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HALPERN-TYPE STRONG CONVERGENCE THEOREMS USING A MULTI-STEP MEAN-VALUED ITERATIVE METHOD IN HILBERT SPACES

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Dedicated to the late Professor Kazimierz Goebel and the late Professor William Art Kirk

Abstract. We prove a Halpern-type strong convergence theorem for finding common attractive and fixed points of commutative nonlinear mappings using a mean-valued iterative method. Summable errors are also incorporated. The mappings we assume are in the class called normally 2-generalized hybrid mappings, which includes nonexpansive mappings, generalized hybrid mappings, and 2-generalized hybrid mappings as special cases. One highlight of this article is that our approach yields various types of iterative methods that are effective to approximate common attractive and fixed points of commutative mappings. Three-step and more general multi-step iterative methods are derived as special cases of our result.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. For a mapping $T : C \to H$, the sets of fixed points and attractive points of T are denoted by

$$F(T) = \{x \in C : Tx = x\} \text{ and} \tag{1.1}$$

$$A(T) = \{x \in H : \|Ty - x\| \leq \|y - x\| \text{ for all } y \in C\}, \tag{1.2}$$

respectively, where C be a nonempty subset of H. For fixed point theorems and their applications, excellent monographs by Goebel and Kirk [9] and Goebel [8] are available. The concept of attractive points was introduced by Takahashi and Takeuchi [43]. For various results concerning attractive points, see, e.g., the studies of [3, 29–31].

A mapping $T : C \to H$ is said to be *nonexpansive* if it satisfies the condition

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C. \tag{1.3}$$

Approximation methods for finding fixed points of nonexpansive mappings have been investigated by many researchers because they have broad range of applications. The following iteration is called the Mann-type [35]:

$$\begin{aligned} &x_1 \in C \text{ is given,} \\ &x_{n+1} = \lambda_n x_n + (1 - \lambda_n) T x_n \end{aligned} \tag{1.4}$$

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for all $n \in \mathbb{N}$, where $\lambda_n \in [0,1]$ satisfies some conditions. Reich [39] demonstrated that sequences generated by Mann iteration (1.4) converge weakly to a fixed point of T in a framework of Banach spaces.

In 1974, Ishikawa proposed the following 2-step approximation method for finding a fixed point of a nonlinear mapping T:

$$\begin{aligned} x_1 &\in C \text{ is given}, \\ y_n &= \mu_n x_n + (1-\mu_n) T x_n, \\ x_{n+1} &= \lambda_n x_n + (1-\lambda_n) T y_n \end{aligned} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $\lambda_n, \mu_n \in [0,1]$ satisfy certain conditions. Various results have been established using Ishikawa iteration, see, e.g., the studies of [6, 21, 46, 48]. Setting $\mu_n = 1$ in (1.5) yields the Mann iteration (1.4). Therefore, the Ishikawa iterative method is an extension of the Mann-type. The Ishikawa method has been further extended to three-step versions; see Noor [37], Phuengrattana and Suantai [38], and Chugh *et al.* [7]. Wittmann [47] proved a strong convergence theorem for finding a fixed point of a nonexpansive mapping T using Halpern-type iteration [10]:

$$\begin{aligned} x_1 &\in C \text{ is given}, \\ x_{n+1} &= \lambda_n x + (1-\lambda_n) T x_n \end{aligned}$$

for all $n \in \mathbb{N}$, where $\lambda_n, \mu_n \in [0,1]$ satisfy certain conditions. For this type of iterative methods, see the articles [3, 14, 15, 30–32, 34, 41, 45].

Subsequent studies have extended the class of mappings for which various types of iterative schemes are available to approximate fixed points. In 2010, Kocourek *et al.* [18] defined a new type of mappings. A mapping $T : C \to H$ is called *generalized hybrid* [18] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1-\alpha) \|x - Ty\|^2 \le \beta \|Tx - y\|^2 + (1-\beta) \|x - y\|^2 \tag{1.6}$$

for all $x, y \in C$. Setting $\alpha = 1$ and $\beta = 0$ in (1.6), we have the condition (1.3) of nonexpansive mappings. Hence, a class of generalized hybrid mappings contains nonexpansive mappings as special cases. Furthermore, the class of generalized hybrid mappings includes *nonspreading mappings* [20], *hybrid mappings* [42], and λ-*hybrid mappings* [1] as special cases. For various types of convergence theorems for finding fixed and attractive points of generalized hybrid mappings, see, e.g., the studies of [14, 18, 43, 45].

