergence theorems for Berinde nonexpansive mappings

In this section, we prove strong convergence theorems for Berinde nonexpansive mappings in a complete CAT(0) space using the Mann iteration process. We begin with the following lemma.

Lemma 3.1. *Let (X, d) be a complete CAT(0) space. Let $T : C \to C$ be a mapping defined on a nonempty subset C of X with $A(T) \neq \emptyset$. If the sequence $\{x_n\}$ is defined by Mann iteration, i.e, $x_1 \in C$ and*

$$x_{n+1} = (1 - \gamma_n)x_n \oplus \gamma_n T x_n, \quad \text{for all} \;\; n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$. Then

(1) $d(x_{n+1}, u) \leq d(x_n, u)$ *for all* $n \in \mathbb{N}$ *and for all* $u \in A(T)$,
(2) $\lim_{n\to\infty} d(x_n, u)$ *exists for all* $u \in A(T)$,
(3) $d(x_{n+1}, A(T)) \leq d(x_n, A(T))$ *for all* $n \in \mathbb{N}$,
(4) $\lim_{n\to\infty} d(x_n, A(T))$ *exists.*

Proof. Let $u \in A(T)$, then $d(Tx, u) \leq d(x, u)$ for all $x \in C$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, u) &= d((1 - \gamma_n)x_n \oplus \gamma_n T x_n, u) \\ &\leq (1 - \gamma_n)d(x_n, u) + \gamma_n d(T x_n, u) \\ &\leq (1 - \gamma_n)d(x_n, u) + \gamma_n d(x_n, u) \\ &= d(x_n, u). \end{aligned} \tag{3.1}$$

This means that $\{d(x_n, u)\}$ is nonincreasing and bounded below for all $u \in A(T)$. Thus, $\lim_{n\to\infty} d(x_n, u)$ exists. Moreover, we have from (3.1) that

$$d(x_{n+1}, A(T)) = \inf\{d(x_{n+1}, u) : u \in A(T)\}$$

$$\leq \inf\{d(x_n, u) : u \in A(T)\}$$
$$= d(x_n, A(T)), \quad \forall n \in \mathbb{N}.$$

Hence $\{d(x_n, A(T))\}$ is nonincreasing and bounded below.

Therefore, $\lim_{n\to\infty} d(x_n, A(T))$ exists. □

Senter and Dotson [19] introduced a condition on mappings, called *Condition* (I). Let C be a nonempty subset of a complete CAT(0) space (X, d). A mapping $T : C \to C$ is said to satisfy *Condition* (I) if there is a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(r) > 0$ for all $r > 0$ such that

$$0 \leq h(d(x, F(T))) \leq d(x, Tx), \quad \forall x \in C,$$

where $d(x, F(T)) = \inf\{d(x, y) : y \in F(T)\}$. Examples of mapping satisfying *Condition* (I) were given in [19] and the authors used this condition to guarantee the convergence to a fixed point of some mappings.

In this paper, we apply *Condition* (I) to the case of attractive point sets, termed *Condition* $(\bar{I})$, that is, a mapping T is said to satisfy *Condition* $(\bar{I})$ if there is a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(r) > 0$ for all $r > 0$ such that

$$0 \leq h(d(x, A(T))) \leq d(x, Tx), \quad \forall x \in C,$$

where $d(x, A(T)) = \inf\{d(x, y) : y \in A(T)\}$.

We provide an example of a mapping that satisfies *Condition* $(\bar{I})$ as follows:

Example 3.2. Let $X = \mathbb{R}$ and $C = \left(0, \frac{1}{2}\right]$. Define the mapping $T : C \to C$ by $Tx = \frac{x}{2}$ for all $x \in C$. Then $A(T) = (-\infty, 0]$. Moreover, define a function $h : [0, \infty) \to [0, \infty)$ by $h(x) = x^2$ for all $x \in [0, \infty)$. Hence, h is a nondecreasing function satisfying $h(0) = 0$ and $h(r) > 0$ for all $r > 0$. Since $d(x, A(T)) = \inf\{d(x, y) : y \in A(T)\} = d(x, 0)$ for all $x \in C$, we find that

$$0 \leq h(d(x, A(T))) = h(d(x, 0)) = x^2 \leq \frac{x}{2} = d(x, Tx),$$

for all $x \in C$. Thus, T satisfies *Condition* $(\bar{I})$.

Subsequently, by utilizing the existence of an attractive point, we prove a strong convergence theorem for fixed points and attractive points of Berinde nonexpansive mappings satisfying *Condition* $(\bar{I})$.

Theorem 3.3. *Let (X, d) be a complete CAT(0) space. Let $T : C \to C$ be a Berinde nonexpansive mapping defined on a nonempty closed convex subset C of X with $A(T) \neq \emptyset$ and satisfying Condition $(\bar{I})$. Suppose that the sequence $\{x_n\}$ is defined by Mann iteration such that $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$. Then $\{x_n\}$ converges strongly to $w \in F(T)$. Moreover, $w \in A(T)$.*

Proof. Let $u \in A(T)$. For all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d^2(x_{n+1}, u) &= d^2((1-\gamma_n)x_n \oplus \gamma_n Tx_n, u) \\ &\leq (1-\gamma_n)d^2(x_n, u) + \gamma_n d^2(Tx_n, u) - (1-\gamma_n)\gamma_n d^2(x_n, Tx_n) \\ &\leq (1-\gamma_n)d^2(x_n, u) + \gamma_n d^2(x_n, u) - (1-\gamma_n)\gamma_n d^2(x_n, Tx_n) \\ &= d^2(x_n, u) - (1-\gamma_n)\gamma_n d^2(x_n, Tx_n). \qquad (3.2) \end{aligned}$$

Then

$$(1-\gamma_n)\gamma_n d^2(x_n, Tx_n) \leq d^2(x_n, u) - d^2(x_{n+1}, u), \quad \forall n \in \mathbb{N}.$$

By assumption on the control sequence $\{\gamma_n\}$, there exist $n_0 \in \mathbb{N}$ and $\eta, \kappa \in (0,1)$ such that $0 < \eta \leq \gamma_n \leq \kappa < 1$ for all $n \geq n_0$. Then

$$0 \leq (1-\kappa)\eta d^2(x_n, Tx_n) \leq (1-\gamma_n)\gamma_n d^2(x_n, Tx_n) \leq d^2(x_n, u) - d^2(x_{n+1}, u).$$

Since $\lim_{n\to\infty} d(x_n, u)$ exists,

$$\lim_{n\to\infty} (1-\kappa)\eta d^2(x_n, Tx_n) = 0.$$

It follows that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.3}$$

Moreover, the *Condition* $(\bar{I})$ implies that there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(r) > 0$ for all $r > 0$ such that

$$0 \leq h(d(x_n, A(T))) \leq d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

By using (3.3) and (3.4), we can conclude that

$$\lim_{n\to\infty} h(d(x_n, A(T))) = 0. \tag{3.5}$$

Hence $\lim_{n\to\infty} d(x_n, A(T)) = 0$. Indeed, suppose that $\lim_{n\to\infty} d(x_n, A(T)) > 0$. Then there exist a positive number ϵ and $l_0 \in \mathbb{N}$ such that

$$d(x_n, A(T)) > \epsilon, \quad \forall n \geq l_0.$$

Hence

$$h(d(x_n, A(T))) \geq h(\epsilon), \quad \forall n \geq l_0.$$

Taking the limit on both sides, we obtain

$$h(\epsilon) \leq \lim_{n\to\infty} h(d(x_n, A(T))).$$

Since h is a nondecreasing function satisfying $h(0) = 0$ and $h(r) > 0$ for all $r > 0$, we obtain

$$0 < h(\epsilon) \leq \lim_{n\to\infty} h(d(x_n, A(T))),$$

which contradicts to (3.5). Thus,

$$\lim_{n\to\infty} d(x_n, A(T)) = 0. \tag{3.6}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. From (3.6), for any given $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$d(x_n, A(T)) < \frac{\epsilon}{2}, \quad \forall n \geq m_0.$$

Therefore, there is $u \in A(T)$ such that $d(x_{m_0}, u) < \frac{\epsilon}{2}$. By Lemma 3.1, we find that $\{d(x_n, u)\}$ is nonincreasing; hence, for $m > n \geq m_0$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, u) + d(u, x_n) \\ &\leq d(x_{m_0}, u) + d(u, x_{m_0}) \\ &< \epsilon. \end{aligned}$$

This means that $\{x_n\}$ is a Cauchy sequence in a closed set C. Thus there exists $w \in C$ such that

$$\lim_{n\to\infty} d(x_n, w) = 0. \tag{3.7}$$

Berinde nonexpansiveness of T implies that

$$\begin{aligned} d(w,Tw) &\le d(w,x_n) + d(x_n,Tx_n) + d(Tx_n,Tw) \\ &\le d(w,x_n) + (x_n,Tx_n) + d(x_n,w) + Ld(w,Tx_n) \\ &\le d(w,x_n) + (x_n,Tx_n) + d(x_n,w) + Ld(w,x_n) + Ld(x_n,Tx_n) \\ &= (2+L)d(w,x_n) + (1+L)d(x_n,Tx_n), \quad \forall n \in \mathbb{N}. \end{aligned}$$

From (3.3) and (3.7), we obtain $d(w,Tw) = 0$, and hence $w = Tw$.

Moreover, by using *Condition* $(\bar{I})$, we find that

$$0 \le h(d(w,A(T))) \le d(w,Tw) = 0.$$

It follows that $d(w,A(T)) = 0$, that is, $w \in A(T)$.

□

Motivated by the convexity of attractive point sets of mappings in complete CAT(0) spaces that satisfy the $(\overline{Q_4})$ *condition* and as well as the work of Takahashi and Toyoda [22], we prove the following results:

Theorem 3.4. *Let (X,d) be a complete CAT(0) space satisfying the $(\overline{Q_4})$ condition. Let $T : C \to C$ be a mapping defined on a nonempty closed convex subset C of X with $A(T) \neq \emptyset$. Let P be a metric projection of C onto $A(T)$. Suppose that the sequence $\{x_n\}$ is defined by the Mann iteration . Then $\{Px_n\}$ converges strongly to an attractive point of T.*

Proof. Since the complete CAT(0) space X satisfies the $(\overline{Q_4})$ *condition*, we apply Lemma 2.6 to imply that $A(T)$ is closed and convex in X. Let $u \in A(T)$. By Lemma 3.1, we obtain

$$d(x_{n+1}, u) \le d(x_n, u),$$

for all $n \in \mathbb{N}$. By applying Lemma 2.5, the proof is complete. □

Using Theorem 3.3 and Theorem 3.4, we can prove the following strong convergence theorem, which deals with a metric projection mapping onto the set of attractive points.

Theorem 3.5. *Let (X,d) be a complete CAT(0) space satisfying the $(\overline{Q_4})$ condition. Let $T : C \to C$ be a Berinde nonexpansive mapping defined on a nonempty closed convex subset C of X, with $A(T) \neq \emptyset$ and satisfying Condition $(\bar{I})$. Let P be a metric projection of C onto $A(T)$. Suppose that the sequence $\{x_n\}$ is defined by the Mann iteration such that $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$. Then $\{x_n\}$ converges strongly to $w \in F(T)$, where $w = \lim_{n\to\infty} Px_n$.*

Proof. By Theorem 3.4, the sequence $\{Px_n\}$ converges strongly to an element $v \in A(T)$. From Lemma 3.1, we establish that $\lim_{n\to\infty} d(x_n, u)$ exists for all $u \in A(T)$, and hence $\lim_{n\to\infty} d(x_n, v)$ exists. By Lemma 3.1 and the definition of Px_{n+1}, we find that

$$d(x_{n+1}, Px_{n+1}) \le d(x_{n+1}, Px_n) \le d(x_n, Px_n) \quad \text{for all } n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} d(x_n, Px_n)$ exists. Moreover,

$$d(x_n, v) \leq d(x_n, Px_n) + d(Px_n, v) \leq d(x_n, v) + d(Px_n, v) \quad \text{for all } n \in \mathbb{N}.$$

Since $\{Px_n\}$ converges strongly to v, we obtain

$$\lim_{n\to\infty} d(x_n, v) \leq \lim_{n\to\infty} d(x_n, Px_n) \leq \lim_{n\to\infty} d(x_n, v).$$

This implies that

$$\lim_{n\to\infty} d(x_n, v) = \lim_{n\to\infty} d(x_n, Px_n). \tag{3.8}$$

It follows from Theorem 3.3 that $\{x_n\}$ converges strongly to an element $w \in F(T)$, i.e,

$$\lim_{n\to\infty} d(x_n, w) = 0. \tag{3.9}$$

Moreover, we can conclude that $w \in A(T)$. Furthermore, we apply Lemma 2.3 to imply that

$$\langle \overrightarrow{Px_n x_n}, \overrightarrow{wPx_n} \rangle \geq 0.$$

Consequently, we can show that

$$\langle \overrightarrow{x_n Px_n}, \overrightarrow{Px_n w} \rangle \geq 0.$$

From (2.1), we obtain

$$\begin{aligned} 0 \leq \langle \overrightarrow{x_n Px_n}, \overrightarrow{Px_n w} \rangle &= \frac{1}{2}\Big(d^2(x_n, w) + d^2(Px_n, Px_n) - d^2(x_n, Px_n) - d^2(Px_n, w)\Big) \\ &= \frac{1}{2}\Big(d^2(x_n, w) - d^2(x_n, Px_n) - d^2(Px_n, w)\Big). \end{aligned}$$

Hence

$$d^2(Px_n, w) \leq d^2(x_n, w) - d^2(x_n, Px_n).$$

Letting $n \to \infty$, we get

$$d^2(v, w) \leq d^2(v, w) - \lim_{n\to\infty} d^2(x_n, Px_n).$$

From (3.8) and (3.9), we obtain

$$d^2(v, w) \leq 0.$$

Hence $v = w$. Therefore, $w = \lim_{n\to\infty} Px_n$.

□

4. Conclusions

In this paper, we use the Mann iterative process and the attractive point concept, defined in (1.1), to prove strong convergence theorems for fixed points and attractive points of Berinde nonexpansive mappings under some conditions in complete CAT(0) spaces. Moreover, since the set of fixed points of Berinde nonexpansive mappings may be nonconvex, we apply the convexity of the set of attractive points to prove strong convergence theorems dealing with a metric projection mapping onto this set.

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