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THE SECOND-ORDER DIFFERENTIAL EQUATION METHOD FOR SOLVING SOCCVI PROBLEM

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Abstract. In this paper, we explore the second-order differential equation (2-ODE) method to solve the second-order cone constrained variational inequality (SOCCVI). Its convergence and comparison with the first-order differential equation (1-ODE) method are reported. The main idea is employing the complementary function to reformulate the Karush-Kuhn-Tucker (KKT) conditions of the SOCCVI into a smooth system of equations, and then transform into an unconstrained optimization problem. In addition, we do numerical experiments to demonstrate the effectiveness of the approach.

1. Introduction

As an effective mathematical tool for describing general system equilibrium phenomena, variational inequalities (VI) have been widely used in problems such as equilibrium problems in economics, transport equivalence and modelling of urban transport networks. There are also many approaches for solving VI problems, including projection method, interior point method, nonsmooth equation method, smoothing method and so on.

At the same time, as an important branch of mathematical planning, the second-order cone program (SOCP) also possesses a very broad background and practical significance, and its research directions involve combinatorial optimization, engineering technology, control, machine learning, neural networks, finance and many other fields. However, as a promotion of SOCP, research on the problem of the second-order cone constrained variational inequality (SOCCVI) is still preliminary.

In this paper, we target on the below SOCCVI, which is find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0,\ y \in C, \tag{1.1}$$

where

$$C = \{x \in \mathbb{R}^n \mid h(x) = 0, -g(x) \in \mathcal{K}\}.$$

Here $\langle \cdot, \cdot \rangle$ is the Euclidean inner product; $F : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^l (l \geq 0)$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable; and $\mathbb{R}^n$ denotes n dimensional real column vector space. In particular, $\mathcal{K}$ is the Cartesian product of p second-order cones. In other words,

$$\mathcal{K} = \mathcal{K}^{m_1} \times \mathcal{K}^{m_2} \times \cdots \times \mathcal{K}^{m_p},$$

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where $m_1, m_2, \ldots, m_p \geq 1$, $m_1 + m_2 + \cdots + m_p = m$, and each $\mathcal{K}^{m_i}$ represents a second-order cone given by

$$\mathcal{K}^{m_i} := \left\{ (x_{i1}, x_{i2}, \ldots, x_{im_i})^{\mathrm{T}} \in \mathbb{R}^{m_i} \mid \|(x_{i2}, \ldots, x_{im_i})\| \leq x_{i1} \right\}.$$

In recent years, the utilization of differential equations to solve constrained optimization problems has attracted much attention. The earliest work was by Arrow and Hurwicz [2], and Fiacco and McCormick used differential equations to study the constrained norm problem in optimality conditions. Since 1980, a series of artificial network methods based on differential equation systems have been proposed by Hopfield and Tank [9], and employed to tackle complementarity problems and variational inequality. In 2000, Antipin [1] investigated the problem of variational inequality with coupling constraints, introduced symmetric functions, and proposed a globally convergent approach to differential equations. He and Yang [8] proposed a differential equation system for nonsymmetric linear VI problems based on projection operators and contraction methods. In 2005, Gao, Liao and Qi [7] studied a differential equation model for solving VI with linear and nonlinear constraints based on projection operator theory. In [10–13], there were various neural network methods investigated for the SOCCVI. All of them belong to the first-order differential equation (1-ODE) method.

Inspired by the aforementioned works and the idea by Attouch et al [3–5] which tackles convex optimization problems by using damped inertial gradient dynamical systems, we construct the second-order differential equation system involving two time-dependent parameters to solve the SOCCVI (1.1). In particular, we first adopt the smoothed complementary function to transform the KKT conditions of the SOCCVI (1.1) into a smoothing equation system problem, and introduce the merit function to transform it into an unconstrained optimization problem. In Section 3, a system of second-order differential equations associated with the above unconstrained problem is established to solve the SOCCVI (1.1). Moreover, the differential equation system involves two time-dependent parameters, which are a positive viscous damping coefficient $\gamma(t)$ and a time scale coefficient $\beta(t)$. Then, in Section 4, we consider a perturbed term $g(t)$ of the second-order differential equation system. In Section 5, we compare the second-order differential equation method with the first-order differential equation method for handling the SOCCVI theoretically. At last, two numerical experiments are reported to verify the effectiveness of the second-order differential equation method for solving the SOCCVI.

2. Preliminaries

For subsequent analysis, we write down the KKT conditions of the SOCCVI (1.1),

$$\begin{gathered} L(x, \mu, \lambda) = F(x) + Jh(x)^{\mathrm{T}}\mu + Jg(x)^{\mathrm{T}}\lambda = 0, \\ h(x) = 0, \\ \langle g(x), \lambda \rangle = 0, -g(x) \in \mathcal{K},\ \lambda \in \mathcal{K}, \end{gathered} \tag{2.1}$$

where $L(x, \mu, \lambda)$ is called the VI Lagrangian function, and $\mu \in \mathbb{R}^l$, $\lambda \in \mathbb{R}^m$. In order to convert the KKT conditions of the SOCCVI (1.1) to an unconstrained optimization problem, we shall employ the second-order cone (SOC) complementarity function, $\varphi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, which satisfies $\varphi(x, y) = 0$ if and only if $x \in \mathcal{K}^m, y \in \mathcal{K}^m$ and $\langle x, y \rangle = 0$.

In order to tackle with optimization problems involving second-order cones, it usually need a special decomposition. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, there holds a spectral decomposition:

$$x = \rho_1(x)u_1(x) + \rho_2(x)u_2(x), \tag{2.2}$$

where $\rho_1(x)$ and $\rho_2(x)$ are called the spectral values of x, given by

$$\rho_i(x) = x_1 + (-1)^i \|x_2\| .$$

In addition, $u_1(x)$ and $u_2(x)$ are the spectral vectors of x, given by

$$u_i(x) = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2}{\|x_2\|}\right) & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i \omega\right) & \text{if} \quad x_2 = 0, \end{cases}$$

where ω is an arbitrary unit vector in $\mathbb{R}^{m-1}$. A popular SOC-complementarity function is the natural residual (NR) function, which is semismooth and defined as (see [6, 11] for more details)

$$\varphi_{\mathrm{NR}}(x, y) = x - \Pi_{\mathcal{K}^m}(x - y) .$$

In light of the spectral decomposition (2.2) of x, the projection $\Pi_{\mathcal{K}^m}$ of x onto $\mathcal{K}^m$ is described by

$$\Pi_{\mathcal{K}^m}(x) = \max\{0, \rho_1(x)\} u_1(x) + \max\{0, \rho_2(x)\} u_2(x) .$$

In this paper, we consider a smoothing metric projector function $\phi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying

$$\phi(\varepsilon, c) = \frac{1}{2}\left(c + \sqrt{\varepsilon^2 e + c^2}\right) \quad \forall (\varepsilon, c) \in \mathbb{R}_+ \times \mathbb{R}^m. \tag{2.3}$$

Notice that $\phi(0, c) = \Pi_{\mathcal{K}^m}(c)$. Moreover, ϕ is continuously differentiable on any neighborhood of $(\varepsilon, c) \in \mathbb{R}_+ \times \mathbb{R}^m$ provided that $\left(\varepsilon^2 e + c^2\right)_0 \neq \left\|\varepsilon^2 e + c^2\right\|$. For $\left(\varepsilon^2 e + c^2\right)_0 = \left\|\varepsilon^2 e + c^2\right\|$, it is known that ϕ is nonsmooth at (ε, c), but its B-subdifferential can nevertheless be computed. For more details regarding the function $\phi(0, c)$, please refer to reference [11]. Now, Based on $\phi(0, c)$ given in (2.3), we define the smoothing NR function given as

$$\varphi^{\varepsilon}_{\mathrm{NR}}(x, y) = x - \phi(\varepsilon, x - y) . \tag{2.4}$$

In view of the function $\varphi^{\varepsilon}_{\mathrm{NR}}$ defined in (2.4), the KKT conditions (2.1) of the SOCCVI (1.1) are recast as

$$S(\varepsilon, x, \mu, \lambda) = \begin{pmatrix} \varepsilon \\ L(x, \mu, \lambda) \\ h(x) \\ \varphi^{\varepsilon}_{\mathrm{NR}}(-g(x), \lambda) \end{pmatrix} = 0, \tag{2.5}$$

where

$$\varphi_{\mathrm{NR}}^{\varepsilon}\left(-g\left(x\right),\lambda\right)=\begin{pmatrix}\varphi_{\mathrm{NR}}^{\varepsilon}\left(-g_{m_1}\left(x\right),\lambda_{m_1}\right)\\ \varphi_{\mathrm{NR}}^{\varepsilon}\left(-g_{m_2}\left(x\right),\lambda_{m_2}\right)\\ \vdots\\ \varphi_{\mathrm{NR}}^{\varepsilon}\left(-g_{m_p}\left(x\right),\lambda_{m_p}\right)\end{pmatrix}.$$

with $-g_{m_i}\left(x\right),\lambda_{m_i}\in\mathcal{K}^{m_i}$. Moreover, the merit function Φ for the SOCCVI (1.1) is as below:

$$\min\Phi\left(\varepsilon,x,\mu,\lambda\right):=\frac{1}{2}\left\|S\left(\varepsilon,x,\mu,\lambda\right)\right\|^2. \tag{2.6}$$

It is clear to see that x^* is the solution to the unconstrained optimization problem (2.6) meaning that x^* is the solution to the SOCCVI (1.1). For notational simplicity, we denote $z=(\varepsilon,x,\mu,\lambda)\in\mathrm{I\!R}\times\mathrm{I\!R}^n\times\mathrm{I\!R}^l\times\mathrm{I\!R}^m$, then the unconstrained optimization problem (2.6) can be expressed as

$$\min\Phi(z):=\frac{1}{2}\left\|S\left(z\right)\right\|^2. \tag{2.7}$$

With (2.7), the differential equation approach to solving the SOCCVI (1.1) has been proposed and investigated, see [10–12]. Although there are some variants, mostly the dynamical model possesses the format of

$$\left\{\begin{array}{rcl}\frac{\mathrm{d}z(t)}{\mathrm{d}t} & = & -\rho\nabla\Phi\left(z\left(t\right)\right)\\ z\left(t_0\right) & = & z_0\end{array}\right. \tag{2.8}$$

where $\nabla\Phi\left(z\left(t\right)\right)=\begin{pmatrix}\nabla_\varepsilon S(z)^{\mathrm{T}}S\left(z\right)\\ \nabla_x S(z)^{\mathrm{T}}S\left(z\right)\\ \nabla_\mu S(z)^{\mathrm{T}}S\left(z\right)\\ \nabla_\lambda S(z)^{\mathrm{T}}S\left(z\right)\end{pmatrix}$ and $\rho>0$ is a scaling factor.

3. The second-order differential equation system

In this section, we explore the second-order differential equation method for solving the SOCCVI (1.1), which is different from the model (2.8). Indeed, inspired by Attouch [5], the second-order differential equation system involves two time-dependent parameters, which are a positive viscous damping coefficient $\gamma(t)$ and a time scale coefficient $\beta(t)$. More specifically, the dynamical model is established as below:

$$\begin{pmatrix}\ddot{\varepsilon}\left(t\right)\\ \ddot{x}\left(t\right)\\ \ddot{\mu}\left(t\right)\\ \ddot{\lambda}\left(t\right)\end{pmatrix}+\gamma\left(t\right)\begin{pmatrix}\dot{\varepsilon}\left(t\right)\\ \dot{x}\left(t\right)\\ \dot{\mu}\left(t\right)\\ \dot{\lambda}\left(t\right)\end{pmatrix}+\beta\left(t\right)\begin{pmatrix}\nabla_\varepsilon S(z)^{\mathrm{T}}S\left(z\right)\\ \nabla_x S(z)^{\mathrm{T}}S\left(z\right)\\ \nabla_\mu S(z)^{\mathrm{T}}S\left(z\right)\\ \nabla_\lambda S(z)^{\mathrm{T}}S\left(z\right)\end{pmatrix}=0, \tag{3.1}$$

where $S(\cdot)$ is defined as in (2.5) and $\gamma,\beta\in[t_0,+\infty)$ are nonnegative continuous functions. For convenience, we take for granted the existence of a solution to this system, and hence the corresponding equilibrium point of the system becomes the solution of the SOCCVI (1.1). According to the parameter tuning of the positive viscous damping coefficient $\gamma(t)$ and the time scaling parameter $\beta(t)$, we will analyze the global convergence of the solution trajectory of the second-order differential equation system (3.1).

For the special case, $\Phi(\varepsilon, x, \mu, \lambda) \equiv 0$, by direct integration of a system of second-order differential equations, we obtain

$$p(t) = e^{\int_{t_0}^{t} \gamma(u) du}.$$

Here, it is assumed that the following conditions are satisfied:

$$\mathbf{H_0}: \quad \int_{t_0}^{+\infty} \frac{du}{p(u)} < +\infty. \tag{3.2}$$

Under this assumption, we can define the function Γ on $[t_0, +\infty)$ by

$$\Gamma(t) := p(t) \int_{t}^{+\infty} \frac{du}{p(u)}. \tag{3.3}$$

By differentiating the above (3.3), we obtain the relation

$$\dot{\Gamma}(t) = \gamma(t) \Gamma(t) - 1. \tag{3.4}$$

Then, let us further respectively define the global energy function $W(t)$ as

$$W(t) = \frac{1}{2} \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 + \beta(t) \left[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi \right] \tag{3.5}$$

and the anchor function $h(t)$ as

$$h(t) = \frac{1}{2} \left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \right\|^2, \tag{3.6}$$

where $\begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \in \arg\min \Phi \neq \emptyset$ is given in advance. These pave the basic constitutive blocks of the function $\xi : [t_0, +\infty) \to \mathbb{R}_+$, which is defined by

$$\xi(t) = \Gamma(t)^2 W(t) + h(t) + \Gamma(t) \dot{h}(t). \tag{3.7}$$

Indeed, it is equivalent to

$$\begin{aligned} \xi(t) &= \Gamma(t)^2 \beta(t) \left[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi \right] \\ &\quad + \frac{1}{2} \left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2. \end{aligned}$$

Theorem 3.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function such that* $\arg\min \Phi \neq \emptyset$. *Suppose that $\beta : [t_0, +\infty)$ is a continuous positive function and $\gamma : [t_0, +\infty)$ is a continuous function satisfying the assumptions H_0 given as in (3.2). We also assume that the following growth condition $H_{\gamma,\beta}$ is satisfied linking $\gamma(t)$ and $\beta(t)$:*

$$H_{\gamma,\beta} : \quad \Gamma(t)\dot{\beta}(t) \leq \beta(t)\left[3 - 2\gamma(t)\Gamma(t)\right].$$

Then, for every solution trajectory $(\varepsilon, x, \mu, \lambda) : [t_0, +\infty) \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ of the second-order differential equation system (3.1), the convergence rate of the values satisfies

$$\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - \min \Phi = \mathrm{O}\left(\frac{1}{\beta(\mathrm{t})\Gamma(\mathrm{t})^2}\right)$$

as $t \to +\infty$. In addition, the trajectory of the solution is bounded on $[t_0, +\infty)$.

Proof. For simplicity, we denote $m := \min \Phi$ and compute the derivative of $\xi(t)$ as below:

$$\dot{\xi}(t) = 2\Gamma(t)\dot{\Gamma}(t)W(t) + \Gamma(t)^2\dot{W}(t) + \dot{h}(t) + \dot{\Gamma}(t)\dot{h}(t) + \Gamma(t)\ddot{h}(t).$$

Then, we calculate the derivatives of the principal components, including the derivatives of the global energy function $W(t)$ and the anchor function $h(t)$, which is defined as in (3.5) and (3.6), respectively . First, we have

$$\begin{aligned}
\dot{W}(t) &= \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix} \right\rangle + \beta(t) \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix} \right\rangle \\
&\quad + \dot{\beta}(t)\left[\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - m\right] \\
&= \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix} + \beta(t) \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix} \right\rangle \\
&\quad + \dot{\beta}(t)\left[\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - m\right] \\
&= \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, -\gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\rangle + \dot{\beta}(t)\left[\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - m\right] \\
&= -\gamma(t) \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 + \dot{\beta}(t)\left[\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - m\right].
\end{aligned}$$

On the other hand, the derivative for the anchor function $h(t)$ gives

$$\dot{h}(t) = \left\langle \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle,$$

and

$$\ddot{h}(t) = \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 + \left\langle \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle.$$

Thus, we obtain that

$$\begin{aligned}
& \gamma(t)\dot{h}(t) + \ddot{h}(t) \\
= {} & \left\langle \gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} + \begin{pmatrix} \ddot{\varepsilon}(t) \\ \ddot{x}(t) \\ \ddot{\mu}(t) \\ \ddot{\lambda}(t) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle + \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 \\
= {} & -\beta(t) \left\langle \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix}, \begin{pmatrix} \varepsilon(t) - \varepsilon^* \\ x(t) - x^* \\ \mu(t) - \mu^* \\ \lambda(t) - \lambda^* \end{pmatrix} \right\rangle + \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 \\
\leq {} & -\beta(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m\right) + \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 \\
\leq {} & \left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2
\end{aligned}$$

where the above inequality holds due to the convexity of $\Phi(\varepsilon, x, \mu, \lambda)$. With this, we can compute the derivative of $\xi(t)$ as

$$\begin{aligned}
\dot{\xi}(t) &= 2\Gamma(t)\dot{\Gamma}(t)W(t) + \Gamma(t)^2\dot{W}(t) + \dot{h}(t) + \dot{\Gamma}(t)\dot{h}(t) + \Gamma(t)\ddot{h}(t) \\
&= 2\Gamma(t)\dot{\Gamma}(t)W(t) + \Gamma(t)^2\dot{W}(t) + \Gamma(t)\left(\gamma(t)\dot{h}(t) + \ddot{h}(t)\right) \\
&\leq 2\Gamma(t)\dot{\Gamma}(t)\left(\frac{1}{2}\left\| \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2 + \beta(t)\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& +\Gamma(t)^2\left(-\gamma(t)\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2+\dot{\beta}(t)\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)\right)\\
& +\Gamma(t)\left(-\beta(t)\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)+\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2\right)\\
\leq\ & \Gamma(t)\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2\left(1+\dot{\Gamma}(t)-\gamma(t)\Gamma(t)\right)\\
& +\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)\Gamma(t)\left(\Gamma(t)\dot{\beta}(t)+2\dot{\Gamma}(t)\beta(t)-\beta(t)\right).
\end{aligned}
$$

This together with equation (3.4) yields

$$
\dot{\xi}(t)\leq\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)\Gamma(t)\left(\Gamma(t)\dot{\beta}(t)+2\dot{\Gamma}(t)\beta(t)-\beta(t)\right).
$$

which implies

(3.8)

$$
\dot{\xi}(t)\leq\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)\Gamma(t)\left(\Gamma(t)\dot{\beta}(t)+\beta(t)\left(2\gamma(t)\Gamma(t)-3\right)\right).
$$

Therefore, from the assumed growth conditions $H_{\gamma,\beta}$, it can be inferred that $\dot{\xi}(t)\leq 0$. In other words, the function $\xi(t)$ is decreasing on $[t_0,+\infty)$, which says $\xi(t)\leq\xi(t_0)$. Then, according to the formulation (3.7) of $\xi(t)$, we can deduce that, for all $t\geq t_0$,

$$
\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-\min\Phi\leq\frac{\xi(t_0)}{\beta(t)\Gamma(t)^2},
$$

that is,

$$
\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-\min\Phi=\mathrm{O}\left(\frac{1}{\beta(t)\Gamma(t)^2}\right).
$$

Next, let us show that the trajectory is bounded. First, using the formulation and the decreasing of $\xi(t)$ leads to

$$
\left\|\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}+\Gamma(t)\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2\leq 2\xi(t)\leq 2\xi(t_0)
$$

which indicates

(3.9)

$$
\left\|\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}\right\|^2+2\Gamma(t)\left\langle\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix},\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\rangle\leq 2\xi(t_0).
$$

Setting $q(t) := \int_t^{+\infty} \frac{ds}{p(s)}$ and considering the assumptions $H_0 : \int_{t_0}^{+\infty} \frac{du}{p(u)} < +\infty$, we achieve that the function $q(t) := \int_t^{+\infty} \frac{ds}{p(s)}$ is bounded as $t \geq t_0$. Moreover, we can deduce the following equivalence relation:

$$q(t) = \frac{\Gamma(t)}{p(t)}, \quad \dot{q}(t) = -\frac{1}{p(t)}.$$

Dividing (3.9) by $p(t)$ and applying the above equivalence relation, we establish

$$\frac{1}{p(t)} h(t) + q(t)\dot{h}(t) \leq \frac{C}{p(t)}, \quad \forall t \in [t_0, +\infty).$$

where $C = \xi(t_0)$. This is equivalent to

$$q(t)\dot{h}(t) - \dot{q}(t)(h(t) - C) \leq 0, \quad \forall t \in [t_0, +\infty)$$

where $h(t)$ is the anchor function. Now, dividing by $q(t)^2$ yields

$$\frac{1}{q(t)^2}\left(q(t)\dot{h}(t) - \dot{q}(t)(h(t) - C_1)\right) = \frac{d}{dt}\left(\frac{h(t) - C_1}{q(t)}\right) \leq 0, \quad \forall t \in [t_0, +\infty).$$

Doing integration on this inequality gives $h(t) \leq C_1(1 + q(t))$ for some $C_1 > 0$. Therefore, the solution trajectory is bounded and the proof is complete. □

In order to speed up the convergence of the solution, we introduce ρ and construct a stronger assumptions $H^+_{\gamma,\beta}$ on $t > t_1$, which is

$$H^+_{\gamma,\beta}: \quad \Gamma(t)\dot{\beta}(t) \leq \beta(t)\big[3 - \rho - 2\gamma(t)\Gamma(t)\big].$$

with $\rho > 0$ and $t_1 > t_0$.

Corollary 3.2. *Let* $\Phi : \mathbb{R} \to \mathbb{R}$ *be a convex function such that* $\arg\min \Phi \neq \emptyset$. *For solutions of the second order differential equation satisfying the assumptions* $H^+_{\gamma,\beta}$, *the following integral energy estimates hold:*

$$\int_{t_0}^{+\infty} \Gamma(t) W(t) dt < +\infty,$$

where

$$W(t) = \frac{1}{2}\left\|\begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}\right\|^2 + \beta(t)\big[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi\big].$$

Equivalently,

$$\int_{t_0}^{+\infty} \Gamma(t)\left\|\begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix}\right\|^2 dt < +\infty,$$

$$\int_{t_0}^{+\infty} \Gamma(t)\beta(t)\big[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi\big] dt < +\infty.$$

Proof. For simplicity, denote $m := \min \Phi$. From the proof of Theorem 3.1, we know the derivation for the global energy function $W(t)$ is

$$\dot{W}(t) = -\gamma(t)\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2 + \dot{\beta}(t)\left[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m\right].$$

Multiplying both sides by $\Gamma(t)^2$ on this leads to

$$\Gamma(t)^2\dot{W}(t) + \gamma(t)\Gamma(t)^2\left\|\begin{pmatrix}\dot{\varepsilon}(t)\\ \dot{x}(t)\\ \dot{\mu}(t)\\ \dot{\lambda}(t)\end{pmatrix}\right\|^2 = \dot{\beta}(t)\Gamma(t)^2\left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m\right).$$

Now, integrating the above inequality over the interval (t, t_0), we obtain

$$\Gamma(t)^2 W(t) - \Gamma(t_0)^2 W(t_0) - 2\int_{t_0}^{t}\Gamma(s)\dot{\Gamma}(s)W(s)ds + \int_{t_0}^{t}\gamma(s)\Gamma(s)^2\left\|\begin{pmatrix}\dot{\varepsilon}(s)\\ \dot{x}(s)\\ \dot{\mu}(s)\\ \dot{\lambda}(s)\end{pmatrix}\right\|^2 ds$$

$$= \int_{t_0}^{t}\dot{\beta}(s)\Gamma(s)^2\left(\Phi(\varepsilon(s), x(s), \mu(s), \lambda(s)) - m\right)ds.$$

Simplifying the above equation gives

(3.10)
$$\Gamma(t)^2 W(t) + \int_{t_0}^{t}\left\|\begin{pmatrix} dot\varepsilon(s)\\ \dot{x}(s)\\ \dot{\mu}(s)\\ \dot{\lambda}(s)\end{pmatrix}\right\|^2\left(\gamma(s)\Gamma(s)^2 - \Gamma(s)\dot{\Gamma}(s)\right)ds = \Gamma(t_0)^2 W(t_0)$$
$$+ \int_{t_0}^{t}\left(2\beta(s)\Gamma(s)\dot{\Gamma}(s) + \dot{\beta}(s)\Gamma(s)^2\right)\left(\Phi(\varepsilon(s), x(s), \mu(s), \lambda(s)) - m\right)ds.$$

From (3.4), we have $\gamma(t)\Gamma(t)^2 - \Gamma(t)\dot{\Gamma}(t) = \Gamma(t)$. This together with the assumption $H_{\gamma,\beta}$ implies $\Gamma(t)^2\dot{\beta}(t) + 2\beta(t)\dot{\Gamma}(t)\Gamma(t) \le \beta(t)\Gamma(t)$. Then, the equation (3.10) is equivalent to

$$\begin{aligned}(3.11)\qquad &\Gamma(t)^2 W(t) + \int_{t_0}^{t}\left\|\begin{pmatrix}\dot{\varepsilon}(s)\\ \dot{x}(s)\\ \dot{\mu}(s)\\ \dot{\lambda}(s)\end{pmatrix}\right\|^2\Gamma(s)ds\\ &\le \Gamma(t_0)^2 W(t_0) + \int_{t_0}^{t}\Gamma(s)\beta(s)\left[\Phi(\varepsilon(s), x(s), \mu(s), \lambda(s)) - m\right]ds.\end{aligned}$$

Applying the assumption $H^+_{\gamma,\beta}$, we know that the equation (3.8) can be converted to

$$\dot{\xi}(t) + \rho\Gamma(t)\beta(t)\left[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m\right] \le 0.$$

Integrating the above inequality over the interval $(t_0, +\infty)$, we have

$$-\xi\left(t_0\right)+\rho \int_{t_0}^{+\infty} \Gamma(st)\beta(s)\big[\Phi\left(\varepsilon(s), x(s), \mu(s), \lambda(s)\right)-m\big] \leq 0.$$

It is known that $\xi\left(t\right)$ is non-negative; and hence there holds

$$\int_{t_0}^{+\infty} \Gamma\left(s\right) \beta\left(s\right)\left(\Phi\left(\varepsilon\left(s\right), x\left(s\right), \mu\left(s\right), \lambda\left(s\right)\right)-m\right) \leq \frac{\xi\left(t_0\right)}{\rho}<+\infty. \tag{3.12}$$

Combining the inequality (3.11), we achieve the following inequality:

$$\Gamma(t)^2 W\left(t\right)+\int_{t_0}^{t}\left\|\begin{pmatrix} \dot{\varepsilon}\left(s\right) \\ \dot{x}\left(s\right) \\ \dot{\mu}\left(s\right) \\ \dot{\lambda}\left(s\right) \end{pmatrix}\right\|^2 \Gamma\left(s\right) ds \leq \Gamma(t_0)^2 W\left(t_0\right)+\frac{\xi\left(t_0\right)}{\rho}.$$

that is,

$$\int_{t_0}^{+\infty}\left\|\begin{pmatrix} \dot{\varepsilon}\left(s\right) \\ \dot{x}\left(s\right) \\ \dot{\mu}\left(s\right) \\ \dot{\lambda}\left(s\right) \end{pmatrix}\right\|^2 \Gamma\left(s\right) ds \leq \Gamma(t_0)^2 W\left(t_0\right)+\frac{\xi\left(t_0\right)}{\rho}. \tag{3.13}$$

According to the definition of the global energy function $W(\cdot)$, by adding the inequalities (3.12) and (3.13), we obtain

$$\int_{t_0}^{+\infty} \frac{1}{2}\left\|\begin{pmatrix} \dot{\varepsilon}\left(s\right) \\ \dot{x}\left(s\right) \\ \dot{\mu}\left(s\right) \\ \dot{\lambda}\left(s\right) \end{pmatrix}\right\|^2 \Gamma\left(s\right) ds+\int_{t_0}^{+\infty} \beta(s)\Gamma(s)\left(\Phi\left(\varepsilon(s), x(s), \mu(s), \lambda\left(s\right)\right)-m\right) ds$$
$$\leq \frac{3}{2}\frac{\xi\left(t_0\right)}{\rho}+\frac{1}{2}\Gamma(t_0)^2 W\left(t_0\right).$$

Then, we eventually establish

$$\int_{t_0}^{+\infty} \Gamma(s)W(s)ds \leq \frac{3}{2}\frac{\xi(t_0)}{\rho}+\frac{1}{2}\Gamma(t_0)^2 W\left(t_0\right)<+\infty.$$

which is the desired result. □

4. The second-order differential equation system with perturbations

In this section, we consider a perturbed situation of the second-order differential equation system. The specific dynamical model is

$$\begin{pmatrix} \ddot{\varepsilon}\left(t\right) \\ \ddot{x}\left(t\right) \\ \ddot{\mu}\left(t\right) \\ \ddot{\lambda}\left(t\right) \end{pmatrix}+\gamma\left(t\right)\begin{pmatrix} \dot{\varepsilon}\left(t\right) \\ \dot{x}\left(t\right) \\ \dot{\mu}\left(t\right) \\ \dot{\lambda}\left(t\right) \end{pmatrix}+\beta\left(t\right)\begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_x S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_\mu S(z)^{\mathrm{T}} S\left(z\right) \\ \nabla_\lambda S(z)^{\mathrm{T}} S\left(z\right) \end{pmatrix}=g(t) \tag{4.1}$$

where $g(\cdot)$ is regard as an external effect on the system, a perturbation or a control term. As will be asserted, the perturbation term does not affect the existence and uniqueness of the solution when it is small enough.

Theorem 4.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex function with* $\arg\min \Phi \neq \emptyset$. *Suppose that $\beta : [t_0, +\infty)$ is a continuous positive function, $\gamma : [t_0, +\infty)$ is a continuous function satisfying the assumption H_0 given as in (3.2). In addition, we assume that the function $g : [t_0, +\infty)$ is locally integrable and satisfies*

$$H_g : \quad \int_{t_0}^{+\infty} \Gamma(t) \, \|g(t)\| \, dt < +\infty.$$

Then, for every solution trajectory $(\varepsilon, x, \mu, \lambda) : [t_0, +\infty) \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ of the second-order differential equation system (4.1) with perturbations,

(a): *Under the assumption $H_{\gamma,\beta}$, we deduce that the trajectory of solution $\begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix}$ and $\Gamma(t) \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix}$ are bounded, and as $t \to +\infty$, the convergence rate of the values is satisfied*

$$\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi = \mathrm{O}\left(\frac{1}{\beta(t) \Gamma(t)^2} \right).$$

(b): *Under the assumption $H^{+}_{\gamma,\beta}$, we also have*

$$\int_{t_0}^{+\infty} \beta(t) \Gamma(t) \left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m \right) dt < +\infty. \tag{4.2}$$

Proof. For the proof, we employ the same energy function $\xi(t)$ defined as in (3.7),

$$\begin{aligned} \xi(t) &= \Gamma(t)^2 \beta(t) \big[\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi \big] \\ &\quad + \frac{1}{2} \left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2 . \end{aligned}$$

(a) To proceed, we compute the derivative of $\xi(t)$ as follows.

$$\begin{aligned} \dot\xi(t) &= \left(2\Gamma(t) \dot\Gamma(t) \beta(t) + \Gamma(t)^2 \dot\beta(t) \right) \left(\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - m \right) \\ &\quad + \Gamma(t)^2 \beta(t) \left\langle \begin{pmatrix} \nabla_\varepsilon S(z)^{\mathrm{T}} S(z) \\ \nabla_x S(z)^{\mathrm{T}} S(z) \\ \nabla_\mu S(z)^{\mathrm{T}} S(z) \\ \nabla_\lambda S(z)^{\mathrm{T}} S(z) \end{pmatrix}, \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\rangle \\ &\quad + \left\langle \frac{\mathrm{d}}{\mathrm{dt}} \left(\begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right), \right. \\ &\qquad \left. \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\rangle . \end{aligned}$$

Based on (3.4) and the second-order differential equation system (4.1) with the perturbation term, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}+\Gamma(t)\begin{pmatrix}\dot\varepsilon(t)\\ \dot x(t)\\ \dot\mu(t)\\ \dot\lambda(t)\end{pmatrix}\right)$$
$$=\Gamma(t)\left(g(t)-\beta(t)\begin{pmatrix}\nabla_\varepsilon S(z)^{\mathrm{T}}S(z)\\ \nabla_x S(z)^{\mathrm{T}}S(z)\\ \nabla_\mu S(z)^{\mathrm{T}}S(z)\\ \nabla_\lambda S(z)^{\mathrm{T}}S(z)\end{pmatrix}\right)$$

which says

$$\begin{aligned}\dot\xi(t) &= \left(2\Gamma(t)\dot\Gamma(t)\beta(t)+\Gamma(t)^2\dot\beta(t)\right)\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)\\ &\quad+\left\langle\Gamma(t)g(t),\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}+\Gamma(t)\begin{pmatrix}\dot\varepsilon(t)\\ \dot x(t)\\ \dot\mu(t)\\ \dot\lambda(t)\end{pmatrix}\right\rangle\\ &\quad-\left\langle\Gamma(t)\beta(t)\begin{pmatrix}\nabla_\varepsilon S(z)^{\mathrm{T}}S(z)\\ \nabla_x S(z)^{\mathrm{T}}S(z)\\ \nabla_\mu S(z)^{\mathrm{T}}S(z)\\ \nabla_\lambda S(z)^{\mathrm{T}}S(z)\end{pmatrix},\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}\right\rangle.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and convexity of Φ yields the following inequality conditions:

$$\begin{aligned}\dot\xi(t) &\le \left(2\Gamma(t)\dot\Gamma(t)\beta(t)+\Gamma(t)^2\dot\beta(t)\right)\left(\Phi(\varepsilon(t),x(t),\mu(t),\lambda(t))-m\right)\\ &\quad+\Gamma(t)\|g(t)\|\left\|\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}+\Gamma(t)\begin{pmatrix}\dot\varepsilon(t)\\ \dot x(t)\\ \dot\mu(t)\\ \dot\lambda(t)\end{pmatrix}\right\|.\end{aligned}\tag{4.3}$$

Then, we obtain that

$$\begin{aligned}\dot\xi(t) &\le \Gamma(t)\|g(t)\|\left\|\begin{pmatrix}\varepsilon(t)\\ x(t)\\ \mu(t)\\ \lambda(t)\end{pmatrix}-\begin{pmatrix}\varepsilon^*\\ x^*\\ \mu^*\\ \lambda^*\end{pmatrix}+\Gamma(t)\begin{pmatrix}\dot\varepsilon(t)\\ \dot x(t)\\ \dot\mu(t)\\ \dot\lambda(t)\end{pmatrix}\right\|\\ &\le \sqrt{2}\Gamma(t)\|g(t)\|\sqrt{\xi(t)}.\end{aligned}$$

By integrating the above inequalities and combining the assumption $(H)_g$, we have

$$\sqrt{\xi(t)}\le\sqrt{\xi(t_0)}+\frac{1}{\sqrt{2}}\int_{t_0}^{+\infty}\Gamma(s)\|g(s)\|ds=Cte<+\infty.$$

This indicates that $\xi(t)$ is bounded and $\xi(t) \le \xi(t_0)$. According to the formulation of $\xi(t)$, we can obtain that

$$\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi \le \frac{\xi(t_0)}{\beta(t)\Gamma(t)^2},$$

that is,

$$\Phi(\varepsilon(t), x(t), \mu(t), \lambda(t)) - \min \Phi = \mathrm{O}\left(\frac{1}{\beta(t)\Gamma(t)^2}\right).$$

(b) Returning to the boundedness of $\xi(t)$, with the definition of $\xi(t)$ we also deduce that $\left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\|^2$ is bounded, which gives

$$\left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \right\|^2 + 2\Gamma(t) \left\langle \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix}, \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\rangle \le C.$$

Taking the same anchor function $h(t)$ defined as in (3.6), we know

$$h(t) := \tfrac{1}{2} \left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} \right\|^2,$$
$$\dot{h}(t) := \left\langle \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix}, \begin{pmatrix} \dot{\varepsilon}(t) \\ \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{pmatrix} \right\rangle.$$

With this, we see that the inequality (4.2) is equivalent to

$$h(t) + \Gamma(t)\dot{h}(t) \le \frac{1}{2}C.$$

Consequently,

$$q(t)\dot{h}(t) - \dot{q}(t)\left(h(t) - \frac{1}{2}C\right) \le 0.$$

where $q(t) = \frac{\Gamma(t)}{p(t)}$ and $\dot{q}(t) = -\frac{1}{p(t)}$. Now, mimicking the same arguments as in Section 3. After dividing the above inequality by $q(t)^2$, it gives

$$\frac{1}{q(t)^2}\left(q(t)\dot{h}(t) - \dot{q}(t)\left(h(t) - C_1\right)\right) = \frac{d}{dt}\left(\frac{h(t) - C_1}{q(t)}\right) \le 0.$$

Integration of this inequality yields $h(t) \le C_1(1+q(t))$ for some $C_1 > 0$, which says that $h(t)$ is bounded. Thus, the solution trajectory $x(\cdot)$ is also bounded. Combining the boundedness of $\left\| \begin{pmatrix} \varepsilon(t) \\ x(t) \\ \mu(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} \varepsilon^* \\ x^* \\ \mu^* \\ \lambda^* \end{pmatrix} + \Gamma(t) \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix} \right\|^2$, we conclude that $\Gamma(t) \begin{pmatrix} \dot\varepsilon(t) \\ \dot x(t) \\ \dot\mu(t) \\ \dot\lambda(t) \end{pmatrix}$ is bounded. In addition, we deduce that the inequality (4.3) is equivalent to

$$\begin{aligned} \dot\xi(t) \quad &\le \quad \left(2\Gamma(t)\dot\Gamma(t)\beta(t) + \Gamma(t)^2\dot\beta(t)\right)\left[\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - m\right] \\ &\quad\ + C\,\Gamma(t)\,\|g(t)\| \end{aligned}$$

for some $C > 0$. Then, integrating this inequality on $[t_0, +\infty)$ and combining the assumption $H^+_{\gamma,\beta}$ yields

$$\begin{aligned} &\int_{t_0}^{+\infty} \beta(t)\Gamma(t)\left[\Phi\left(\varepsilon(t), x(t), \mu(t), \lambda(t)\right) - m\right]dt \\ \le\ &\frac{1}{\rho}\left(\xi(t_0) + C\int_{t_0}^{+\infty}\Gamma(t)g(t)dt\right) < +\infty. \end{aligned}$$

which shows the proof of part(b). □

To sum up, we conclude that the convergence rate of the solutions is not affected when the perturbation term g satisfies the assumption $H_g : \int_{t_0}^{+\infty} \Gamma(t)\,\|g(t)\|\,dt < +\infty$, which means the perturbation term g disappears fast enough.

5. Theoretical and numerical comparisons of 1-ODE and 2-ODE

In this section, we demonstrate theoretical and numerical comparisons for the 1-ODE system (2.8) and the 2-ODE system (3.1). First, we summarize the theoretical comparison of the 1-ODE system (2.8) and the 2-ODE system (3.1), shown as in Table 5.1.

Table 1. Table 5.1 The comparison of theories with the 1-ODE system (2.8) and the 2-ODE system (3.1).

Condition	1-ODE	2-ODE
the convexity of Φ		✓
the nonsingularity of $\nabla\Phi$	✓	
$\arg\min\Phi \ne \emptyset$	✓	✓
the compact of $\arg\min\Phi$	✓	
the positive semidefinite of $JF(x^*)$	✓	

Next, we elaborate two examples of the SOCCVI and use a system of second-order differential equations without a perturbation term and a system of second-order differential equations with a perturbation term $g(t)$ to verify the validity of the solutions. The numerical implementation is coded by Matlab 9.0 and the ordinary differential equation solver adopted is ode45. In the following tests, the parameter $\gamma(t) = \frac{\alpha}{t}$ with $\alpha = 25$, $\beta(t) \equiv 1$ and $|g(t)| \leq 0.02$ is a bounded perturbation.

Example 5.1 Consider the problem

$$\left\langle \frac{1}{2}Dx, y - x \right\rangle \geq 0 \quad \forall y \in C,$$

where $C = \{x \in \mathbb{R}^n \,|Ax - a = 0, Bx - b \preceq 0\}$. with $A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{n\times n}$ being a symmetric matrix. Here $a \in \mathbb{R}^l$, $b \in \mathbb{R}^m$ with $l + m \leq n$. Like what was done [10, 11], we denote

$$D = (D_{ij})_{n\times n}, \quad \text{where } D_{ij} = \begin{cases} 2, & i = j, \\ 1, & |i - j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$A = \left[I_{l\times l}, 0_{l\times(n-l)}\right]_{l\times n}$, $B = \left[0_{m\times(n-m)}, I_{m\times m}\right]_{m\times n}$, $a = 0_{l\times 1}$ and $b = \left(e_{m_1}, \ldots, e_{m_p}\right)$, where $e_{m_i} = (1, \ldots, 0)^T \in \mathbb{R}^{m_i}$.

The CPU-time (seconds) to reach termination condition with the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.1 are shown in Table 5.2. Numerical results of Example 5.1 by the 1-ODE system (2.8) and the 2-ODE system (3.1), 2-ODE system(4.1) are summarized in Table 5.3. The solution trajectories $x(t)$ of the 2-ODE system (3.1) from the given initial random points for Example 5.1 are depicted in Figure 1. The comparison of error rates of $\|x(t) - x^*\|_2$ for the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.1 are presented in Figure 2.

TABLE 2. The CPU-time to reach termination condition (seconds) with the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.1.

	1-ODE	2-ODE
CPU-time(s)	0.012012	0.337710

TABLE 3. Numerical results summary of Example 5.1 by the 1-ODE system (2.8) and the 2-ODE system (3.1), (4.1).

l	m	n	$\mathcal{K}$	x^* with the 1-ODE system (2.8)	x^* with the 2-ODE system (3.1)	x^* with the 2-ODE system (4.1)
2	2	4	$\mathcal{K}^2 \times \mathcal{K}^2$	$\begin{pmatrix} 1.1111e-06 \\ -4.0027e-06 \\ 2.7126e-05 \\ -2.5912e-05 \end{pmatrix}$	$\begin{pmatrix} 5.4773e-05 \\ -6.9542e-05 \\ 1.9268e-04 \\ -1.8164e-04 \end{pmatrix}$	$\begin{pmatrix} -1.0000e-04 \\ -1.0000e-04 \\ 9.9942e-04 \\ -9.5387e-04 \end{pmatrix}$
3	3	6	$\mathcal{K}^3 \times \mathcal{K}^3$	$\begin{pmatrix} 5.3051e-07 \\ 9.4351e-07 \\ 1.2045e-07 \\ 9.9093e-07 \\ 2.6553e-07 \\ 1.3725e-07 \end{pmatrix}$	$\begin{pmatrix} 5.4031e-05 \\ 4.6095e-05 \\ 1.6225e-05 \\ 2.1543e-05 \\ 5.4784e-05 \\ 4.7972e-05 \end{pmatrix}$	$\begin{pmatrix} 4.3752e-04 \\ 6.2845e-04 \\ 9.5689e-05 \\ 3.5876e-04 \\ 4.6213e-04 \\ 4.7210e-05 \end{pmatrix}$
4	4	8	$\mathcal{K}^4 \times \mathcal{K}^4$	$\begin{pmatrix} 1.0993e-07 \\ 1.9390e-07 \\ 1.2045e-07 \\ 2.0947e-07 \\ 2.6684e-07 \\ -3.9048e-08 \\ 5.8089e-08 \\ 1.4516e-08 \end{pmatrix}$	$\begin{pmatrix} 1.0651e-05 \\ 1.2191e-05 \\ 1.3912e-05 \\ -4.5016e-04 \\ 6.2000e-04 \\ -6.4000e-04 \\ -3.7177e-05 \\ -1.9000e-04 \end{pmatrix}$	$\begin{pmatrix} -9.8912e-04 \\ -2.0000e-04 \\ -1.8000e-04 \\ -1.6000e-03 \\ 6.3000e-04 \\ -7.2000e-04 \\ -3.0000e-03 \\ -3.9000e-04 \end{pmatrix}$

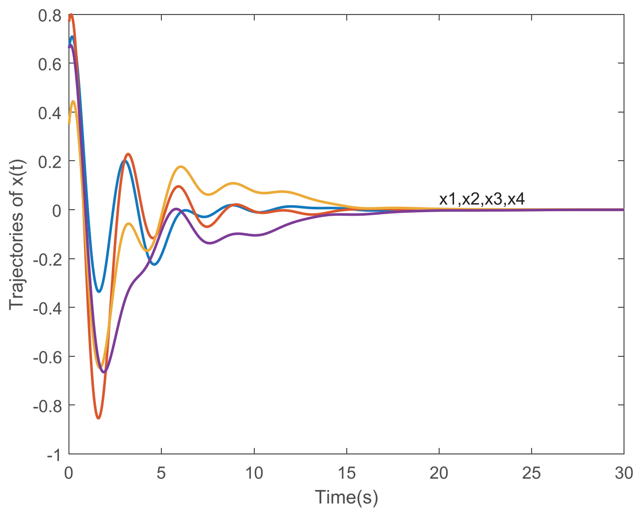
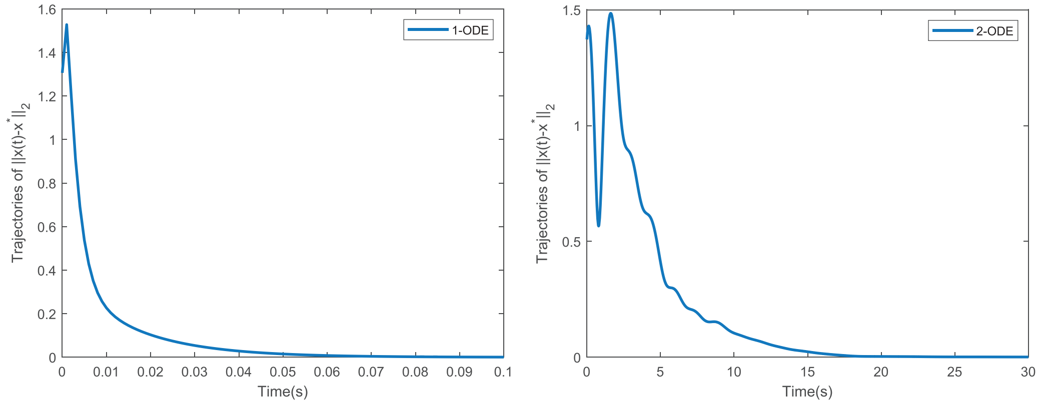


FIGURE 1. Trajectories of $x(t)$ of the 2-ODE system (3.1) for Example 5.1 from four random initial points about x with $l = m = 2$ and $n = 4$.

Example 5.2 Consider the SOCCVI problem

$$\langle F(x^*), y - x^* \rangle \geq 0 \quad y \in C,$$



(a) 1-ODE (b) 2-ODE

(a) 1-ODE (b) 2-ODE

FIGURE 2. Comparison of error rates of $\|x(t)-x^*\|_2$ for the 1-ODE system (2.8) and 2-ODE system (3.1) for Example 5.1.

where

$$C=\left\{x\in \mathbb{R}^8 \,\middle|\, -g\left(x\right)=x\in \mathcal{K}^3\times\mathcal{K}^3\times\mathcal{K}^2\right\}.$$

and

$$F\left(x\right)=\begin{pmatrix} 2x_1+x_2+1 \\ x_1+6x_2-x_3-2 \\ -x_2+3x_3-\frac{6}{5}x_4+3 \\ -\frac{6}{5}x_3+2x_4+\frac{1}{2}\sin x_4\cos x_5\sin x_6+6 \\ \frac{1}{2}\cos x_4\sin x_5\sin x_6+2x_5-\frac{5}{2} \\ -\frac{1}{2}\cos x_4\cos x_5\cos x_6+2x_6+\frac{1}{4}\cos x_6\sin x_7\cos x_8+1 \\ \frac{1}{4}\sin x_6\cos x_7\cos x_8+4x_7-2 \\ -\frac{1}{4}\sin x_6\sin x_7\sin x_8+2x_8+\frac{1}{2} \end{pmatrix}$$

Here, we point out that $x^*=(0,0.3333,0,0,1.2500,0,0.5000,0)$ is one of the optimal solutions.

The CPU-time to reach termination condition (seconds) with the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.2 are shown in Table 5.4. Numerical results of Example 5.2 by the 1-ODE system (2.8), the 2-ODE system (3.1), and the 2-ODE system(4.1) are summarized in Table 5.5. The solution trajectories $x\left(t\right)$ of the 2-ODE system (3.1) from the given initial random points for Example 5.2 are shown in Figure 3. The comparison of error rates of $\|x(t)-x^*\|_2$ for the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.2 are presented in Figure 4.

6. Conclusions

In this paper, we establish the second-order differential equation system (3.1) involving a positive viscous damping coefficient $\gamma(t)$ and a time scale coefficient

TABLE 4. Table 5.4 The CPU-time to reach termination condition (seconds) with the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.2.

	1-ODE	2-ODE
CPU-time(s)	0.062887	1.698720

TABLE 5. Numerical results summary of Example 5.2 by the 1-ODE system (2.8) and the 2-ODE system (3.1), (4.1).

x^* with the 1-ODE system (2.8)	x^* with the 2-ODE system (3.1)	$x^*$with the 2-ODE system (4.1)
$\begin{pmatrix} 1.7059e-06, \\ 0.3333, \\ -2.1409e-05, \\ -1.5039e-05, \\ 1.2500, \\ 4.1031e-11, \\ 0.5000, \\ -4.6562e-13 \end{pmatrix}$	$\begin{pmatrix} 1.9250e-06, \\ 0.3333, \\ -3.4583e-06, \\ 1.2699e-05, \\ 1.2500, \\ 6.4008e-07, \\ 0.5000, \\ 7.7376e-06 \end{pmatrix}$	$\begin{pmatrix} 5.6330e-04, \\ 0.3334, \\ 5.9365e-04, \\ 3.4706e-04, \\ 1.2501, \\ 5.1410e-04, \\ 0.5001, \\ 6.0210e-04 \end{pmatrix}$

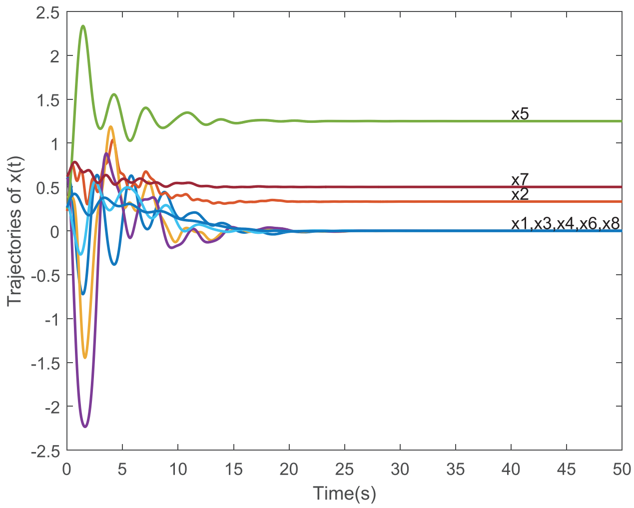
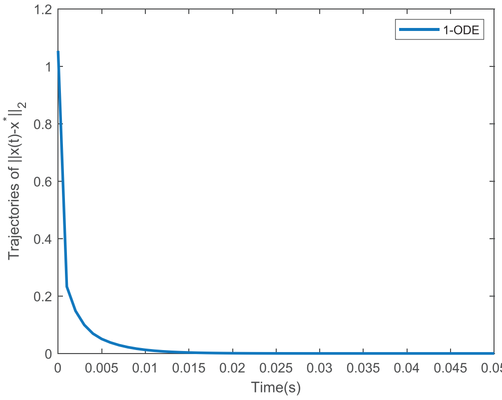
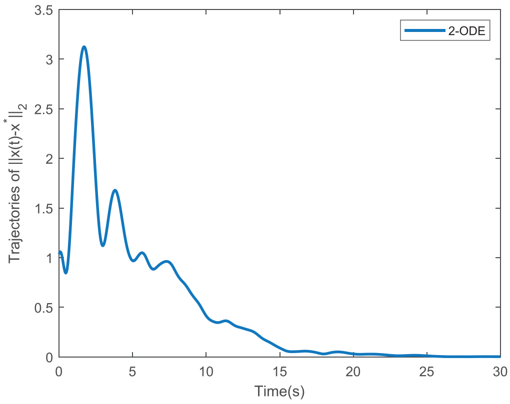


FIGURE 3. Trajectories of $x(t)$ of the 2-ODE system (3.1) for Example 5.2 from eight random initial points about x.

$\beta(t)$ inspired by Attouch et. al [5] to solve the SOCCVI (1.1). We study the asymptotic behavior and convergence rate of the trajectory of the second-order differential equation system (3.1). In addition, the system (4.1) is also used to solve the SOCCVI (1.1) by adding an external perturbation term $g(t)$ to the original system of second-order differential equation (3.1). By comparing the theory and numerical results of the 1-ODE and 2-ODE methods, we find that the constraints of the 2-ODE method are looser than that of the 1-ODE method, but the accuracy of



(a) 1-ODE



(b) 2-ODE

(a) 1-ODE (b) 2-ODE

FIGURE 4. Comparison of error rates of $\|x(t) - x^*\|_2$ for the 1-ODE system (2.8) and the 2-ODE system (3.1) for Example 5.2.

the numerical results of the 2-ODE method is less than that of the 1-ODE method. As far as we know, the study towards the 2-ODE approach for solving the SOCCVI is still very preliminary, it needs to be further investigated and improved.

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