The classes of mappings have been furthermore extended for fixed point and attractive point theorems and convergence theorems to be established. A mapping $T : C \to C$ is called a *normally 2-generalized hybrid mapping* [29] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_2 \left\|T^2 x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ &+\beta_2 \left\|T^2 x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \le 0 \end{aligned} \tag{1.7}$$

for all $x, y \in C$, where the parameters satisfy $\sum_{n=0}^{2} (\alpha_n + \beta_n) \ge 0$ and $\alpha_2 + \alpha_1 + \alpha_0 > 0$. This class of mappings contains generalized hybrid mappings (1.6) as the case of $\alpha_2 = \beta_2 = 0$, $\alpha_1 + \alpha_0 = 1$, and $\beta_1 + \beta_0 = -1$. Furthermore, the class of normally 2-generalized hybrid mappings includes *normally generalized hybrid mappings* [44] and 2-*generalized hybrid mappings* [36] as special cases. For examples of these types of nonlinear mappings, see Kondo [21, 26] and articles cited therein.

In 2018, Hojo *et al.* [12] proved the following strong convergence theorem for commutative normally 2-generalized hybrid mappings using a "mean-valued iterative method":

Theorem 1.1 ([12]). *Let C be a nonempty and convex subset of a real Hilbert space H. Let S and T be commutative normally 2-generalized hybrid mappings from C into itself that satisfy $A(S) \cap A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u \,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C \text{ is given,}$$

$$x_{n+1} = \lambda_n u_n + (1 - \lambda_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \tag{1.8}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\overline{u} = P_{A(S)\cap A(T)} u$ of $A(S) \cap A(T)$, where $P_{A(S)\cap A(T)}$ is the metric projection from H onto $A(S) \cap A(T)$. Additionally, if C is closed in H, then $x_n \to \widehat{u} = P_{F(S)\cap F(T)} u$, where $P_{F(S)\cap F(T)}$ is the metric projection from H onto $F(S) \cap F(T)$.

They also proved other types of weak and strong convergence theorems. Mean-valued iterative methods such as (1.8) was initiated by Shimizu and Takahashi [40] and Atsushiba and Takahashi [4]; see also the classic study by Baillon [5]. In very recent articles, Kondo [25, 27, 28] combined the mean-valued and the three-step iterative methods and showed that various types of iterative methods are effective to approximate fixed points for general classes of nonlinear mappings.

In this article, we develop Theorem 1.1, incorporating multi-step iteration such as Kondo [25, 27, 28]. An error term is also introduced following Kamimura and Takahashi [17] and Kondo and Takahashi [33]. We assume two commutative normally 2-generalized hybrid mappings that have a common attractive point while continuity of mappings is not required. Our approach reveals that various types of iterative methods are effective to approximate common attractive and fixed points. Three-step and more general multi-step iterative methods are derived as special cases of our result. In Section 2, we introduce background knowledge and results. In Section 3, we prove the main theorem that generalizes Theorem 1.1. In Section 4, deduced results are presented as corollaries.

2. Preliminaries

This section provides background information. In a real Hilbert space H, it is known that

$$2\langle x - y,\ y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle x - y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. Following convention, we denote by P_C a *metric projection* from H onto C, which means that $\|x - P_C x\| \leq \|x - h\|$ for any $x \in H$ and $h \in C$, where C is a nonempty, closed, and convex subset of H. A metric projection is nonexpansive.

Concerning the metric projection, the following inequality is frequently employed:

$$\langle x - P_C x,\ P_C x - h \rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $h \in C$.

Weak and strong convergence of a sequence $\{x_n\}$ in H to a point $x\,(\in H)$ are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The following lemma has been used in the literature to prove strong convergence theorems:

Lemma 2.1 ([2]; see also [49])**.** *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{U_n\}$ be a sequence of real numbers such that $\overline{\lim}_{n\to\infty} U_n \leq 0$, and let $\{\eta_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \eta_n < \infty$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. If $X_{n+1} \leq (1-\lambda_n) X_n + \lambda_n U_n + \eta_n$ for all $n \in \mathbb{N}$, then $X_n \to 0$ as $n \to \infty$.*

A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - q\| \leq \|x - q\|$$

for all $x \in C$ and $q \in F(T)$, where C is a nonempty subset of H. It is known that the set of fixed points of a quasi-nonexpansive mapping is closed and convex. Kondo and Takahashi [29] showed that a normally 2-generalized hybrid mapping (1.7) with a fixed point is quasi-nonexpansive:

Lemma 2.2 ([29])**.** *Let $T : C \to C$ be a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$, where C is a nonempty subset of H. Then, T is quasi-nonexpansive.*

As nonexpansive mappings (1.3) and generalized hybrid mappings (1.6) are special cases of normally 2-generalized hybrid mapping, they are also quasi-nonexpansive if they have fixed points.

According to Takahashi and Takeuchi [43], a set of attractive points (1.2) has the following properties:

Lemma 2.3 ([43])**.** *Let T be a mapping from C into H, where C is a nonempty subset of H. Let $A(T)$ be the set that collects all attractive points of T. Then, the following hold true:*

(a) $A(T)$ is a closed and convex subset of H;
(b) $A(T) \cap C \subset F(T)$;
(c) $F(T) \subset A(T)$ if T is quasi-nonexpansive.

Hojo *et al.* [12] proved the following:

Lemma 2.4 ([12]; see also [30])**.** *Let $S, T : C \to C$ be commutative normally 2-generalized hybrid mappings such that $A(S) \cap A(T) \neq \emptyset$, where C is a nonempty subset of H. For a bounded sequence $\{y_n\}$ in C, define*

$$A_n \equiv \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n \,(\in H) \tag{2.3}$$

for each $n \in \mathbb{N}$. Suppose that $A_{n_j} \rightharpoonup p \in H$, where $\{A_{n_j}\}$ is a subsequence of $\{A_n\}$. Then, $p \in A(S) \cap A(T)$. Additionally, if C is closed and convex, then it holds that $p \in F(S) \cap F(T)$ whenever $A_{n_j} \rightharpoonup p$.

In the main theorem of this article, we assume that two commutative normally 2-generalized hybrid mappings have a common attractive point. The next theorem reveals a framework for that assumption to be fulfilled:

Theorem 2.5 ([11]). *Let $S, T : C \to C$ be commutative normally 2-generalized hybrid mappings, where C is a nonempty subset of H. Assume that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is also nonempty.*

For the existence theorems of attractive or fixed points, see also [13, 19, 22, 24, 29, 36, 43, 44].

3. Main results

In this section, we prove the main theorem of this article.

Theorem 3.1. *Let C be a nonempty and convex subset of a real Hilbert space H. Let S and T be commutative normally 2-generalized hybrid mappings from C into itself that satisfy $A(S) \cap A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{\eta_n\}$ be a sequence of nonnegative real numbers that satisfies $\sum_{n=1}^{\infty} \eta_n < \infty$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u\,(\in H)$. Let $\{y_n\}$ be a sequence in C. Define a sequence $\{x_n\}$ in C as follows:*

$$
\begin{aligned}
&x_1 \in C \text{ is given,}\\
&Y_n \in C \text{ such that } \left\| Y_n - \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n \right\| \leq \eta_n,\\
&x_{n+1} = \lambda_n u_n + (1 - \lambda_n) Y_n
\end{aligned}
$$

for all $n \in \mathbb{N}$. Assume that

$$
\|y_n - q\| \leq \|x_n - q\| \tag{3.1}
$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\overline{u}$ of $A(S) \cap A(T)$, where $\overline{u} = P_{A(S) \cap A(T)} u$. Additionally, if C is closed in H, then $x_n \to \widehat{u} = P_{F(S) \cap F(T)} u$.

Proof. First, note that from Lemma 2.3-(a), $A(S) \cap A(T)$ is a closed and convex subset of H. As $A(S) \cap A(T) \neq \emptyset$ is assumed, the metric projection $P_{A(S) \cap A(T)}$ from H onto $A(S) \cap A(T)$ exists. Define

$$
A_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n \,(\in H)
$$

for all $n \in \mathbb{N}$. Then, we have that $\|Y_n - A_n\| \leq \eta_n$. Let us verify that

$$
\|A_n - q\| \leq \|y_n - q\| \tag{3.2}
$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, using the definition of an attractive point (1.2) we have that

$$\begin{aligned}
\|A_n - q\| &= \left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n - q \right\| \\
&= \frac{1}{n^2} \left\| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n - n^2 q \right\| = \frac{1}{n^2} \left\| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left(S^k T^l y_n - q \right) \right\| \\
&\le \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left\| S^k T^l y_n - q \right\| \le \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left\| S^{k-1} T^l y_n - q \right\| \le \cdots \\
&\le \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left\| T^l y_n - q \right\| \le \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left\| T^{l-1} y_n - q \right\| \le \cdots \\
&\le \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \|y_n - q\| = \|y_n - q\|
\end{aligned}$$

as claimed. Using (3.2) and (3.1), we have that

$$\|A_n - q\| \le \|y_n - q\| \le \|x_n - q\| \text{ and hence,} \tag{3.3}$$

$$\begin{aligned}
\|Y_n - q\| &\le \|Y_n - A_n\| + \|A_n - q\| \\
&\le \eta_n + \|x_n - q\|
\end{aligned} \tag{3.4}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$.

Observe that $\{x_n\}$ is bounded. This fact can be ascertained as follows: Let $q \in A(S) \cap A(T)$ and define

$$M = \max \left\{ \sup_{n \in \mathbb{N}} \|u_n - q\|, \ \|x_1 - q\| \right\}.$$

As $\{u_n\}$ is bounded, M is a real number. We show that

$$\|x_n - q\| \le M + \sum_{i=1}^{n-1} \eta_i$$

using a mathematical induction, where $\sum_{i=1}^{0} \eta_i \equiv 0$. (i) For $n = 1$, the desired result holds true. (ii) As a next step, assume that $\|x_k - q\| \le M + \sum_{i=1}^{k-1} \eta_i$, where $k \in \mathbb{N}$. It follows from (3.4) that

$$\begin{aligned}
\|x_{k+1} - q\| &= \|\lambda_k u_k + (1 - \lambda_k) Y_k - q\| \\
&\le \lambda_k \|u_k - q\| + (1 - \lambda_k) \|Y_k - q\| \\
&\le \lambda_k \|u_k - q\| + (1 - \lambda_k) (\|x_k - q\| + \eta_k) \\
&\le \lambda_k M + (1 - \lambda_k) \left(M + \sum_{i=1}^{k} \eta_i \right) \\
&\le M + \sum_{i=1}^{k} \eta_i.
\end{aligned}$$

Therefore, it holds that $\|x_n - q\| \leq M + \sum_{i=1}^{\infty} \eta_i$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is bounded as claimed. From (3.3), $\{y_n\}$ and $\{A_n\}$ are also bounded.

Next, we verify that

$$\|x_{n+1} - A_n\| \to 0. \tag{3.5}$$

As $\{\eta_n\} \subset [0, \infty)$ and $\sum_{n=1}^{\infty} \eta_n < \infty$, it follows that $\eta_n \to 0$. As $\{u_n\}$ and $\{A_n\}$ are bounded, using $\lambda_n \to 0$, we have

$$\begin{aligned} \|x_{n+1} - A_n\| &= \|\lambda_n u_n + (1 - \lambda_n) Y_n - A_n\| \\ &\leq \lambda_n \|u_n - A_n\| + (1 - \lambda_n) \|Y_n - A_n\| \\ &\leq \lambda_n \|u_n - A_n\| + (1 - \lambda_n) \eta_n \to 0. \end{aligned}$$

Define $X_n = \|x_n - \overline{u}\|^2$, where $\overline{u} = P_{A(S) \cap A(T)} u$. Our goal is to demonstrate that $X_n \to 0$. Using (2.1) and (3.4), we have

$$\begin{aligned} X_{n+1} &= \|x_{n+1} - \overline{u}\|^2 \\ &= \|\lambda_n (u_n - \overline{u}) + (1 - \lambda_n)(Y_n - \overline{u})\|^2 \\ &\leq (1 - \lambda_n)^2 \|Y_n - \overline{u}\|^2 + 2\lambda_n \langle u_n - \overline{u},\ x_{n+1} - \overline{u}\rangle \\ &\leq (1 - \lambda_n)(\|x_n - \overline{u}\| + \eta_n)^2 + 2\lambda_n \langle u_n - \overline{u},\ x_{n+1} - \overline{u}\rangle \\ &\leq (1 - \lambda_n)\|x_n - \overline{u}\|^2 + \eta_n (2\|x_n - \overline{u}\| + \eta_n) \\ &\quad + 2\lambda_n \langle u_n - \overline{u},\ x_{n+1} - \overline{u}\rangle \\ &\leq (1 - \lambda_n)\|x_n - \overline{u}\|^2 + K\eta_n + 2\lambda_n \langle u_n - \overline{u},\ x_{n+1} - \overline{u}\rangle \\ &\leq (1 - \lambda_n) X_n + K\eta_n \\ &\quad + 2\lambda_n (\langle u_n - u,\ x_{n+1} - \overline{u}\rangle + \langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle) \end{aligned}$$

where

$$K = \sup_{n \in \mathbb{N}} (2\|x_n - \overline{u}\| + \eta_n).$$

As $\{x_n\}$ and $\{\eta_n\}$ are bounded, K is a real number. