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COMMON ATTRACTORS OF GENERALIZED ITERATED FUNCTION SYSTEMS IN GENERALIZED METRIC SPACES

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Abstract. In this paper, we establish a new common fractal with the assistance of a finite family of generalized contractive mappings, which belong to a special class of mappings defined on a generalized metric space. Consequently, we acquire different results for G-iterated function systems satisfying a different set of generalized contractive conditions. We present an example to reinforce the results proved herein.

1. Introduction

Over the past decades, the study of metrical fixed point theory has attracted much attention with a vast range of applications both within and beyond mathematics [5,10,17,18,24,25]. There are several useful and important applications of this work in the algorithm design for optimization based problems and inverse problems. Some useful results related to application of inverse problem in elliptic partial differential equations and iterative regularization methods for ill-posed inverse problems are presented in [16,29].

Mustafa and Sims [23] generalized metric space by introducing the notion of a G-metric space. Many authors obtained some fixed point theorems for mappings satisfying a variety of contractive conditions in G-metric space [20–22,24]. Abbas and Rhoades [3] motivated the study of a common fixed point theory in generalized metric spaces.

In his 1981 seminal work, Hutchinson [11] laid the mathematical foundations for iterated function systems (IFS). He proved that the Hutchinson operator defined on $\mathbb{R}^k$ has as its fixed point, a subset of $\mathbb{R}^k$ which is closed and bounded, known as an attractor of iterated function system [6,7]. Recently, Strobin [28] established the results of contractive iterated function systems enriched with nonexpansive maps. Miculescu and Mihail [19] studied some results related to the diameter diminishing to zero iterated function system. Several useful results related to the iterated function systems are established in [1,8,14,15,26,27].

Our primary objective in this paper is the construction of a fractal set of generalized iterated function system of generalized contractions in the setup of a G-metric space. We observe that the Hutchinson operator defined on a finite family of contractive mappings on a complete G-metric space is itself a generalized contractive mapping on a family of compact subsets of Y. By successive application of a generalized Hutchinson operator, a final fractal is obtained, and a presentation of a

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nontrivial example shall follow in support of the proven result. We also apply results to obtain the existence of solutions of functional equations arising in the dynamic programming.

Throughout this paper, $\mathbb{N}$ will denote a set of natural numbers, $\mathbb{R}$ a set of real numbers, $\mathbb{R}^+$ a set of nonnegative real numbers and $\mathbb{R}^k$ a set of k-tuples of real numbers. Consistent with Mustafa and Sims [23], we state the following preliminary results.

Definition 1.1. Let Y be a non-empty set. A function $G : Y \times Y \times Y \to [0, +\infty)$ is called a G-metric if the following conditions are satisfied:

(1) $G(u_1, v_1, w_1) = 0$ if $u_1 = v_1 = w_1$ (coincidence),
(2) $0 < G(u_1, u_1, v_1)$ for all $u_1, v_1 \in Y$, with $u_1 \neq v_1$,
(3) $G(u_1, u_1, v_1) \leq G(u_1, v_1, w_1)$ for all $u_1, v_1, w_1 \in Y$, with $v_1 \neq w_1$,
(4) $G(u_1, v_1, w_1) = G(p\{u_1, v_1, w_1\})$, where p is a permutation of u_1, v_1, w_1 (symmetry),
(5) $G(u_1, v_1, w_1) \leq G(u_1, b, b) + G(b, v_1, w_1)$ for all $u_1, v_1, w_1, b \in Y$.

A G-metric is said to be symmetric if $G(u_1, v_1, v_1) = G(v_1, u_1, u_1)$ for all $u_1, v_1 \in Y$.

The pair (Y, G) is called a G-metric space if the function G is a G-metric on Y.

Let (Y, G) be a G-metric space and define the function $d_G : Y \times Y \to [0, +\infty)$, by

$$d_G(u_1, v_1) = G(u_1, v_1, v_1) + G(v_1, u_1, u_1) \text{ for all } u_1, v_1 \in Y,$$

then (d_G, Y) is a metric space.

Definition 1.2 ([4])**.** Let (Y, G) be a G-metric space and $\{y_n\}$ be a sequence in Y. Then

a) $\{y_n\} \subset Y$ is a G-convergent sequence if, for any $\varepsilon > 0$, there is a point $y \in Y$ and a natural number N such that for all $n, m \geq N$, $G(y, y_n, y_m) < \varepsilon$;
b) $\{y_n\} \subset Y$ is a G-Cauchy sequence if, for any $\varepsilon > 0$, there exist an $N \in \mathbb{N}$ such that for all $n, m, l \geq N$, $G(y_n, y_m, y_l) < \varepsilon$;
c) (Y, G) is G-complete if every G-Cauchy sequence in a G-metric space is convergent in Y. $\{y_n\}$ converges to $y \in Y$ if and only if $G(y_m, y_n, y) \to 0$ as $m, n \to \infty$ and $\{y_n\}$ is Cauchy if and only if $G(y_m, y_n, y_l) \to 0$ as $m, n, l \to \infty$.

Definition 1.3 ([4])**.** Let (Y, G) and (Y', G') be two G-metric spaces. Map $h : (Y, G) \to (Y', G')$ is G-continuous at a point $b \in Y$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $u, v \in Y$ and $G(b, u, v) < \delta$ implies $G'(h(b), h(u), h(v)) < \varepsilon$. A map h is G-continuous on Y if and only if it is G-continuous at any point $b \in Y$.

Proposition 1.4 ([4])**.** *Let (Y, G) be a G-metric space. Then:*

(1) *$G(u, v, w)$ is simultaneously continuous in all three of its variables,*
(2) *$G(u, v, v) \leq 2G(v, u, u)$.*

Consider next the following families of subsets of a G-metric space (Y,G).

$N(Y)=\{W : W \text{ is a non-empty subset of } Y\}.$

$CB(Y)=\{W : W \text{ is a non-empty closed and bounded subset of } Y\}.$

$\mathcal{C}^G(Y)=\{W : W \text{ is a non-empty compact subset of } Y\}.$

Definition 1.5 ([12])**.** Let (Y,G) be a G-metric space. A mapping $H_G : CB(Y)\times CB(Y)\times CB(Y)\to[0,+\infty)$ defined as

$$H_G(D,E,F)=\max\left\{\sup_{u\in D}G(u,E,F),\sup_{v\in E}G(v,F,D),\sup_{w\in F}G(w,D,E)\right\}$$

for all $D,E,F\in CB(Y)$, where $G(u,E,F)=\inf\{G(u,v,w) : v\in E, w\in F\}$ is called a Hausdorff G-metric on $CB(Y)$.

If (Y,G) is a G-complete metric space, then the pair $(CB(Y),H_G)$ is also an H_G-complete metric space.

Lemma 1.6. *Let (Y,G) be a G- metric space. Then for all $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{S},\mathcal{U},\mathcal{V}\in\mathcal{C}^G(Y)$, the following conditions are true:*

(a) *If $\mathcal{Q}\subseteq\mathcal{R}$, then* $\sup_{k\in\mathcal{P}}G(k,\mathcal{R},\mathcal{R})\le\sup_{k\in\mathcal{P}}G(k,\mathcal{Q},\mathcal{Q})$;

(b) $\sup_{x\in\mathcal{P}\cup\mathcal{Q}}G(x,\mathcal{R},\mathcal{R})=\max\{\sup_{k\in\mathcal{P}}G(k,\mathcal{R},\mathcal{R}),\sup_{\ell\in\mathcal{Q}}G(\ell,\mathcal{R},\mathcal{R})\}$;

(c) $H_G(\mathcal{P}\cup\mathcal{Q},\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V})\le\max\{H_G(\mathcal{P},\mathcal{R},\mathcal{U}),H_G(\mathcal{Q},\mathcal{S},\mathcal{V})\}$.

Proof. (a) Since $\mathcal{Q}\subseteq\mathcal{R}$, for all $k\in\mathcal{P}$, we have

$$G(k,\mathcal{R},\mathcal{R})=\inf\{G(k,\mu,\mu):\mu\in\mathcal{R}\}\le\inf\{G(k,\ell,\ell):\ell\in\mathcal{Q}\}=G\left(k,\mathcal{Q},\mathcal{Q}\right),$$

this implies that

$$\sup_{k\in\mathcal{P}}G(k,\mathcal{R},\mathcal{R})\le\sup_{k\in\mathcal{P}}G(k,\mathcal{Q},\mathcal{Q}).$$

(b) Note that

$$\begin{aligned}\sup_{x\in\mathcal{P}\cup\mathcal{Q}}G(x,\mathcal{R},\mathcal{R})&=\max\{\sup\{G\left(x,\mathcal{R},\mathcal{R}\right):x\in\mathcal{P}\},\sup\{G\left(x,\mathcal{R},\mathcal{R}\right):x\in\mathcal{Q}\}\}\\&=\max\left\{\sup_{k\in\mathcal{P}}G\left(k,\mathcal{R},\mathcal{R}\right),\sup_{\ell\in\mathcal{Q}}G\left(\ell,\mathcal{R},\mathcal{R}\right)\right\}.\end{aligned}$$

(c) We note that

$$\begin{aligned}&\sup_{x\in\mathcal{P}\cup\mathcal{Q}}G(x,\mathcal{R}\cup\ \mathcal{S},\mathcal{U}\cup\mathcal{V})\\&\le\max\left\{\sup_{k\in\mathcal{P}}G(k,\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V}),\sup_{\ell\in\mathcal{Q}}G(\ell,\mathcal{Q}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V})\right\}\ \text{(from (b))}\\&\le\max\left\{\sup_{k\in\mathcal{P}}G(k,\mathcal{R},\mathcal{U}),\sup_{\ell\in\mathcal{Q}}G(\ell,\mathcal{S},\mathcal{V})\right\}\qquad\text{(from (a))}\\&\le\max\left\{\max\left\{\sup_{k\in\mathcal{P}}G(k,\mathcal{R},\mathcal{U}),\sup_{\mu\in\mathcal{R}}G(\mu,\mathcal{P},\mathcal{U})\right\},\right.\\&\qquad\left.\max\left\{\sup_{\ell\in\mathcal{Q}}G(\ell,\mathcal{S},\mathcal{V}),\sup_{\eta\in\mathcal{S}}G(\eta,\mathcal{Q},\mathcal{V})\right\}\right\}\\&=\max\left\{H_G\left(\mathcal{P},\mathcal{R},\mathcal{U}\right),H_G\left(\mathcal{Q},\mathcal{S},\mathcal{V}\right)\right\}.\end{aligned}$$

Similarly,

$$\sup_{v\in\mathcal{R}\cup\mathcal{S}} G(v,\mathcal{Q}\cup\mathcal{P},\mathcal{U}\cup\mathcal{V})\leq\max\left\{H_G\left(\mathcal{P},\mathcal{R},\mathcal{U}\right),H_G\left(\mathcal{Q},\mathcal{S},\ \mathcal{V}\right)\right\}.$$

Hence

$$\begin{aligned}
&H_G(\mathcal{P}\cup\mathcal{Q},\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V})\\
&\quad=\max\left\{\sup_{v\in\mathcal{P}\cup\ \mathcal{Q}} G(v,\mathcal{R}\cup\mathcal{S},\mathcal{U}\cup\mathcal{V}),\ \sup_{t\in\mathcal{R}\cup\mathcal{S}} G(t,\mathcal{P}\cup\mathcal{Q},\mathcal{U}\cup\mathcal{V})\right\}\\
&\quad\leq\max\left\{H_G\left(\mathcal{P},\mathcal{R},\mathcal{U}\right),H_G\left(\mathcal{Q},\mathcal{S},\mathcal{V}\right)\right\}.
\end{aligned}$$

□

Theorem 1.7 ([23]). *In a complete G-metric space (Y,G) consider a self-map $h:Y\to Y$. Then h is a G-contraction on Y, i.e. for all $u_1,v_1,w_1\in Y$,*

$$G\left(hu_1,hv_1,hw_1\right)\leq\kappa G\left(u_1,v_1,w_1\right)$$

holds, where $\kappa\in[0,1)$.

Mustafa et al. [21] obtained the following useful result of a unique fixed point of generalized G-contraction on Y in G-metric space (Y,G).

Theorem 1.8 ([21]). *In a complete G-metric space (Y,G), let $h:Y\to Y$ be a generalized G-contraction on Y, that is, for all $u_1,v_1,w_1\in Y$, either*

$$\begin{aligned}
G\left(hu_1,hv_1,hw_1\right)\leq{}&\kappa_1G\left(u_1,v_1,w_1\right)+\kappa_2G\left(u_1,hu_1,hu_1\right)\\
&+\kappa_3G\left(v_1,hv_1,hv_1\right)+\kappa_4G\left(w_1,hw_1,hw_1\right)
\end{aligned}$$

or

$$\begin{aligned}
G\left(hu_1,hv_1,hw_1\right)\leq{}&\kappa_1G\left(u_1,v_1,w_1\right)+\kappa_2G\left(u_1,u_1,hu_1\right)\\
&+\kappa_3G\left(v_1,v_1,hv_1\right)+\kappa_4G\left(w_1,w_1,hw_1\right),
\end{aligned}$$

where $\kappa_j\geq0$ for $j\in\{1,2,3,4\}$ with $0\leq\kappa_1+\kappa_2+\kappa_3+\kappa_4<1$. Then h has a unique fixed point, say u^* in Y. Moreover, for any choice $v_0\in Y$, the sequence of iterates $\{v_0,hv_0,h^2v_0,h^3v_0,...\}$ converges to u^*. Furthermore, h is G-continuous.

Theorem 1.9. *In a G-metric space (Y,G) consider a G-contraction map, $h:Y\to Y$. Then*

a) *h maps elements in $\mathcal{C}^G(Y)$ to elements in $\mathcal{C}^G(Y)$.*
b) *If for any $\mathcal{R}\in\mathcal{C}^G(Y)$,*

$$h(\mathcal{R})=\{h(u_1):u_1\in\mathcal{R}\},$$

then $h:\mathcal{C}^G(Y)\to\mathcal{C}^G(Y)$ is a G-contraction on $(\mathcal{C}^G(Y),H_G)$.

Proof. (a) We observe that every generalized contraction mapping is continuous. Moreover, under every continuous map $h:Y\to Y$, the image of a compact set is also compact, that is, if

$$\mathcal{R}\in\mathcal{C}^G(Y),\text{ then }h(\mathcal{R})\in\mathcal{C}^G(Y).$$

(b) Let $\mathcal{Q},\mathcal{R},\mathcal{S}\in\mathcal{C}^G(Y)$ and $h:Y\to Y$ be a generalized contraction mapping, then

$$G\left(hu_1,h\left(\mathcal{R}\right),h\left(\mathcal{S}\right)\right)=\inf\{G\left(hu_1,hv_1,hw_1\right):v_1\in\mathcal{R},w_1\in\mathcal{S}\}$$

$$\begin{aligned}&\leq \inf\{\kappa G\left(u_1, v_1, w_1\right) : v_1 \in \mathcal{R}, w_1 \in \mathcal{S}\}\\ &= \kappa \inf\{G\left(u_1, v_1, w_1\right) : v_1 \in \mathcal{R}, w_1 \in \mathcal{S}\}\\ &= \kappa G\left(u_1, \mathcal{R}, \mathcal{S}\right).\end{aligned}$$

Also

$$\begin{aligned}G\left(hw_1, h\left(\mathcal{R}\right), h\left(\mathcal{Q}\right)\right) &= \inf\{G\left(hw_1, hv_1, hu_1\right) : v_1 \in \mathcal{R}, u_1 \in \mathcal{Q}\}\\ &\leq \inf\{\kappa G\left(w_1, v_1, u_1\right) : v_1 \in \mathcal{R}, w_1 \in \mathcal{Q}\}\\ &= \kappa \inf\{G\left(w_1, v_1, u_1\right) : v_1 \in \mathcal{R}, w_1 \in \mathcal{Q}\}\\ &= \kappa G\left(w_1, \mathcal{R}, \mathcal{Q}\right).\end{aligned}$$

And

$$\begin{aligned}G\left(hv_1, h\left(\mathcal{Q}\right), h\left(\mathcal{S}\right)\right) &= \inf\{G\left(hv_1, hu_1, hw_1\right) : u_1 \in \mathcal{Q}, w_1 \in \mathcal{S}\}\\ &\leq \inf\{\kappa G\left(v_1, u_1, w_1\right) : u_1 \in \mathcal{Q}, w_1 \in \mathcal{S}\}\\ &= \kappa \inf\{G\left(v_1, u_1, w_1\right) : u_1 \in \mathcal{Q}, w_1 \in \mathcal{S}\}\\ &= \kappa G\left(v_1, \mathcal{Q}, \mathcal{S}\right).\end{aligned}$$

Now

$$\begin{aligned}&H_G\left(h\left(\mathcal{R}\right), h\left(\mathcal{S}\right), h\left(\mathcal{Q}\right)\right)\\ &\quad = \max\Big\{\sup_{u_1\in\mathcal{Q}} G\left(hu_1, h\left(\mathcal{R}\right), h\left(\mathcal{S}\right)\right), \sup_{w_1\in\mathcal{S}} G\left(hw_1, h\left(\mathcal{R}\right), h\left(\mathcal{Q}\right)\right),\\ &\qquad \sup_{v_1\in\mathcal{R}} G\left(hv_1, h\left(\mathcal{Q}\right), h\left(\mathcal{S}\right)\right)\Big\}\\ &\leq \max\Big\{\sup_{u_1\in\mathcal{Q}} \kappa G\left(u_1, \mathcal{R}, \mathcal{S}\right), \sup_{w_1\in\mathcal{S}} \kappa G\left(w_1, \mathcal{R}, \mathcal{Q}\right), \sup_{v_1\in\mathcal{R}} \kappa G\left(v_1, \mathcal{Q}, \mathcal{S}\right)\Big\}\\ &= \kappa \max\Big\{\sup_{u_1\in\mathcal{Q}} G\left(u_1, \mathcal{R}, \mathcal{S}\right), \sup_{w_1\in\mathcal{S}} G\left(w_1, \mathcal{R}, \mathcal{Q}\right), \sup_{v_1\in\mathcal{R}} G\left(v_1, \mathcal{Q}, \mathcal{S}\right)\Big\}\\ &= \kappa H_G\left(\mathcal{R}, \mathcal{S}, \mathcal{Q}\right).\end{aligned}$$

Thus $h : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ is a G-contraction. □

Theorem 1.10. *Consider a G-metric space (Y, G). Let $\{h_k : k = 1, 2, ..., q\}$ be a finite family of G-contractions on Y with contraction constants $\kappa_1, \kappa_2, ..., \kappa_q$, respectively. Define $\Psi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ by*

$$\Psi(\mathcal{R}) = \cup_{k=1}^{q} h_k(\mathcal{R}),$$

for every $\mathcal{R} \in \mathcal{C}^G(Y)$. Then Ψ is also a G-contraction mapping on $\mathcal{C}^G(Y)$ with contraction constant $\kappa = \max\{\kappa_1, \kappa_2, ..., \kappa_q\}$.

Proof. We demonstrate the assertion for $q = 2$. Let $h_1, h_2 : Y \to Y$ be two contractions. Take $\mathcal{R}, \mathcal{S}, \mathcal{Q} \in \mathcal{C}^G(Y)$. From Lemma 1.6. (c), we have

$$\begin{aligned}H_G(\Psi(\mathcal{R}), \Psi(\mathcal{S}), \Psi(\mathcal{Q})) &= H_G(h_1(\mathcal{R}) \cup h_2(\mathcal{R}), h_1(\mathcal{S}) \cup h_2(\mathcal{S}), h_1(\mathcal{Q}) \cup h_2(\mathcal{Q}))\\ &\leq \max\{H_G(h_1(\mathcal{R}), h_1(\mathcal{S}), h_1(\mathcal{Q})), H_G(h_2(\mathcal{R}), h_2(\mathcal{S}), h_2(\mathcal{Q}))\}\\ &\leq \max\{\kappa_1 H_G(\mathcal{R}, \mathcal{S}, \mathcal{Q}), \kappa_2 H_G(\mathcal{R}, \mathcal{S}, \mathcal{Q})\}\\ &\leq \kappa H_G(\mathcal{R}, \mathcal{S}, \mathcal{Q}),\end{aligned}$$

where $\kappa = \max\{\kappa_1, \kappa_2\}$. □

Theorem 1.11. *In a complete G-metric space (Y, G), let $\{h_k : k = 1, 2, ..., q\}$ be a finite family of G-contraction mappings on Y. Define a mapping Ψ on $\mathcal{C}^G(Y)$ by*

$$\Psi(\mathcal{R}) = \cup_{k=1}^{q} h_k(\mathcal{R}),$$

for each $\mathcal{R} \in \mathcal{C}^G(Y)$. Then

(i) $\Psi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$.

(ii) *Ψ has a distinct fixed point $U_1 \in \mathcal{C}^G(Y)$, that is, $U_1 = \Psi(U_1) = \cup_{k=1}^{q} h_k(U_1)$.*

(iii) *for any set $\mathcal{R}_0 \in \mathcal{C}^G(Y)$, the sequence*

$$\{\mathcal{R}_0, \Psi(\mathcal{R}_0), \Psi^2(\mathcal{R}_0), ...\}$$

converges to U_1.

Proof. (i) Since each h_k is a G-contraction mapping, the conclusion follows, from the definition of Ψ and Theorem 1.9.

(ii) Using Theorem 1.11 we note that $\Psi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ is also a G-contraction mapping. Thus if (Y, G) is a complete G-metric space, then $(\mathcal{C}^G(Y), H_G)$ is complete. Consequently, we deduce (ii) and (iii) from Theorem 1.8. □

Now we define G-iterated function system in the setup of G-metric space.

Definition 1.12. Let (Y, G) be a G-metric space. If $h_k : Y \to Y$, $k = 1, 2, ..., q$ are G-contraction mappings, then $\{Y; h_k, k = 1, 2, \ldots, q\}$ is a G-iterated function system (G-IFS).

It follows that the G-iterated function system is composed of a G-metric space and a finite family of G-contractions on Y.

The next definition is about the attractor of a G-iterated function system.

Definition 1.13. Let (Y, G) be a G-metric space with $\mathcal{R} \in \mathcal{C}^G(Y)$, then $\mathcal{R}$ is called an attractor of the G-iterated function system if

(i) $\Psi(\mathcal{R}) = \mathcal{R}$ and

(ii) there exists an open set $V_1 \subseteq Y$ such that $\mathcal{R} \subseteq V_1$ and $\lim_{k\to\infty} \Psi^k(\mathcal{S}) = \mathcal{R}$ for any compact set $\mathcal{S} \subseteq V_1$, where the limit is taken with respect to the G-Hausdorff metric.

The maximal open set V_1 such that (ii) is satisfied is known as a basin of attraction.

2. Generalized iterated function system in G-metric spaces

Recently, some results on generalized iterated function system for multi-valued mapping in a metric space are appeared in [9]. We discuss a generalized iterated function system in the setup of G-metric spaces. To begin with, we define a generalized contraction self-mapping which some preliminary results will follow.

Definition 2.1. In a G-metric space (Y, G), let $f, g, h : Y \to Y$ be three self-mappings. A triplet (f, g, h) is called a generalized G-contraction mappings if

$$G(fu_1, gv_1, hw_1) \leq \lambda G(u_1, v_1, w_1)$$

for all $u_1, v_1, w_1 \in Y$, where $\lambda \in [0, 1)$.

Definition 2.2. Consider a G-metric space (Y, G) and let $f, g, h : Y \to Y$ be continuous mappings. If the triplet of mappings (f, g, h) is a generalized G-contraction with $\lambda \in [0, 1)$. Then

(1) the elements in $\mathcal{C}^G(Y)$ are mapped to elements in $\mathcal{C}^G(Y)$ under f, g and h;
(2) if for an arbitrary $U \in \mathcal{C}^G(Y)$, the mappings $f, h, g : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ are defined as

$$\begin{aligned} f(U) &= \{f(u_1) : u_1 \in U\}, \\ g(U) &= \{g(v_1) : v_1 \in U\}, \\ h(U) &= \{h(w_1) : w_1 \in U\}, \end{aligned}$$

then the triplet (f, g, h) is a generalized G-contraction on $(\mathcal{C}^G(Y), H_G)$.

Proof. To prove (1): Since f is a continuous mapping and the image of a compact subset under a continuous mapping, $f : Y \to Y$ is compact, then

$$U \in \mathcal{C}^G(Y) \text{ implies that } f(U) \in \mathcal{C}^G(Y).$$

Similarly,

$$U \in \mathcal{C}^G(Y) \text{ implies that } g(U) \in \mathcal{C}^G(Y) \text{ and } h(U) \in \mathcal{C}^G(Y).$$

To prove (2): Let $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$. Since the triplet (f, g, h) is a generalized G-contraction mappings on Y, so we have

$$G(fu_1, gv_1, hw_1) \le \lambda G(u_1, v_1, w_1) \text{ for all } u_1, v_1, w_1 \in Y,$$

where $\lambda \in [0, 1)$.

Now

$$\begin{aligned} G(fu_1, g(\mathcal{R}), h(\mathcal{N})) &= \inf\{G(fu_1, gv_1, hw_1) : v_1 \in \mathcal{R}, w_1 \in \mathcal{N}\} \\ &\le \inf\{\lambda G(u_1, v_1, w_1) : v_1 \in \mathcal{R}, w_1 \in \mathcal{N}\} \\ &= \lambda G(u_1, \mathcal{R}, \mathcal{N}). \end{aligned}$$

In the same manner,

$$\begin{aligned} G(gv_1, f(\mathcal{Q}), h(\mathcal{N})) &= \inf\{G(gv_1, fu_1, htw_1) : u_1 \in \mathcal{Q}, w_1 \in \mathcal{N}\} \\ &\le \inf\{\lambda G(v_1, u_1, w_1) : u_1 \in \mathcal{Q}, w_1 \in \mathcal{N}\} \\ &= \lambda G(v_1, \mathcal{Q}, \mathcal{N}) \end{aligned}$$

and

$$\begin{aligned} G(hw_1, f(\mathcal{Q}), g(\mathcal{R})) &= \inf\{G(hw_1, fu_1, gv_1) : u_1 \in \mathcal{Q}, v_1 \in \mathcal{R}\} \\ &\le \inf\{\lambda G(w_1, u_1, v_1) : u_1 \in \mathcal{Q}, v_1 \in \mathcal{R}\} \\ &= \lambda G(w_1, \mathcal{Q}, \mathcal{R}). \end{aligned}$$

Now

$$\begin{aligned} H_G(f(\mathcal{Q}), g(\mathcal{R}), h(\mathcal{N})) = \max\Big\{ &\sup_{u_1 \in \mathcal{L}} G(fu_1, g(\mathcal{R}), h(\mathcal{N})), \\ &\sup_{v_1 \in \mathcal{M}} G(gv_1, f(\mathcal{Q}), h(\mathcal{N})), \sup_{w_1 \in \mathcal{N}} G(hw_1, f(\mathcal{Q}), g(\mathcal{R}))\Big\} \\ \le \max\Big\{ &\sup_{u_1 \in \mathcal{L}} \lambda G(u_1, \mathcal{R}, \mathcal{N}), \sup_{v_1 \in \mathcal{M}} \lambda G(v_1, \mathcal{Q}, \mathcal{N}), \sup_{w_1 \in \mathcal{N}} \lambda G(w_1, \mathcal{Q}, \mathcal{R})\Big\} \end{aligned}$$

$$= \lambda \max\left\{ \sup_{u_1 \in \mathcal{L}} G(u_1, \mathcal{R}, \mathcal{N}), \sup_{v_1 \in \mathcal{M}} G(v_1, \mathcal{Q}, \mathcal{N}), \sup_{w_1 \in \mathcal{N}} G(w_1, \mathcal{Q}, \mathcal{R}) \right\}$$
$$= \lambda H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}).$$

Hence, the triplet (f, g, h) is a generalized G-contraction mappings on $(\mathcal{C}^G(Y), H_G)$. □

Proposition 2.3. *In a G-metric space (Y, G), suppose the mappings $f_k, g_k, h_k : Y \to Y$ for $k = 1, 2, \ldots, q$ are continuous and satisfy*

$$G(f_k u_1, g_k v_1, h_k w_1) \leq \lambda_k G(u_1, v_1, w_1) \text{ for all } u_1, v_1, w_1 \in Y,$$

where $\lambda_k \in [0, 1)$ for each $k \in \{1, 2, \ldots, q\}$. Then the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ defined by

$$\Upsilon(\mathcal{Q}) = \cup_{k=1}^{q} f_k(\mathcal{Q}), \text{ for each } \mathcal{Q} \in \mathcal{C}^G(Y),$$

$$\Psi(\mathcal{R}) = \cup_{k=1}^{q} g_k(\mathcal{R}), \text{ for each } \mathcal{R} \in \mathcal{C}^G(Y)$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^{q} h_k(\mathcal{N}), \text{ for each } \mathcal{N} \in \mathcal{C}^G(Y)$$

satisfy

$$H_G(\Upsilon\mathcal{Q}, \Psi\mathcal{R}, \Phi\mathcal{N}) \leq \lambda_* H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}) \text{ for all } \mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y),$$

where $\lambda_* = \max\{\lambda_k : k = 1, 2, ..., q\}$, that is, the triplet (Υ, Ψ, Φ) is a generalized G-contraction on $\mathcal{C}^G(Y)$.

Proof. We give a proof for $q = 2$. Let $f_k, g_k, h_k, : Y \to Y$, $k \in \{1, 2\}$ be self-mappings such that (f_1, g_1, h_1) and (f_2, g_2, h_2) are triplets of generalized G-contractions. For $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$ and from Lemma 1.7 (c),

$$\begin{aligned} H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) &= H_G(f_1(\mathcal{Q}) \cup f_2(\mathcal{Q}), g_1(\mathcal{R}) \cup g_2(\mathcal{R}), h_1(\mathcal{N}) \cup h_2(\mathcal{N})) \\ &\leq \max\{H_G(f_1(\mathcal{Q}), g_1(\mathcal{R}), h_1(\mathcal{N})), H_G(f_2(\mathcal{Q}), g_2(\mathcal{R}), h_2(\mathcal{N}))\} \\ &\leq \max\{\lambda_1 H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}), \lambda_2 H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N})\} \\ &\leq \lambda_* H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}). \end{aligned}$$

□

Definition 2.4. In a G-metric space (Y, G), let $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$. The mappings (Υ, Ψ, Φ) are called

(I) generalized G-Hutchinson contractive operators (type I) if for any $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$,

$$H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) \leq Z_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N})$$

holds, where

$$\begin{aligned} Z_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = {} & \alpha H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}) + \beta H_G(\mathcal{Q}, \Upsilon(\mathcal{Q}), \Upsilon(\mathcal{Q})) \\ & + \gamma H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})) + \eta H_G(\mathcal{N}, \Phi(\mathcal{N}), \Phi(\mathcal{N})), \end{aligned}$$

with $\alpha, \beta, \gamma, \eta \geq 0$ and $\alpha + \beta + \gamma + \eta < 1$.

(II) generalized G-Hutchinson contractive operators (type II) if for any $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$,

$$H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) \leq R_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N})$$

holds, where

$$\begin{aligned} R_{\Upsilon,\Psi,\Phi}(\mathcal{Q}, \mathcal{R}, \mathcal{N}) = {} & \lambda_1 H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}) + \lambda_2[H_G(\mathcal{Q}, \mathcal{Q}, \Upsilon(\mathcal{Q})) \\ & + H_G(\mathcal{R}, \mathcal{R}, \Psi(\mathcal{R})) + H_G(\mathcal{N}, \mathcal{N}, \Phi(\mathcal{N}))] \\ & + \lambda_3[H_G(\Upsilon(\mathcal{Q}), \mathcal{R}, \mathcal{N}) + H_G(\mathcal{Q}, \Psi(\mathcal{R}), \mathcal{N}) \\ & + H_G(\mathcal{Q}, \mathcal{R}, \Phi(\mathcal{N}))], \end{aligned}$$

with $\lambda_j \geq 0$ for $j \in \{1, 2, 3\}$ and $\lambda_1 + 3\lambda_2 + 4\lambda_3 < 1$.

Note that if the mappings (Υ, Ψ, Φ) defined as in Proposition 2.3 are generalized G-contractions on $\mathcal{C}^G(Y)$, then (Υ, Ψ, Φ) are generalized G-Hutchinson contractive operators.

Definition 2.5. In a complete G-metric space (Y, G), if $f_k, g_k, h_k : Y \to Y$, $k = 1, 2, ..., q$ are continuous mappings such that each triplet (f_k, g_k, h_k) for $k = 1, 2, ..., q$ is a generalized G-contraction, then $\{Y; (f_k, g_k, h_k), k = 1, 2, \ldots, q\}$ is called the generalized G-iterated function system.

Consequently, the generalized G-iterated function system consists of a G-metric space and a finite collection of generalized G-contraction mappings on Y.

Definition 2.6. Let (Y, G) be a complete G-metric space and $U \subseteq Y$ a non-empty compact set. Then U is the unique common attractor of the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ if

i) $\Upsilon(U) = \Psi(U) = \Phi(U) = U$ and
ii) there exists an open set $V_1 \subseteq Y$ such that $U \subseteq V_1$ and $\lim_{k\to+\infty} \Upsilon^k(\mathcal{Q}) = \lim_{k\to+\infty} \Psi^k(\mathcal{R}) = \lim_{k\to+\infty} \Phi^k(\mathcal{N}) = U$ for any compact sets $\mathcal{Q}, \mathcal{R}, \mathcal{N} \subseteq V_1$, where the limit is taken relative to the G-Hausdorff metric.

Thus, the maximal open set V_1 such that (ii) is satisfied is called a basin of common attraction.

3. Main results

We state and prove some theorems on the existence and uniqueness of a common attractor of generalized G-Hutchinson contractive operators in the setup of G-metric space.

Theorem 3.1. *In a complete G-metric space (Y, G), let $\{Y; (f_k, g_k, h_k), k = 1, 2, \ldots, q\}$ be the generalized G-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ by*

$$\Upsilon(\mathcal{Q}) = \cup_{k=1}^q f_k(\mathcal{Q}),$$
$$\Psi(\mathcal{R}) = \cup_{k=1}^q g_k(\mathcal{R}),$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^q h_k(\mathcal{N})$$

for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$. If the mappings (Υ, Ψ, Φ) are generalized G-Hutchinson contractive operators (type I), then Υ, Ψ and Φ have a unique common attractor $U^* \in \mathcal{C}^G(Y)$, that is,

$$U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*).$$

Additionally, for any arbitrarily chosen initial set $\mathcal{R}_0 \in \mathcal{C}^G(Y)$, the sequence

$$\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), ...\}$$

of compact sets converges to the common attractor U^*.

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $U^* \in \mathcal{C}^G(Y)$ is such that $\Upsilon(U^*) = U^*$. We need to show that $U^* = \Psi(U^*) = \Phi(U^*)$. As the mappings (Υ, Ψ, Φ) are generalized G-Hutchinson contractive operators (type I), we get

$$\begin{aligned} H_G(U^*, \Psi(U^*), \Phi(U^*)) &= H_G(\Upsilon(U^*), \Psi(U^*), \Phi(U^*)) \\ &\leq \alpha H_G(U^*, U^*, U^*) + \beta H_G(U^*, \Upsilon(U^*), \Upsilon(U^*)) \\ &\quad + \gamma H_G(U^*, \Psi(U^*), \Psi(U^*)) + \eta H_G(U^*, \Phi(U^*), \Phi(U^*)) \\ &= \gamma H_G(U^*, \Psi(U^*), \Psi(U^*)) + \eta H_G(U^*, \Phi(U^*), \Phi(U^*)) \\ &\leq (\gamma + \eta) H_G(U^*, \Psi(U^*), \Phi(U^*)), \end{aligned}$$

thus

$$H_G(U^*, \Psi(U^*), \Phi(U^*)) \leq \lambda H_G(U^*, \Psi(U^*), \Phi(U^*)),$$

where $\lambda = \gamma + \eta < 1$, which implies that $H_G(U^*, \Psi(U^*), \Phi(U^*)) = 0$ a so $U^* = \Psi(U^*) = \Phi(U^*)$. In an analogous manner, for $U^* = \Phi(U^*)$ or for $U^* = \Psi(U^*)$, we obtain that U^* is the common attractor of Υ, Ψ and Φ.

We proceed by showing that Υ, Ψ and Φ have a unique common attractor. Let $\mathcal{R}_0 \in \mathcal{C}^G(Y)$ be chosen randomly. Define a sequence $\{\mathcal{R}_k\}$ by $\mathcal{R}_{3k+1} = \Upsilon(\mathcal{R}_{3k})$, $\mathcal{R}_{3k+2} = \Psi(\mathcal{R}_{3k+1})$ and $\mathcal{R}_{3k+3} = \Phi(\mathcal{R}_{3k+2})$, $k = 0, 1, 2, \ldots$. If $\mathcal{R}_k = \mathcal{R}_{k+1}$ for some k, with $k = 3n$, then $U^* = \mathcal{R}_{3k}$ is an attractor of Υ and from the Proof above, U^* is a common attractor for Υ, Ψ and Φ. The same is true for $k = 3n+1$ or $k = 3n+2$. We assume that $\mathcal{R}_k \neq \mathcal{R}_{k+1}$ for all $k \in \mathbb{N}$, then

$$\begin{aligned} &H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) = H_G(\Upsilon(\mathcal{R}_{3k}), \Psi(\mathcal{R}_{3k+1}), \Phi(\mathcal{R}_{3k+2})) \\ &\quad \leq \alpha H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + \beta H_G(\mathcal{R}_{3k}, \Upsilon(\mathcal{R}_{3k}), \Upsilon(\mathcal{R}_{3k})) \\ &\qquad + \gamma H_G(\mathcal{R}_{3k+1}, \Psi(\mathcal{R}_{3k+1}), \Psi(\mathcal{R}_{3k+1})) + \eta H_G(\mathcal{R}_{3k+2}, \Phi(\mathcal{R}_{3k+2}), \Phi(\mathcal{R}_{3k+2})) \\ &\quad = \alpha H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + \beta H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+1}) \\ &\qquad + \gamma H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+2}) + \eta H_G(\mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}, \mathcal{R}_{3k+3}) \\ &\quad \leq \alpha H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + \beta H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) \\ &\qquad + \gamma H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) + \eta H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}). \end{aligned}$$

Thus, we have

$$(1 - \gamma - \eta) H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) \leq (\alpha + \beta) H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}).$$

Hence,

$$H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) \leq \lambda H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}),$$

where $\lambda = \dfrac{\alpha+\beta}{1-\gamma-\eta}$, with $0<\lambda<1$. Similarly, one can show that

$$H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+4}) \leq \lambda H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3})$$

and

$$H_G(\mathcal{R}_{3k+3},\mathcal{R}_{3k+4},\mathcal{R}_{3k+5}) \leq \lambda H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+4}).$$

Thus, for all k,

$$\begin{aligned} H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}) &\leq \lambda H_G\left(\mathcal{R}_k,\mathcal{R}_{k+1},\mathcal{R}_{k+2}\right) \\ &\leq \cdots \leq \lambda^{k+1} H_G\left(\mathcal{R}_0,\mathcal{R}_1,\mathcal{R}_2\right). \end{aligned}$$

Now, for l,m,k, with $l>m>k$,

$$\begin{aligned} H_G(\mathcal{R}_k,\mathcal{R}_m,\mathcal{R}_l) &\leq H_G(\mathcal{R}_k,\mathcal{R}_{k+1},\mathcal{R}_{k+1}) + H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+2}) \\ &\quad + \cdots + H_G(\mathcal{R}_{l-1},\mathcal{R}_{l-1},\mathcal{R}_l) \\ &\leq H_G(\mathcal{R}_k,\mathcal{R}_{k+1},\mathcal{R}_{k+2}) + H_G(\mathcal{R}_{k+1},\mathcal{R}_{k+2},\mathcal{R}_{k+3}) \\ &\quad + \cdots + H_G(\mathcal{R}_{l-2},\mathcal{R}_{l-1},\mathcal{R}_l) \\ &\leq [\lambda^k+\lambda^{k+1}+\cdots+\lambda^{l-2}] H_G\left(\mathcal{R}_0,\mathcal{R}_1,\mathcal{R}_2\right) \\ &\leq \frac{\lambda^k}{1-\lambda} H_G(\mathcal{R}_0,\mathcal{R}_1,\mathcal{R}_2). \end{aligned}$$

Note that if $l=m>k$, we get identical results and if $l>m=k$, then

$$H_G(\mathcal{R}_k,\mathcal{R}_m,\mathcal{R}_l) \leq \frac{\lambda^{k-1}}{1-\lambda} H_G(\mathcal{R}_0,\mathcal{R}_1,\mathcal{R}_2).$$

and so $\lim_{k,m,l\to+\infty} H_G(\mathcal{R}_k,\mathcal{R}_m,\mathcal{R}_l)=0$. Thus $\{\mathcal{R}_k\}$ is a G-Cauchy sequence in $\mathcal{C}^G(Y)$. Since $(\mathcal{C}^G(Y),H_G)$ is a complete G-metric space, there exists $U_1\in\mathcal{C}^G(Y)$ such that $\lim_{k\to+\infty}\mathcal{R}_k=U_1$, that is, $\lim_{k\to+\infty} H_G\left(\mathcal{R}_k,\mathcal{R}_k,U_1\right)=0$.

To prove that $\Upsilon\left(U_1\right)=U_1$, we have claim in the contrary

$$\begin{aligned} &H_G(\Upsilon\left(U_1\right),\mathcal{R}_{3k+2},\mathcal{R}_{3k+3}) H_G(\Upsilon\left(U_1\right),\Psi\left(\mathcal{R}_{3k+1}\right),\Phi\left(\mathcal{R}_{3k+2}\right)) \\ &\quad\leq \alpha H_G(\Upsilon\left(U_1\right),\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + \beta H_G(U_1,\Upsilon\left(U_1\right),\Upsilon\left(U_1\right)) \\ &\qquad + \gamma H_G(\mathcal{R}_{3k+1},\Psi\left(\mathcal{R}_{3k+1}\right),\Psi\left(\mathcal{R}_{3k+1}\right)) + \eta H_G(\mathcal{R}_{3k+2},\Phi\left(\mathcal{R}_{3k+2}\right),\Phi\left(\mathcal{R}_{3k+2}\right)) \\ &\leq \alpha H_G(U_1,\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + \beta H_G(\Upsilon\left(U_1\right),U_1,\mathcal{R}_{3k+1}) \\ &\qquad + \gamma H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+2}) + \eta H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+3},\mathcal{R}_{3k+3}), \end{aligned}$$

taking the limit as $k\to+\infty$, yields

$$H_G(\Upsilon\left(U_1\right),U_1,U_1) \leq \beta H_G(\Upsilon\left(U_1\right),U_1,U_1),$$

which is a contradiction as $\beta<1$. Thus $\Upsilon\left(U_1\right)=U_1$. Follows the conclusion above, U^* is a common attractor of Υ, Ψ and Φ.

For uniqueness, we suppose that V_1 is another common attractor of Υ, Ψ and Φ. Then

$$\begin{aligned} H_G(U_1,V_1,V_1) &= H_G(\Upsilon(U_1),\Psi(V_1),\Phi(V_1)) \\ &\leq \alpha H_G(U_1,V_1,V_1) + \beta H_G(U_1,\Upsilon\left(U_1\right),\Upsilon\left(U_1\right)) \\ &\quad + \gamma H_G(V_1,\Psi\left(V_1\right),\Psi\left(V_1\right)) + \eta H_G(V_1,\Phi\left(V_1\right),\Phi\left(V_1\right)) \end{aligned}$$

$$= \alpha H_G(U_1, V_1, V_1) + \beta H_G(U_1, U_1, U_1)$$
$$+ \gamma H_G(V_1, V_1, V_1) + \eta H_G(V_1, V_1, V_1)$$
$$= \alpha H_G(U_1, V_1, V_1)$$

from which we conclude that $H_G(U_1, V_1, V_1) = 0$ and thus $U_1 = V_1$. Hence U_1 is the unique common attractor of Υ, Ψ and Φ. □

Theorem 3.2 (Generalized Collage I)**.** *In a complete G-metric space (Y, G), let $\{Y; (f_k, g_k, h_k), k = 1, 2, \ldots, q\}$ be the generalized G-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ by*

$$\Upsilon(\mathcal{Q}) = \cup_{k=1}^{q} f_k(\mathcal{Q}),$$
$$\Psi(\mathcal{R}) = \cup_{k=1}^{q} g_k(\mathcal{R}),$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^{q} h_r(\mathcal{N})$$

for $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$. Suppose that the mappings (Υ, Ψ, Φ) are generalized G-Hutchinson contractive operators (type I) and $U_1 \in \mathcal{C}^G(Y)$ is the common attractor for Υ, Ψ and Φ. If for any $\mathcal{R} \in \mathcal{C}^G(Y)$ such that

(a) $H_G(\mathcal{R}, \Upsilon(\mathcal{R}), \Upsilon(\mathcal{R})) \leq \varepsilon$, *then*

$$H_G(\mathcal{R}, U_1, U_1) \leq \frac{\varepsilon\,(1+\beta)}{1-\alpha}.$$

(b) $H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})) \leq \varepsilon$, *then*

$$H_G(\mathcal{R}, U_1, U_1) \leq \frac{\varepsilon\,(1+\gamma)}{1-\alpha}.$$

(c) $H_G(\mathcal{R}, \Phi(\mathcal{R}), \Phi(\mathcal{R})) \leq \varepsilon$, *then*

$$H_G(\mathcal{R}, U_1, U_1) \leq \frac{\varepsilon\,(1+\eta)}{1-\alpha}.$$

Proof. To prove (a): Let $H_G(\mathcal{R}, \Upsilon(\mathcal{R}), \Upsilon(\mathcal{R})) \leq \varepsilon$ for any $\mathcal{R} \in \mathcal{C}^G(Y)$, then

$$\begin{aligned}
H_G(\mathcal{R}, U_1, U_1) &\leq H_G(\mathcal{R}, \Upsilon(\mathcal{R}), \Upsilon(\mathcal{R})) + H_G(\Upsilon(\mathcal{R}), U_1, U_1)\\
&= H_G(\mathcal{R}, \Upsilon(\mathcal{R}), \Upsilon(\mathcal{R})) + H_G(\Upsilon(\mathcal{R}), \Psi(U_1), \Phi(U_1))\\
&\leq \varepsilon + \alpha H_G(\mathcal{R}, U_1, U_1) + \beta H_G(\mathcal{R}, \Upsilon(\mathcal{R}), \Upsilon(\mathcal{R}))\\
&\quad + \gamma H_G(U_1, \Psi(U_1), \Psi(U_1)) + \eta H_G(U_1, \Phi(U_1), \Phi(U_1))\\
&= \varepsilon + \alpha H_G(\mathcal{R}, U_1, U_1) + \beta H_G(\mathcal{R}, \Upsilon(\mathcal{R}), \Upsilon(\mathcal{R})),
\end{aligned}$$

which further implies that

$$H_G(\mathcal{R}, U_1, U_1) \leq \frac{\varepsilon\,(1+\beta)}{1-\alpha}.$$

To prove (b): Assume that $H_G(\mathcal{R}, \Psi\,(\mathcal{R})\,, \Psi(\mathcal{R})) \leq \varepsilon$ for any $\mathcal{R} \in \mathcal{C}^G(Y)$. Then,

$$\begin{aligned}
H_G(\mathcal{R}, U_1, U_1) &\leq H_G(\mathcal{R}, \Psi(\mathcal{R}), \Psi(\mathcal{R})) + H_G(\Psi(\mathcal{R}), U_1, U_1)\\
&\leq \varepsilon + H_G(\Upsilon(U_1), \Psi(\mathcal{R}), \Phi(U_1))\\
&\leq \varepsilon + \alpha H_G(U_1, \mathcal{R}, U_1) + \beta H_G(U_1, \Upsilon(U_1), \Upsilon(U_1))
\end{aligned}$$

$$+\gamma H_G(\mathcal{R},\Psi(\mathcal{R}),\Psi(\mathcal{R}))+\eta H_G(U_1,\Phi(U_1),\Phi(U_1))$$
$$=\varepsilon+\alpha H_G(\mathcal{R},U_1,U_1)+\gamma H_G(\mathcal{R},\Psi(\mathcal{R}),\Psi(\mathcal{R})),$$

which further implies that

$$H_G(\mathcal{R},\mathcal{R},U_1)\le\frac{\varepsilon\,(1+\gamma)}{1-\alpha}.$$

To prove (c): Assuming that $H_G(\mathcal{R},\Phi(\mathcal{R}),\Phi(\mathcal{R}))\le\varepsilon$ for any $\mathcal{R}\in\mathcal{C}^G(Y)$, we have

$$\begin{aligned}H_G(\mathcal{R},U_1,U_1)&\le H_G(\mathcal{R},\Phi(\mathcal{R}),\Phi(\mathcal{R}))+H_G(\Phi(\mathcal{R}),U_1,U_1)\\&\le\varepsilon+H_G(\Upsilon(U_1),\Psi(U_1),\Phi(\mathcal{R}))\\&\le\varepsilon+\alpha H_G(U_1,U_1,\mathcal{R})+\beta H_G(U_1,\Upsilon(U_1),\Upsilon(U_1))\\&\quad+\gamma H_G(U_1,\Psi(U_1),\Psi(U_1))+\eta H_G(\mathcal{R},\Phi(\mathcal{R}),\Phi(\mathcal{R}))\\&=\varepsilon+\alpha H_G(\mathcal{R},U_1,U_1)+\eta H_G(\mathcal{R},\Phi(\mathcal{R}),\Phi(\mathcal{R})),\end{aligned}$$

from which we have

$$H_G(\mathcal{R},U_1,U_1)\le\frac{\varepsilon\,(1+\eta)}{1-\alpha}.$$

□

Theorem 3.3. ***(Generalized Collage II)*** *Consider a complete G-metric space (Y,G). Let $\mathcal{R}\in\mathcal{C}^G(Y)$ and $\varepsilon\ge 0$ be given. If a generalized G-iterated function system $\{Y;(f_k,g_k,h_k),k=1,2,\dots,q\}$ with contractive constant $\lambda\in[0,1)$, such that either*

$$H_G(\mathcal{R},\mathcal{R},\Upsilon(\mathcal{R}))\le\varepsilon$$

or

$$H_G(\mathcal{R},\mathcal{R},\Psi(\mathcal{R}))\le\varepsilon$$

or

$$H_G(\mathcal{R},\mathcal{R},\Phi(\mathcal{R}))\le\varepsilon,$$

where $\Upsilon(\mathcal{R})=\cup_{k=1}^q f_k(\mathcal{R})$, $\Psi(\mathcal{R})=\cup_{k=1}^q g_k(\mathcal{R})$ and $\Phi(\mathcal{R})=\cup_{k=1}^q h_k(\mathcal{R})$. Then,

$$H_G(\mathcal{R},\mathcal{R},U_1)\le\frac{\varepsilon}{1-\lambda},$$

where $U_1\in\mathcal{C}^G(Y)$ is the common attractor for Υ, Ψ and Φ.

Proof. Assume that $H_G(\mathcal{R},\mathcal{R},\Upsilon(\mathcal{R}))\le\varepsilon$ for any $\mathcal{R}\in\mathcal{C}^G(Y)$, then

$$\begin{aligned}H_G(\mathcal{R},\mathcal{R},U_1)&\le H_G(\mathcal{R},\mathcal{R},\Upsilon(\mathcal{R}))+H_G(\Upsilon(\mathcal{R}),\Upsilon(\mathcal{R}),U_1)\\&\le H_G(\mathcal{R},\mathcal{R},\Upsilon(\mathcal{R}))+H_G(\Upsilon(\mathcal{R}),\Psi(\mathcal{R}),\Phi(U_1))\\&\le\varepsilon+\lambda H_G(\mathcal{R},\mathcal{R},U_1),\end{aligned}$$

which further implies that

$$H_G(\mathcal{R},\mathcal{R},U_1)\le\frac{\varepsilon}{1-\lambda}.$$

Similarly, if we assume that $H_G(\mathcal{R},\mathcal{R},\Psi(\mathcal{R}))\le\varepsilon$ for any $\mathcal{R}\in\mathcal{C}^G(Y)$. Then,

$$\begin{aligned}H_G(\mathcal{R},\mathcal{R},U_1)&\le H_G(\mathcal{R},\mathcal{R},\Psi(\mathcal{R}))+H_G(\Psi(\mathcal{R}),\Psi(\mathcal{R}),U_1)\\&\le H_G(\mathcal{R},\mathcal{R},\Psi(\mathcal{R}))+H_G(\Upsilon(\mathcal{R}),\Psi(\mathcal{R}),\Phi(U_1))\end{aligned}$$

$$\leq \varepsilon + \lambda H_G(\mathcal{R}, \mathcal{R}, U_1),$$

giving us

$$H_G(\mathcal{R}, \mathcal{R}, U_1) \leq \frac{\varepsilon}{1-\lambda}.$$

Lastly by assuming that $H_G(\mathcal{R}, \mathcal{R}, \Phi(\mathcal{R})) \leq \varepsilon$ for any $\mathcal{R} \in \mathcal{C}^G(Y)$, we get

$$\begin{aligned} H_G(\mathcal{R}, \mathcal{R}, U_1) &\leq H_G(\mathcal{R}, \mathcal{R}, \Phi(\mathcal{R})) + H_G(\Phi(\mathcal{R}), \Phi(\mathcal{R}), U_1) \\ &\leq H_G(\mathcal{R}, \mathcal{R}, \Phi(\mathcal{R})) + H_G(\Phi(\mathcal{R}), \Psi(\mathcal{R}), \Upsilon(U_1)) \\ &\leq \varepsilon + \lambda H_G(\mathcal{R}, \mathcal{R}, U_1), \end{aligned}$$

from which we have

$$H_G(\mathcal{R}, \mathcal{R}, U_1) \leq \frac{\varepsilon}{1-\lambda}.$$

□

Remark 3.4. In Theorem 3.1, take the collection $\mathcal{S}^G(Y)$, of all singleton subsets of the given space Y, then $\mathcal{S}^G(Y) \subseteq \mathcal{C}^G(Y)$. Furthermore, if we take the mappings $(f_k, g_k, h_k) = (f, g, h)$ for each k, where $f = f_1$, $g = g_1$ and $h = h_1$, then the operators (Υ, Ψ, Φ) become

$$(\Upsilon(v_1), \Psi(v_2), \Phi(v_3)) = (f(v_1), g(v_2), h(v_3)).$$

The following common fixed point result is obtained.

Corollary 3.5. *Let $\{Y; (f_k, g_k, h_k), k = 1, 2, \dots, q\}$ be a generalized G-iterated function system in a complete G-metric space (Y, G) and define the mappings $f, g, h : Y \to Y$ as in Remark 3.4. If some $\alpha, \beta, \gamma, \eta \geq 0$ exist with $\alpha + \beta + \gamma + \eta < 1$ such that for any $v_1, v_2, v_3 \in Y$, the following holds*

$$\begin{aligned} G(fv_1, gv_2, hv_3) \leq{}& \alpha G(v_1, v_2, v_3) + \beta G(v_1, \Upsilon(v_1), \Upsilon(v_1)) \\ &+ \gamma G(v_2, \Psi(v_2), \Psi(v_2)) + \eta G(v_3, \Phi(v_3), \Phi(v_3)). \end{aligned}$$

Then f, g and h have a unique common fixed point $u \in Y$. Additionally, for an arbitrary element $u_0 \in Y$, the sequence $\{u_0, fu_0, gfu_0, hgfu_0, fhgfu_0, \dots\}$ converges to the common fixed point of f, g and h.

Corollary 3.6. *Let $\{Y; (f_k, g_k, h_k), k = 1, 2, \dots, q\}$ be a generalized G-iterated function system in a complete G-metric space (Y, G) and define the mappings $f, g, h : Y \to Y$ as in Remark 3.4. If a triplet (f, g, h) is a generalized G-contraction mappings, then the triple (Υ, Ψ, Φ) defined on $\mathcal{C}^G(Y)$ as in Theorem 3.1 has exactly one common fixed point in $\mathcal{C}^G(Y)$. Moreover, for any initial set $\mathcal{R}_0 \in \mathcal{C}^G(Y)$, $\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), \dots\}$ converges to the common fixed point of Υ, Ψ and Φ.*

Example 3.7. Let $Y=[0,1]$ and a G-metric on Y be given by $G(y_1,y_2,y_3)=\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}$. Define $f_k,g_k,h_k:Y\to Y$, $k=1,2$ by

$$f_1(y_1)=\begin{cases}\dfrac{y_1}{18} & \text{if } 0\le y_1<\frac{1}{2}\\[2ex] \dfrac{y_1}{16} & \text{if } \frac{1}{2}\le y_1\le 1,\end{cases}\qquad f_2(y_1)=\begin{cases}\dfrac{y_1}{14} & \text{if } 0\le y_1<\frac{1}{2}\\[2ex] \dfrac{y_1}{12} & \text{if } \frac{1}{2}\le y_1\le 1,\end{cases}$$

$$g_1(y_1)=\begin{cases}\dfrac{y_1}{10} & \text{if } 0\le y_1<\frac{1}{2}\\[2ex] \dfrac{y_1}{8} & \text{if } \frac{1}{2}\le y_1\le 1,\end{cases}\qquad g_2(y_1)=\begin{cases}\dfrac{y_1}{6} & \text{if } 0\le y_1<\frac{1}{2}\\[2ex] \dfrac{y_1}{4} & \text{if } \frac{1}{2}\le y_1\le 1,\end{cases}$$

$$h_1(y_1)=\begin{cases}\dfrac{y_1}{9} & \text{if } 0\le y_1<\frac{1}{2}\\[2ex] \dfrac{y_1}{7} & \text{if } \frac{1}{2}\le y_1\le 1,\end{cases}\qquad h_2(y_1)=\begin{cases}\dfrac{y_1}{5} & \text{if } 0\le y_1<\frac{1}{2}\\[2ex] \dfrac{y_1}{3} & \text{if } \frac{1}{2}\le y_1\le 1.\end{cases}$$

We observe that the maps f_1,f_2,g_1,g_2,h_1 and h_2 are not G-continuous. Moreover,

$$f_1g_1\left(\tfrac{1}{2}\right)=f_1\left(\tfrac{1}{16}\right)=\tfrac{1}{224},\qquad g_1f_1\left(\tfrac{1}{2}\right)=g_1\left(\tfrac{1}{32}\right)=\tfrac{1}{320},$$

$$f_2g_2\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{8}\right)=\tfrac{1}{112},\qquad g_2f_2\left(\tfrac{1}{2}\right)=g_2\left(\tfrac{1}{24}\right)=\tfrac{1}{48},$$

$$g_1h_1\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{14}\right)=\tfrac{1}{140},\qquad h_1g_1\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{16}\right)=\tfrac{1}{144},$$

$$g_2h_2\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{6}\right)=\tfrac{1}{36},\qquad h_2g_2\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{8}\right)=\tfrac{1}{40},$$

$$f_1h_1\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{14}\right)=\tfrac{1}{252},\qquad h_1f_1\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{32}\right)=\tfrac{1}{288},$$

$$f_2h_2\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{6}\right)=\tfrac{1}{84},\qquad h_2f_2\left(\tfrac{1}{2}\right)=f_2\left(\tfrac{1}{24}\right)=\tfrac{1}{120},$$

and so the mappings f_k, g_k and h_k for $k=1,2$ do not commute.

Now, for $y_1,y_2,y_3\in[0,\frac{1}{2}]$, we have

$$\begin{aligned}G(y_1,y_2,y_3)&=\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\},\\ G(y_1,f_1y_1,f_1y_1)&=\max\{\left|y_1-\tfrac{y_1}{18}\right|,\left|\tfrac{y_1}{18}-\tfrac{y_1}{18}\right|,\left|\tfrac{y_1}{18}-y_1\right|\}=\tfrac{17y_1}{18},\\ G(y_1,f_2y_1,f_2y_1)&=\max\{\left|y_1-\tfrac{y_1}{14}\right|,\left|\tfrac{y_1}{14}-\tfrac{y_1}{14}\right|,\left|\tfrac{y_1}{14}-y_1\right|\}=\tfrac{13y_1}{14},\\ G(y_2,g_1y_2,g_1y_2)&=\max\{\left|y_2-\tfrac{y_2}{10}\right|,\left|\tfrac{y_2}{10}-\tfrac{y_2}{10}\right|,\left|\tfrac{y_2}{10}-y_2\right|\}=\tfrac{9y_2}{10},\\ G(y_2,g_2y_2,g_2y_2)&=\max\{\left|y_2-\tfrac{y_2}{6}\right|,\left|\tfrac{y_2}{6}-\tfrac{y_2}{6}\right|,\left|\tfrac{y_2}{6}-y_2\right|\}=\tfrac{5y_2}{6},\\ G(y_3,h_1y_3,h_1y_3)&=\max\{\left|y_3-\tfrac{y_3}{9}\right|,\left|\tfrac{y_3}{9}-\tfrac{y_3}{9}\right|,\left|\tfrac{y_3}{9}-y_3\right|\}=\tfrac{8y_3}{9},\\ G(y_3,h_2y_3,h_2y_3)&=\max\{\left|y_3-\tfrac{y_3}{5}\right|,\left|\tfrac{y_3}{5}-\tfrac{y_3}{5}\right|,\left|\tfrac{y_3}{5}-y_3\right|\}=\tfrac{4y_3}{5}.\end{aligned}$$

Thus

$$\begin{aligned}&G(f_1y_1,g_1y_2,h_1y_3)=\max\{\left|\tfrac{y_1}{18}-\tfrac{y_2}{10}\right|,\left|\tfrac{y_2}{10}-\tfrac{y_3}{9}\right|,\left|\tfrac{y_3}{9}-\tfrac{y_1}{18}\right|\}\\ &\quad=\tfrac{1}{10}\max\left\{\left|\tfrac{5y_1}{9}-y_2\right|,\left|y_2-\tfrac{10y_3}{9}\right|,\left|\tfrac{10y_3}{9}-\tfrac{5y_1}{9}\right|\right\}\end{aligned}$$

$$\begin{aligned}
&\leq \tfrac{1}{10}\left[\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+y_1+y_2+y_3\right]\\
&= \tfrac{1}{10}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+\tfrac{7}{65}\left(\tfrac{13y_1}{14}\right)+\tfrac{1}{9}\left(\tfrac{9y_2}{10}\right)+\tfrac{7}{60}\left(\tfrac{6y_3}{7}\right)\\
&= \alpha_1 G(y_1,y_2,y_3)+\beta_1 G(y_1,fy_1,fy_1)\\
&\quad +\gamma_1 G(y_2,gy_2,gy_2)+\eta_1 G(y_3,hy_3,hy_3)
\end{aligned}$$

and

$$\begin{aligned}
&G(f_2y_1,g_2y_2,h_2y_3)=\max\{\left|\tfrac{y_1}{14}-\tfrac{y_2}{6}\right|,\left|\tfrac{y_2}{6}-\tfrac{y_3}{5}\right|,\left|\tfrac{y_3}{5}-\tfrac{y_1}{14}\right|\}\\
&= \tfrac{1}{6}\max\left\{\left|\tfrac{3y_1}{7}-y_2\right|,\left|y_2-\tfrac{6y_3}{5}\right|,\left|\tfrac{6y_3}{5}-\tfrac{3y_1}{7}\right|\right\}\\
&\leq \tfrac{1}{6}\left[\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+y_1+y_2+y_3\right]\\
&= \tfrac{1}{6}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+\tfrac{7}{39}\left(\tfrac{13y_1}{14}\right)+\tfrac{1}{5}\left(\tfrac{5y_2}{6}\right)+\tfrac{5}{24}\left(\tfrac{4y_3}{5}\right)\\
&= \alpha_2 G(y_1,y_2,y_3)+\beta_2 G(y_1,fy_1,fy_1)\\
&\quad +\gamma_2 G(y_2,gy_2,gy_2)+\eta_2 G(y_3,hy_3,hy_3).
\end{aligned}$$

Therefore

$$\begin{aligned}
G(f_ky_1,g_ky_2,h_ky_3)=\alpha G(y_1,y_2,y_3)+\beta G(y_1,f_ky_1,f_ky_1)+\gamma G(y_2,g_ky_2,g_ky_2)\\
+\eta G(y_3,h_ky_3,h_ky_3)
\end{aligned}$$

for $k=1,2$, where $0<\alpha+\beta+\gamma+\eta=0.755<1$ and

$$\begin{aligned}
\alpha&=\max\{\alpha_1,\alpha_2\}=\max\{\tfrac{1}{10},\tfrac{1}{6}\}=\tfrac{1}{6},\\
\beta&=\max\{\beta_1,\beta_2\}=\max\{\tfrac{1}{85},\tfrac{7}{39}\}=\tfrac{7}{39},\\
\gamma&=\max\{\gamma_1,\gamma_2\}=\max\{\tfrac{1}{9},\tfrac{1}{5}\}=\tfrac{1}{5},\\
\eta&=\max\{\eta_1,\eta_2\}=\max\{\tfrac{9}{80},\tfrac{5}{24}\}=\tfrac{5}{24}.
\end{aligned}$$

For $y_1,y_2,y_3\in[\frac{1}{2},1]$,

$$\begin{aligned}
G(y_1,y_2,y_3)&=\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\},\\
G(y_1,f_1y_1,f_1y_1)&=\max\{\left|y_1-\tfrac{y_1}{16}\right|,\left|\tfrac{y_1}{16}-\tfrac{y_1}{16}\right|,\left|\tfrac{y_1}{16}-y_1\right|\}=\tfrac{15y_1}{16},\\
G(y_1,f_2y_1,f_2y_1)&=\max\{\left|y_1-\tfrac{y_1}{12}\right|,\left|\tfrac{y_1}{12}-\tfrac{y_1}{12}\right|,\left|\tfrac{y_1}{12}-y_1\right|\}=\tfrac{11y_1}{12},\\
G(y_2,g_1y_2,g_1y_2)&=\max\{\left|y_2-\tfrac{y_2}{8}\right|,\left|\tfrac{y_2}{8}-\tfrac{y_2}{8}\right|,\left|\tfrac{y_2}{8}-y_2\right|\}=\tfrac{7y_2}{8},\\
G(y_2,g_2y_2,g_2y_2)&=\max\{\left|y_2-\tfrac{y_2}{4}\right|,\left|\tfrac{y_2}{4}-\tfrac{y_2}{4}\right|,\left|\tfrac{y_2}{4}-y_2\right|\}=\tfrac{3y_2}{4},\\
G(y_3,h_1y_3,h_1y_3)&=\max\{\left|y_3-\tfrac{y_3}{7}\right|,\left|\tfrac{y_3}{7}-\tfrac{y_3}{7}\right|,\left|\tfrac{y_3}{7}-y_3\right|\}=\tfrac{6y_3}{7},\\
G(y_3,h_2y_3,h_2y_3)&=\max\{\left|y_3-\tfrac{y_3}{3}\right|,\left|\tfrac{y_3}{3}-\tfrac{y_3}{3}\right|,\left|\tfrac{y_3}{3}-y_3\right|\}=\tfrac{2y_3}{3}.
\end{aligned}$$

Thus

$$\begin{aligned}
&G(f_1y_1,g_1y_2,h_1y_3)=\max\{\left|\tfrac{y_1}{16}-\tfrac{y_2}{8}\right|,\left|\tfrac{y_2}{8}-\tfrac{y_3}{7}\right|,\left|\tfrac{y_3}{7}-\tfrac{y_1}{16}\right|\}\\
&= \tfrac{1}{8}\max\{\left|\tfrac{y_1}{2}-y_2\right|,\left|y_2-\tfrac{8y_3}{7}\right|,\left|\tfrac{8y_3}{7}-\tfrac{y_1}{2}\right|\}\\
&\leq \tfrac{1}{8}\left[\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+y_1+y_2+y_3\right]\\
&= \tfrac{1}{8}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+\tfrac{2}{15}\left(\tfrac{15y_1}{16}\right)+\tfrac{1}{7}\left(\tfrac{7y_2}{8}\right)+\tfrac{7}{48}\left(\tfrac{6y_3}{7}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_1 G(y_1, y_2, y_3) + \beta_1 G(y_1, f_1 y_1, f_1 y_1) \\
&\quad + \gamma_1 G(y_2, g_1 y_2, g_1 y_2) + \eta_1 G(y_3, h_1 y_3, h_1 y_3)
\end{aligned}
$$

and

$$
\begin{aligned}
G(f_2 y_1, g_2 y_2, h_2 y_3) &= \max\{\left|\tfrac{y_1}{12} - \tfrac{y_2}{4}\right|, \left|\tfrac{y_2}{4} - \tfrac{y_3}{3}\right|, \left|\tfrac{y_3}{3} - \tfrac{y_1}{12}\right|\} \\
&= \tfrac{1}{12}\max\{|y_1 - 3y_2|, |3y_2 - 4y_3|, |4y_3 - y_1|\} \\
&\leq \tfrac{1}{12}\left[\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\} + y_1 + y_2 + y_3\right] \\
&= \tfrac{1}{12}\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\} + \tfrac{1}{11}\left(\tfrac{11}{12}y_1\right) + \tfrac{1}{9}\left(\tfrac{3}{4}y_2\right) + \tfrac{1}{8}\left(\tfrac{2}{3}y_3\right) \\
&= \alpha_2 G(y_1, y_2, y_3) + \beta_2 G(y_1, f_2 y_1, f_2 y_1) \\
&\quad + \gamma_2 G(y_2, g_2 y_2, g_2 y_2) + \eta_2 G(y_3, h_2 y_3, h_2 y_3).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G(f_k y_1, g_k y_2, h_k y_3) &= \alpha G(y_1, y_2, y_3) + \beta G(y_1, f_k y_1, f_k y_1) + \gamma G(y_2, g_k y_2, g_k y_2) \\
&\quad + \eta G(y_3, h_k y_3, h_k y_3)
\end{aligned}
$$

for $k = 1, 2$ where $0 < \alpha + \beta + \gamma + \eta = 0.547 < 1$ and

$$
\begin{aligned}
\alpha &= \max\{\alpha_1, \alpha_2\} = \max\{\tfrac{1}{8}, \tfrac{1}{12}\} = \tfrac{1}{8}, \\
\beta &= \max\{\beta_1, \beta_2\} = \max\{\tfrac{2}{15}, \tfrac{1}{11}\} = \tfrac{2}{15}, \\
\gamma &= \max\{\gamma_1, \gamma_2\} = \max\{\tfrac{1}{7}, \tfrac{1}{9}\} = \tfrac{1}{7}, \\
\eta &= \max\{\eta_1, \eta_2\} = \max\{\tfrac{7}{48}, \tfrac{1}{8}\} = \tfrac{7}{48}.
\end{aligned}
$$

For $y_1 \in [0, \frac{1}{2}), y_2, y_3 \in [\frac{1}{2}, 1]$,

$$
\begin{aligned}
G(y_1, y_2, y_3) &= \max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\}, \\
G(y_1, f_1 y_1, f_1 y_1) &= \max\{\left|y_1 - \tfrac{y_1}{18}\right|, \left|\tfrac{y_1}{18} - \tfrac{y_1}{18}\right|, \left|\tfrac{y_1}{18} - y_1\right|\} = \tfrac{17y_1}{18}, \\
G(y_1, f_2 y_1, f_2 y_1) &= \max\{\left|y_1 - \tfrac{y_1}{14}\right|, \left|\tfrac{y_1}{14} - \tfrac{y_1}{14}\right|, \left|\tfrac{y_1}{14} - y_1\right|\} = \tfrac{13y_1}{14}, \\
G(y_2, g_1 y_2, g_1 y_2) &= \max\{\left|y_2 - \tfrac{y_2}{8}\right|, \left|\tfrac{y_2}{8} - \tfrac{y_2}{8}\right|, \left|\tfrac{y_2}{8} - y_2\right|\} = \tfrac{7y_2}{8}, \\
G(y_2, g_2 y_2, g_2 y_2) &= \max\{\left|y_2 - \tfrac{y_2}{4}\right|, \left|\tfrac{y_2}{4} - \tfrac{y_2}{4}\right|, \left|\tfrac{y_2}{4} - y_2\right|\} = \tfrac{3y_2}{4}, \\
G(y_3, h_1 y_3, h_1 y_3) &= \max\{\left|y_3 - \tfrac{y_3}{7}\right|, \left|\tfrac{y_3}{7} - \tfrac{y_3}{7}\right|, \left|\tfrac{y_3}{7} - y_3\right|\} = \tfrac{6y_3}{7}, \\
G(y_3, h_2 y_3, h_2 y_3) &= \max\{\left|y_3 - \tfrac{y_3}{3}\right|, \left|\tfrac{y_3}{3} - \tfrac{y_3}{3}\right|, \left|\tfrac{y_3}{3} - y_3\right|\} = \tfrac{2y_3}{3}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
G(f_1 y_1, g_1 y_2, h_1 y_3) &= \max\{\left|\tfrac{y_1}{18} - \tfrac{y_2}{8}\right|, \left|\tfrac{y_2}{8} - \tfrac{y_3}{7}\right|, \left|\tfrac{y_3}{7} - \tfrac{y_1}{18}\right|\} \\
&= \tfrac{1}{8}\max\{\left|\tfrac{4y_1}{9} - y_2\right|, \left|y_2 - \tfrac{8y_3}{7}\right|, \left|\tfrac{8y_3}{7} - \tfrac{4y_1}{9}\right|\} \\
&\leq \tfrac{1}{8}\left[\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\} + y_1 + y_2 + y_3\right] \\
&= \tfrac{1}{8}\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\} + \tfrac{9}{68}\left(\tfrac{17}{18}y_1\right) + \tfrac{1}{7}\left(\tfrac{7}{8}y_2\right) + \tfrac{7}{48}\left(\tfrac{6y_3}{7}\right) \\
&= \alpha_1 G(y_1, y_2, y_3) + \beta_1 G(y_1, f_1 y_1, f_1 y_1) \\
&\quad + \gamma_1 G(y_2, g_1 y_2, g_1 y_2) + \eta_1 G(y_3, h_1 y_3, h_1 y_3)
\end{aligned}
$$

and

$$\begin{aligned}G\left(f_2y_1,g_2y_2,h_2y_3\right) &= \max\{\left|\tfrac{y_1}{14}-\tfrac{y_2}{4}\right|,\left|\tfrac{y_2}{4}-\tfrac{y_3}{3}\right|,\left|\tfrac{y_3}{3}-\tfrac{y_1}{14}\right|\}\\
&= \tfrac{1}{14}\max\{\left|y_1-\tfrac{7y_2}{2}\right|,\left|\tfrac{7y_2}{2}-\tfrac{14y_3}{3}\right|,\left|\tfrac{14y_3}{3}-y_1\right|\}\\
&\le \tfrac{1}{14}\left[\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+y_1+y_2+y_3\right]\\
&= \tfrac{1}{14}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+\tfrac{y_1}{14}+\tfrac{y_2}{14}+\tfrac{y_3}{14}\\
&= \tfrac{1}{14}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+\tfrac{1}{13}\left(\tfrac{13y_1}{14}\right)+\tfrac{2}{21}\left(\tfrac{3y_2}{4}\right)+\tfrac{3}{28}\left(\tfrac{2y_3}{3}\right)\\
&= \alpha_2G\left(y_1,y_2,y_3\right)+\beta_2G\left(y_1,f_2y_1,f_2y_1\right)\\
&\quad+\gamma_2G\left(y_2,g_2y_2,g_2y_2\right)+\eta_2G\left(y_3,h_2y_3,h_2y_3\right).\end{aligned}$$

Therefore

$$\begin{aligned}G\left(f_ky_1,g_ky_2,h_ky_3\right) = \alpha G\left(y_1,y_2,y_3\right)+\beta G\left(y_1,f_ky_1,f_ky_1\right)+\gamma G\left(y_2,g_ky_2,g_ky_2\right)\\
+\eta G\left(y_3,h_ky_3,h_ky_3\right)\end{aligned}$$

for $k=1,2$, where $0<\alpha+\beta+\gamma+\eta=0.546<1$ with

$$\begin{aligned}\alpha &= \max\{\alpha_1,\alpha_2\}=\max\{\tfrac{1}{8},\tfrac{1}{14}\}=\tfrac{1}{8},\\
\beta &= \max\{\beta_1,\beta_2\}=\max\{\tfrac{9}{68},\tfrac{1}{13}\}=\tfrac{9}{68},\\
\gamma &= \max\{\gamma_1,\gamma_2\}=\max\{\tfrac{1}{7},\tfrac{2}{21}\}=\tfrac{1}{7},\\
\eta &= \max\{\eta_1,\eta_2\}=\max\{\tfrac{7}{48},\tfrac{3}{28}\}=\tfrac{7}{48}.\end{aligned}$$

For $y_1,y_2\in[0,\frac{1}{2})$ and $y_3\in[\frac{1}{2},1]$,

$$\begin{aligned}G\left(y_1,y_2,y_3\right) &= \max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\},\\
G\left(y_1,f_1y_1,f_1y_1\right) &= \max\{\left|y_1-\tfrac{y_1}{18}\right|,\left|\tfrac{y_1}{18}-\tfrac{y_1}{18}\right|,\left|\tfrac{y_1}{18}-y_1\right|\}=\tfrac{17y_1}{18},\\
G\left(y_1,f_2y_1,f_2y_1\right) &= \max\{\left|y_1-\tfrac{y_1}{14}\right|,\left|\tfrac{y_1}{14}-\tfrac{y_1}{14}\right|,\left|\tfrac{y_1}{14}-y_1\right|\}=\tfrac{13y_1}{14},\\
G\left(y_2,g_1y_2,g_1y_2\right) &= \max\{\left|y_2-\tfrac{y_2}{10}\right|,\left|\tfrac{y_2}{10}-\tfrac{y_2}{10}\right|,\left|\tfrac{y_2}{10}-y_2\right|\}=\tfrac{9y_2}{10},\\
G\left(y_2,g_2y_2,g_2y_2\right) &= \max\{\left|y_2-\tfrac{y_2}{6}\right|,\left|\tfrac{y_2}{6}-\tfrac{y_2}{6}\right|,\left|\tfrac{y_2}{6}-y_2\right|\}=\tfrac{5y_2}{6},\\
G\left(y_3,h_1y_3,h_1y_3\right) &= \max\{\left|y_3-\tfrac{y_3}{7}\right|,\left|\tfrac{y_3}{7}-\tfrac{y_3}{7}\right|,\left|\tfrac{y_3}{7}-y_3\right|\}=\tfrac{6y_3}{7},\\
G\left(y_3,h_2y_3,h_2y_3\right) &= \max\{\left|y_3-\tfrac{y_3}{3}\right|,\left|\tfrac{y_3}{3}-\tfrac{y_3}{3}\right|,\left|\tfrac{y_3}{3}-y_3\right|\}=\tfrac{2y_3}{3}.\end{aligned}$$

Thus

$$\begin{aligned}G\left(f_1y_1,g_1y_2,h_1y_3\right) &= \max\{\left|\tfrac{y_1}{18}-\tfrac{y_2}{10}\right|,\left|\tfrac{y_2}{10}-\tfrac{y_3}{7}\right|,\left|\tfrac{y_3}{7}-\tfrac{y_1}{18}\right|\}\\
&= \tfrac{1}{10}\max\{\left|\tfrac{5y_1}{9}-y_2\right|,\left|y_2-\tfrac{10y_3}{7}\right|,\left|\tfrac{10y_3}{7}-\tfrac{5y_1}{9}\right|\}\\
&\le \tfrac{1}{10}\left[\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+y_1+y_2+y_3\right]\\
&= \tfrac{1}{10}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\}+\tfrac{9}{85}\left(\tfrac{17}{18}y_1\right)+\tfrac{1}{9}\left(\tfrac{9y_2}{10}\right)+\tfrac{7}{60}\left(\tfrac{6y_3}{7}\right)\\
&= \alpha_1G\left(y_1,y_2,y_3\right)+\beta_1G\left(y_1,f_1y_1,f_1y_1\right)\\
&\quad+\gamma_1G\left(y_2,g_1y_2,g_1y_2\right)+\eta_1G\left(y_3,h_1y_3,h_1y_3\right)\end{aligned}$$

and

$$\begin{aligned}
G\left(f_2y_1, g_2y_2, h_2y_3\right) &= \max\{\left|\tfrac{y_1}{14}-\tfrac{y_2}{6}\right|, \left|\tfrac{y_2}{6}-\tfrac{y_3}{3}\right|, \left|\tfrac{y_3}{3}-\tfrac{y_1}{14}\right|\} \\
&= \tfrac{1}{14}\max\{\left|y_1-\tfrac{7y_2}{3}\right|, \left|\tfrac{7y_2}{3}-\tfrac{14y_3}{3}\right|, \left|\tfrac{14y_3}{3}-y_1\right|\} \\
&\leq \tfrac{1}{14}\left[\max\{|y_1-y_2|, |y_2-y_3|, |y_3-y_1|\}+y_1+y_2+y_3\right] \\
&= \tfrac{1}{14}\max\{|y_1-y_2|, |y_2-y_3|, |y_3-y_1|\}+\tfrac{1}{13}\left(\tfrac{13y_1}{14}\right)+\tfrac{3}{35}\left(\tfrac{5y_2}{6}\right)+\tfrac{3}{28}\left(\tfrac{2y_3}{3}\right) \\
&= \alpha_2G\left(y_1, y_2, y_3\right)+\beta_2G\left(y_1, f_2y_1, f_2y_1\right) \\
&\quad +\gamma_2G\left(y_2, g_2y_2, g_2y_2\right)+\eta_2G\left(y_3, h_2y_3, h_2y_3\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
G\left(f_ky_1, g_ky_2, h_ky_3\right) = \alpha G\left(y_1, y_2, y_3\right)+\beta G\left(y_1, f_ky_1, f_ky_1\right)+\gamma G\left(y_2, g_ky_2, g_ky_2\right) \\
+\eta G\left(y_3, h_ky_3, h_ky_3\right)
\end{aligned}$$

for $k=1,2$, where $0<\alpha+\beta+\gamma+\eta=0.426<1$ with

$$\begin{aligned}
\alpha &= \max\{\alpha_1, \alpha_2\} = \max\{\tfrac{1}{10}, \tfrac{1}{14}\} = \tfrac{1}{10}, \\
\beta &= \max\{\beta_1, \beta_2\} = \max\{\tfrac{9}{85}, \tfrac{1}{13}\} = \tfrac{9}{85}, \\
\gamma &= \max\{\gamma_1, \gamma_2\} = \max\{\tfrac{1}{9}, \tfrac{3}{35}\} = \tfrac{1}{9}, \\
\eta &= \max\{\eta_1, \eta_2\} = \max\{\tfrac{7}{60}, \tfrac{3}{28}\} = \tfrac{3}{28}.
\end{aligned}$$

We observe that 0 is the unique common fixed point of f, g and h.

Let $\{Y; (f_1, f_2, g_1, g_2, h_1, h_2)\}$ be the generalized G-iterated function system with the mappings $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ defined by

$$\begin{aligned}
\Upsilon(\mathcal{Q}) &= f_1(\mathcal{Q}) \cup f_2(\mathcal{Q}), \\
\Psi(\mathcal{R}) &= g_1(\mathcal{R}) \cup g_2(\mathcal{R}), \\
\Phi(\mathcal{N}) &= h_1(\mathcal{N}) \cup h_2(\mathcal{N})
\end{aligned}$$

for all $\mathcal{Q}, \mathcal{R}, \mathcal{N} \in \mathcal{C}^G(Y)$. From Proposition 2.3, we have that

$$H_G(\Upsilon(\mathcal{Q}), \Psi(\mathcal{R}), \Phi(\mathcal{N})) \leq \kappa H_G(\mathcal{Q}, \mathcal{R}, \mathcal{N}),$$

where $\kappa = \max\{0.755, 0.547, 0.546, 0.426\} = 0.755$. Thus, all the conditions of Theorem 3.1 are satisfied, and moreover, for any initial set $\mathcal{R}_0 \in \mathcal{C}^G(Y)$, the sequence $\{\mathcal{R}_0, \Upsilon\left(\mathcal{R}_0\right), \Psi\Upsilon\left(\mathcal{R}_0\right), \Phi\Psi\Upsilon\left(\mathcal{R}_0\right), \Upsilon\Phi\Psi\Upsilon\left(\mathcal{R}_0\right), \dots\}$ of compact sets is convergent and has a limit, the common attractor of Υ, Ψ and Φ.

Theorem 3.8. *In a complete G-metric space (Y, G), let $\{Y; (f_k, g_k, h_k), k = 1, 2, \dots, q\}$ be the generalized G-iterated function system. Define $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ by*

$$\Upsilon(\mathcal{Q}) = \cup_{k=1}^{q} f_k(\mathcal{Q}),$$

$$\Psi(\mathcal{R}) = \cup_{k=1}^{q} g_k(\mathcal{R})$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^{q} h_k(\mathcal{N})$$

for $\mathcal{Q},\mathcal{R},\mathcal{N} \in \mathcal{C}^G(Y)$. If the mappings (Υ,Ψ,Φ) are generalized G-Hutchinson contractive operators (type II), then Υ,Ψ and Φ have a unique common attractor $U^* \in \mathcal{C}^G(Y)$, that is,

$$U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*).$$

Moreover, for an arbitrarily chosen initial set $\mathcal{R}_0 \in \mathcal{C}^G(Y)$, the sequence

$$\{\mathcal{R}_0, \Upsilon(\mathcal{R}_0), \Psi\Upsilon(\mathcal{R}_0), \Phi\Psi\Upsilon(\mathcal{R}_0), \Upsilon\Phi\Psi\Upsilon(\mathcal{R}_0), ...\}$$

of compact sets converges to the common attractor U^*.

Proof. We show that any attractor of Υ is an attractor of Ψ and Φ. To that end, we assume that $U^* \in \mathcal{C}^G(Y)$ is such that $\Upsilon(U^*) = U^*$. We need to show that $U^* = \Psi(U^*) = \Phi(U^*)$. As

$$\begin{aligned}
&H_G(U^*,\Psi(U^*),\Phi(U^*)) = H_G(\Upsilon(U^*),\Psi(U^*),\Phi(U^*))\\
&\quad\le \lambda_1 H_G(U^*,U^*,U^*) + \lambda_2[H_G(U^*,U^*,\Upsilon(U^*)\\
&\qquad + H_G(U^*,U^*,\Psi(U^*)) + H_G(U^*,U^*,\Phi(U^*))]\\
&\qquad + \lambda_3[H_G(\Upsilon(U^*),U^*,U^*) + H_G(U^*,\Psi(U^*),U^*) + H_G(U^*,U^*,\Phi(U^*))]\\
&= (\lambda_2+\lambda_3)[H_G(U^*,\Psi(U^*),U^*) + H_G(U^*,U^*,\Phi(U^*))]\\
&\le (\lambda_2+\lambda_3)[H_G(U^*,\Psi(U^*),\Phi(U^*)) + H_G(U^*,\Psi(U^*),\Phi(U^*))]\\
&= 2(\lambda_2+\lambda_3) H_G(U^*,\Psi(U^*),\Phi(U^*)),
\end{aligned}$$

that is, $(1-2(\lambda_2+\lambda_3)) H_G(U^*,\Psi(U^*),\Phi(U^*)) \le 0$ and so $H_G(U^*,\Psi(U^*),\Phi(U^*)) = 0$ as $2(\lambda_2+\lambda_3) < 1$. Thus $U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*)$. Similarly, if we take $U^* = \Phi(U^*)$ or $U^* = \Psi(U^*)$, then we conclude that $U^* = \Upsilon(U^*) = \Psi(U^*) = \Phi(U^*)$.

We show that Υ,Ψ, and Φ have a unique common attractor. Let $\mathcal{R}_0 \in \mathcal{C}^G(Y)$ be an arbitrary point. Define $\{\mathcal{R}_k\}$ by $\mathcal{R}_{3k+1} = \Upsilon(\mathcal{R}_{3k})$, $\mathcal{R}_{3k+2} = \Psi(\mathcal{R}_{3k+1}), \mathcal{R}_{3k+3} = \Psi(\mathcal{R}_{3k+2})$, $k = 0,1,2,\ldots$. If $\mathcal{R}_k = \mathcal{R}_{k+1}$ for some k, with $k = 3n$, then $U^* = \mathcal{R}_{3k}$ is an attractor of Υ and from the proof above, U^* is a common attractor for Υ,Ψ and Φ. The same is true for $k = 3n+1$ or $k = 3n+2$. We assume that $\mathcal{R}_k \neq \mathcal{R}_{k+1}$ for all $k \in \mathbb{N}\cup\{0\}$, then

$$\begin{aligned}
&H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3}) = H_G(\Upsilon(\mathcal{R}_{3k}),\Psi(\mathcal{R}_{3k+1}),\Phi(\mathcal{R}_{3k+2}))\\
&\quad\le \lambda_1 H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + \lambda_2[H_G(\mathcal{R}_{3k},\mathcal{R}_{3k},\Upsilon(\mathcal{R}_{3k})\\
&\qquad + H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+1},\Psi(\mathcal{R}_{3k+1})) + H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+2},\Phi(\mathcal{R}_{3k+2}))]\\
&\qquad + \lambda_3[H_G(\Upsilon(\mathcal{R}_{3k}),\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k},\Psi(\mathcal{R}_{3k+1}),\mathcal{R}_{3k+2})\\
&\qquad + H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\Phi(\mathcal{R}_{3k+2}))]\\
&= \lambda_1 H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + \lambda_2[H_G(\mathcal{R}_{3k},\mathcal{R}_{3k},\mathcal{R}_{3k+1})\\
&\qquad + H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k+2},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3})]\\
&\qquad + \lambda_3[H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+2},\mathcal{R}_{3k+2})\\
&\qquad + H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+3})]\\
&\quad\le \lambda_1 H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + \lambda_2[H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2})\\
&\qquad + H_G(\mathcal{R}_{3k},\mathcal{R}_{3k+1},\mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k+1},\mathcal{R}_{3k+2},\mathcal{R}_{3k+3})]
\end{aligned}$$

$$+ \lambda_3[H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2})$$
$$+ \{H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3})\}].$$

Thus, we have

$$(1 - \lambda_2 - \lambda_3)H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) \leq (\lambda_1 + 2\lambda_2 + 3\lambda_3)H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}).$$

Hence,

$$H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) \leq \lambda H_G(\mathcal{R}_{3k}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}),$$

where $\lambda = \frac{\lambda_1+2\lambda_2+3\lambda_3}{1-\lambda_2-\lambda_3}$, with $0 < \lambda < 1$. In a similar manner, it can be proved that

$$H_G(\mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}, \mathcal{R}_{3k+4}) \leq \lambda H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3})$$

and

$$H_G(\mathcal{R}_{3k+3}, \mathcal{R}_{3k+4}, \mathcal{R}_{3k+5}) \leq \lambda H_G(\mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}, \mathcal{R}_{3k+4}).$$

Thus, for all k,

$$H_G(\mathcal{R}_{k+1}, \mathcal{R}_{k+2}, \mathcal{R}_{k+3}) \leq \lambda H_G\left(\mathcal{R}_k, \mathcal{R}_{k+1}, \mathcal{R}_{k+2}\right).$$

Follow the similar steps as in the Proof of Theorem 3.1, we obtain that $\{\mathcal{R}_k\}$ is a G-Cauchy sequence in $\mathcal{C}^G(Y)$. Since $(\mathcal{C}^G(Y), H_G)$ is a complete G-metric space, there exists $U_1 \in \mathcal{C}^G(Y)$ such that $\lim_{k\to+\infty} H_G\left(\mathcal{R}_k, \mathcal{R}_k, U_1\right) = 0$.

To show that $\Upsilon\left(U_1\right) = U_1$, we have claim in the contrary

$$\begin{aligned}
H_G(\Upsilon\left(U_1\right), \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3}) &= H_G(\Upsilon\left(U_1\right), \Psi\left(\mathcal{R}_{3k+1}\right), \Phi\left(\mathcal{R}_{3k+2}\right)) \\
&\leq \lambda_1 H_G(U_1, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + \lambda_2[H_G(U_1, U_1, \Upsilon(U_1) \\
&\quad + H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+1}, \Psi\left(\mathcal{R}_{3k+1}\right)) + H_G(\mathcal{R}_{3k+2}, \mathcal{R}_{3k+2}, \Phi\left(\mathcal{R}_{3k+2}\right))] \\
&\quad + \lambda_3[H_G(\Upsilon(U_1), \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + H_G(U_1, \Psi\left(\mathcal{R}_{3k+1}\right), \mathcal{R}_{3k+2}) \\
&\quad + H_G(U_1, \mathcal{R}_{3k+1}, \Phi\left(\mathcal{R}_{3k+2}\right))] \\
&= \lambda_1 H_G(U_1, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + \lambda_2[H_G(U_1, U_1, \Upsilon(U_1) \\
&\quad + H_G(\mathcal{R}_{3k+1}, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + H_G(\mathcal{R}_{3k+2}, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+3})] \\
&\quad + \lambda_3[H_G(\Upsilon(U_1), \mathcal{R}_{3k+1}, \mathcal{R}_{3k+2}) + H_G(U_1, \mathcal{R}_{3k+2}, \mathcal{R}_{3k+2}) \\
&\quad + H_G(U_1, \mathcal{R}_{3k+1}, \mathcal{R}_{3k+3})]
\end{aligned}$$

and as $k \to +\infty$, we gives

$$H_G(\Upsilon\left(U_1\right), U_1, U_1) \leq \left(\lambda_2 + \lambda_3\right) H_G(\Upsilon(U_1), U_1, U_1)$$

which is a contradiction as $(\lambda_2 + \lambda_3) < 1$. Thus $\Upsilon\left(U_1\right) = U_1$. Likewise, we can show that $\Psi\left(U_1\right) = U_1$ and $\Phi\left(U_1\right) = U_1$. To prove uniqueness, suppose that V_1 is also a common attractor of Υ, Ψ and Φ. Then

$$\begin{aligned}
H_G(U_1, V_1, V_1) &= H_G(\Upsilon\left(U_1\right), \Psi\left(V_1\right), \Phi\left(V_1\right)) \\
&\leq \lambda_1 H_G(U_1, V_1, V_1) + \lambda_2[H_G(U_1, U_1, \Upsilon(U_1) + H_G(V_1, V_1, \Psi\left(V_1\right)) \\
&\quad + H_G(V_1, V_1, \Phi\left(V_1\right))] + \lambda_3[H_G(\Upsilon(U_1), V_1, V_1) \\
&\quad + H_G(U_1, \Psi\left(V_1\right), V_1) + H_G(U_1, V_1, \Phi\left(V_1\right))] \\
&= \lambda_1 H_G(U_1, V_1, V_1) + \lambda_2[H_G(U_1, U_1, U_1) + H_G(V_1, V_1, V_1) \\
&\quad + H_G(V_1, V_1, V_1)] + \lambda_3[H_G(U_1, V_1, V_1) + H_G(U_1, V_1, V_1) \\
&\quad + H_G(U_1, V_1, V_1)]
\end{aligned}$$

$$= (\lambda_1 + 3\lambda_3)\, H_G(U_1, V_1, V_1)$$

from which we conclude that $H_G(U_1, V_1, V_1) = 0$ and thus $U_1 = V_1$. Hence U_1 is a unique common attractor of Υ, Ψ and Φ. □

Corollary 3.9. *In a complete G-metric space (Y, G), let $\{Y; f_k, g_k, h_k, k=1, 2, \ldots, q\}$ be a generalized iterated function system and define the mappings $f, g, h : Y \to Y$ as in Remark 3.4. If there exist $\lambda_j \geq 0$ for $j \in \{1, 2, 3\}$ with $\lambda_1 + 3\lambda_2 + 4\lambda_3 < 1$ such that for any $w_1, w_2, w_3 \in \mathcal{C}^G(Y)$, the following holds:*

$$G(fw_1, gw_2, hw_3) \leq R_{f,g,h}(w_1, w_2, w_3),$$

where

$$\begin{aligned} R_{f,g,h}(w_1, w_2, w_3) = {} & \lambda_1 G(w_1, w_2, w_3) + \lambda_2 [G(w_1, w_1, f(w_1)) \\ & + G(w_2, w_2, g(w_2)) + G(w_3, w_3, h(w_3))] \\ & + \lambda_3 [G(f(w_1), w_2, w_3) + G(w_1, g(w_2), w_3) \\ & + G(w_1, w_2, h(w_3))]. \end{aligned}$$

Then f, g and h have a unique common fixed point. In addition, for a randomly chosen $v_0 \in Y$, the sequence $\{v_0, fv_0, gfv_0, hgfv_0, fhgfv_0, ...\}$ converges to a common fixed point of f, g and h.

Now we examine the application of the derived result in Theorem 3.10. We apply results to obtain the existence of solutions of functional equations arising in the dynamic programming.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces with $U \subseteq \mathcal{B}_1$ and $V \subseteq \mathcal{B}_2$. Suppose that

$$\kappa\colon U \times V \longrightarrow U, \quad \zeta\colon U \times V \longrightarrow \mathbb{R}, \quad \sigma_1, \sigma_2\colon U \times V \times \mathbb{R} \longrightarrow \mathbb{R}.$$

If we consider U and V as the state and decision spaces respectively, then the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p_1(x) = \sup_{y \in V}\{\zeta(x, y) + \sigma_1(x, y, p_1(\kappa(x, y)))\} \text{ for } x \in U \tag{3.1}$$

$$p_2(x) = \sup_{y \in V}\{\zeta(x, y) + \sigma_2(x, y, p_2(\kappa(x, y)))\} \text{ for } x \in U. \tag{3.2}$$

We study the existence and uniqueness of the bounded solution of the functional equations (3.1) and (3.2) arising in dynamic programming in the setup of G-metric spaces. Let $B(U)$ denotes the set of all bounded real valued functions on U. For an arbitrary $\eta \in B(U)$, define $\|\eta\| = \sup_{t \in U} |\eta(t)|$. Then $(B(U), \|\cdot\|)$ is a Banach space. Now consider

$$G_B(\eta, \xi, \omega) = \sup_{t \in U} |\eta(t) - \xi(t)| + \sup_{t \in U} |\xi(t) - \omega(t)| + \sup_{t \in U} |\omega(t) - \eta(t)|,$$

where $\eta, \xi, \omega \in B(U)$. Then G_B is a complete G-metric on $B(U)$.

Also, we assume that:

(T_1): ζ, σ_1 and σ_2 are bounded and continuous.

(T_2): For $x \in U$, $\eta \in B(U)$, take $\Psi, \Phi : B(U) \to B(U)$ as

$$\Psi\eta(x) = \sup_{y \in V}\{\zeta(x, y) + \sigma_1(x, y, \eta(\kappa(x, y)))\}, \text{ for } x \in U, \tag{3.3}$$

$$\Phi\eta(x) = \sup_{y\in V}\{\zeta(x,y) + \sigma_2(x,y,\eta(\kappa(x,y)))\}, \text{ for } x \in U. \tag{3.4}$$

Moreover, for every $(x, y) \in U \times V$, $\eta, \xi \in B(U)$ and $t \in U$ implies

$$|\sigma_1(x,y,\eta(t)) - \sigma_2(x,y,\xi(t))| \leq R_{\Psi,\Phi}(\eta(t), \xi(t), \xi(t)), \tag{3.5}$$

where

$$\begin{aligned} R_{\Psi,\Phi}(\eta(t), \xi(t), \xi(t)) = {} & \lambda_1 G_B(\eta(t), \xi(t), \xi(t)) \\ & + \lambda_2[G_B(\eta(t), \eta(t), \Psi(\eta(t)) + 2G_B(\xi(t), \xi(t), \Phi(\xi(t)))] \\ & + \lambda_3[G_B(\Psi(\eta(t)), \xi(t), \xi(t)) + 2G_B(\eta(t), \xi(t), \Phi(\xi(t)))]. \end{aligned}$$

Theorem 3.10. *Assume that the conditions (T_1) and (T_2) hold. Then, the functional equations (3.1) and (3.2) have a unique common and bounded solution in $B(U)$.*

Proof. By (T_1), Ψ and Φ are self-mappings of $B(U)$. And by (3.3) and (3.4) in (T_2), it follows that for any $\eta, \xi \in B(U)$, we can find $x \in U$ and $y_1, y_2 \in V$ such that

$$2\Psi\eta < \zeta(x,y_1) + \sigma_1(x,y_1,\eta(\kappa(x,y_1))), \tag{3.6}$$

$$2\Phi\xi < \zeta(x,y_2) + \sigma_2(x,y_2,\xi(\kappa(x,y_2))), \tag{3.7}$$

which further implies that

$$2\Psi\eta \geq \zeta(x,y_2) + \sigma_1(x,y_2,\eta(\kappa(x,y_2))), \tag{3.8}$$

$$2\Phi\xi \geq \zeta(x,y_1) + \sigma_2(x,y_1,\xi(\kappa(x,y_1))). \tag{3.9}$$

From (3.6) and (3.9) together with (3.5) implies

$$\begin{aligned} 2\Psi\eta(t) - 2\Phi\xi(t) & < \sigma_1(x,y_1,\eta(\kappa(x,y_1))) - \sigma_2(x,y_1,\xi(\kappa(x,y_1))) \\ & \leq |\sigma_1(x,y_1,\eta(\kappa(x,y_1))) - \sigma_2(x,y_1,\xi(\kappa(x,y_1)))| \\ & \leq R_{\Psi,\Phi}(\eta(t), \xi(t), \xi(t)). \end{aligned} \tag{3.10}$$

From (3.7) and (3.8) together with (3.5) implies

$$\begin{aligned} 2\Phi\xi(t) - 2\Psi\eta(t) & < \sigma_2(x,y_2,\xi(\kappa(x,y_2))) - \sigma_1(x,y_2,\eta(\kappa(x,y_2))) \\ & \leq |\sigma_1(x,y_2,\eta(\kappa(x,y_2))) - \sigma_2(x,y_2,\xi(\kappa(x,y_2)))| \\ & \leq R_{\Psi,\Phi}(\eta(t), \xi(t), \xi(t)). \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we get

$$2\,|\Psi\eta(t) - \Phi\xi(t)| \leq R_{\Psi,\Phi}(\eta(t), \xi(t), \xi(t)). \tag{3.12}$$

The inequality (3.12) implies that

$$G_B(\Psi\eta(t), \Phi\xi(t), \Phi\xi(t)) \leq R_{\Psi,\Phi}(\eta(t), \xi(t), \xi(t)), \tag{3.13}$$

where

$$\begin{aligned}&R_{\Psi,\Phi}(\eta(t),\xi(t),\xi(t))\\&\quad=\lambda_1 G_B(\eta(t),\xi(t),\xi(t))\\&\qquad+\lambda_2[G_B(\eta(t),\eta(t),\Psi(\eta(t))+2G_B(\xi(t),\xi(t),\Phi(\xi(t)))]\\&\qquad+\lambda_3[G_B(\Psi(\eta(t)),\xi(t),\xi(t))+2G_B(\eta(t),\xi(t),\Phi(\xi(t)))].\end{aligned}$$

Therefore, all the hypothesis of a Corollary 3.9 are fulfilled. Thus, there exists a unique $\eta^* \in B(U)$, such that $\eta^*(t)$ is a common solution of functional equations (3.1) and (3.2). □

4. Well-posedness

Now, we consider the well-posedness of attractor-based problems of generalized Hutchinson contractive operators (type I) and generalized Hutchinson contractive operators (type II) given in Definition 2.4 in the framework of Hausdorff G-metric spaces. Some useful results of well-posedness of fixed point problems appear in [2,13].

Definition 4.1. A common attractor-based problem of mappings $\Upsilon,\Psi,\Phi:\mathcal{C}^G(Y)\to\mathcal{C}^G(Y)$ is said to be well-posed if the triplet (Υ,Ψ,Φ) has a unique common attractor $\Lambda_*\in\mathcal{C}^G(Y)$ and for any sequence $\{\Lambda_k\}$ in $\mathcal{C}^G(Y)$ such that $\lim_{k\to+\infty}H_G(\Upsilon(\Lambda_k),\Upsilon(\Lambda_k),\Lambda_k)=0$, $\lim_{k\to+\infty}H_G(\Psi(\Lambda_k),\Psi(\Lambda_k),\Lambda_k)=0$ and $\lim_{k\to+\infty}H_G(\Phi(\Lambda_k),\Phi(\Lambda_k),\Lambda_k)=0$ implies that $\lim_{k\to+\infty}H_G(\Lambda_k,\Lambda_k,\Lambda_*)=0$, that is, $\lim_{k\to+\infty}\Lambda_k=\Lambda_*$.

Theorem 4.2. *Let (Y,G) be a complete G-metric space and $\Upsilon,\Psi,\Phi:\mathcal{C}^G(Y)\to\mathcal{C}^G(Y)$ be defined as in Theorem 3.1. Then the mappings Υ,Ψ,Φ have a well-posed common attractor-based problem.*

Proof. From Theorem 3.1, we deduce that the mappings Υ,Ψ and Φ have a unique common attractor $\mathcal{B}_*$, say.

Let a sequence $\{\mathcal{B}_k\}$ in $\mathcal{C}^G(Y)$ be such that $\lim_{k\to+\infty}H_G(\Upsilon(\mathcal{B}_k),\Upsilon(\mathcal{B}_k),\mathcal{B}_k)=0$, $\lim_{k\to+\infty}H_G(\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k),\mathcal{B}_k)=0$ and $\lim_{k\to+\infty}H_G(\Phi(\mathcal{B}_k),\Phi(\mathcal{B}_k),\mathcal{B}_k)=0$.

We show that $\mathcal{B}_*=\lim_{k\to+\infty}\mathcal{B}_k$. As the mappings (Υ,Ψ,Φ) are generalized G-Hutchinson contractive operators (type I), so that

$$\begin{aligned}H_G(\mathcal{B}_k,\mathcal{B}_k,\mathcal{B}_*)&\le H_G(\mathcal{B}_k,\mathcal{B}_k,\Psi(\mathcal{B}_k))+H_G(\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k),\mathcal{B}_*)\\&\le 2H_G(\mathcal{B}_k,\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k))+H_G(\Upsilon(\mathcal{B}_*),\Psi(\mathcal{B}_k),\Phi(\mathcal{B}_*))\\&\le 2H_G(\mathcal{B}_k,\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k))+\alpha H_G(\mathcal{B}_*,\mathcal{B}_k,\mathcal{B}_k)\\&\quad+\beta H_G(\mathcal{B}_*,\Upsilon(\mathcal{B}_*),\Upsilon(\mathcal{B}_*))+\gamma H_G(\mathcal{B}_k,\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k))\\&\quad+\eta H_G(\mathcal{B}_k,\Phi(\mathcal{B}_k),\Phi(\mathcal{B}_k)).\end{aligned}$$

Thus

$$\begin{aligned}H_G(\mathcal{B}_k,\mathcal{B}_k,\mathcal{B}_*)\le\frac{1}{1-\alpha}&[2H_G(\mathcal{B}_k,\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k))+\gamma H_G(\mathcal{B}_k,\Psi(\mathcal{B}_k),\Psi(\mathcal{B}_k))\\&+\eta H_G(\mathcal{B}_k,\Phi(\mathcal{B}_k),\Phi(\mathcal{B}_k))].\end{aligned}$$

Taking limit on both sides implies that $\lim_{k\to+\infty}\mathcal{B}_k=\mathcal{B}_*$. □

Theorem 4.3. *Let* (Y, G) *be a complete* G*-metric space and* $\Upsilon, \Psi, \Phi : \mathcal{C}^G(Y) \to \mathcal{C}^G(Y)$ *be defined as in Theorem 3.8. Then the mappings* Υ, Ψ, Φ *have a well-posed common attractor-based problem.*

Proof. From Theorem 3.8, it follows that the mappings Υ, Ψ and Φ have a unique common attractor $\mathcal{B}_*$, say.

Let a sequence $\{\mathcal{B}_k\}$ in $\mathcal{C}^G(Y)$ be such that $\lim_{k\to+\infty} H_G(\Upsilon(\mathcal{B}_k), \Upsilon(\mathcal{B}_k), \mathcal{B}_k) = 0$, $\lim_{k\to+\infty} H_G(\Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k), \mathcal{B}_k) = 0$ and $\lim_{k\to+\infty} H_G(\Phi(\mathcal{B}_k), \Phi(\mathcal{B}_k), \mathcal{B}_k) = 0$.

We want to show that $\mathcal{B}_* = \lim_{k\to+\infty} \mathcal{B}_k$. As the mappings (Υ, Ψ, Φ) are generalized G-Hutchinson contractive operators (type II), so that

$$\begin{aligned}
H_G(\mathcal{B}_k, \mathcal{B}_k, \mathcal{B}_*) &\leq H_G(\mathcal{B}_k, \mathcal{B}_k, \Psi(\mathcal{B}_k)) + H_G(\Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k), \mathcal{B}_*) \\
&\leq 2H_G(\mathcal{B}_k, \Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k)) + H_G(\Upsilon(\mathcal{B}_*), \Psi(\mathcal{B}_k), \Phi(\mathcal{B}_*)) \\
&\leq 2H_G(\mathcal{B}_k, \Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k)) + \lambda_1 H_G(\mathcal{B}_*, \mathcal{B}_k, \mathcal{B}_k) + \lambda_2 [H_G(\mathcal{B}_*, \mathcal{B}_*, \Upsilon(\mathcal{B}_*) \\
&\quad + H_G(\mathcal{B}_k, \mathcal{B}_k, \Psi(\mathcal{B}_k)) + H_G(\mathcal{B}_k, \mathcal{B}_k, \Phi(\mathcal{B}_k))] \\
&\quad + \lambda_3 [H_G(\Upsilon(\mathcal{B}_*), \mathcal{B}_k, \mathcal{B}_k) + H_G(\mathcal{B}_*, \Psi(\mathcal{B}_k), \mathcal{B}_k) + H_G(\mathcal{B}_*, \mathcal{B}_k, \Phi(\mathcal{B}_k))] \\
&\leq 2H_G(\mathcal{B}_k, \Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k)) + \lambda_1 H_G(\mathcal{B}_*, \mathcal{B}_k, \mathcal{B}_k) + 2\lambda_2 [H_G(\mathcal{B}_k, \Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k)) \\
&\quad + H_G(\mathcal{B}_k, \Phi(\mathcal{B}_k), \Phi(\mathcal{B}_k))] + \lambda_3 [3H(\mathcal{B}_*, \mathcal{B}_k, \mathcal{B}_k) \\
&\quad + 2H_G(\mathcal{B}_k, \Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k)) + 2H_G(\mathcal{B}_k, \Phi(\mathcal{B}_k), \Phi(\mathcal{B}_k))].
\end{aligned}$$

Thus

$$\begin{aligned}
H_G(\mathcal{B}_k, \mathcal{B}_k, \mathcal{B}_*) \leq \frac{1}{1-\lambda_1-3\lambda_3} &[2(1+\lambda_2+\lambda_3) H_G(\mathcal{B}_k, \Psi(\mathcal{B}_k), \Psi(\mathcal{B}_k)) \\
&+ 2(\lambda_2+\lambda_3) H_G(\mathcal{B}_k, \Phi(\mathcal{B}_k), \Phi(\mathcal{B}_k))].
\end{aligned}$$

Taking limit on both side implies that $\lim_{k\to+\infty} \mathcal{B}_k = \mathcal{B}_*$. □

Conclusion

This article dealt with the existence of common attractors of generalized Hutchinson operator defined on a finite family of generalized contractive mappings on a complete G-metric spaces. We also acquire different results for G-iterated function systems satisfying a different set of generalized contractive conditions. Moreover, we consider the well-posedness of attractor-based problems of generalized Hutchinson contractive operators. One can consider the results in this paper for further study in the setup of more general spaces like quasi metric spaces and controlled metric spaces. In quasi metric spaces, the problem of Smyth completeness via the existence of common attractors of finite family of generalized contractive mappings would be worth doing.

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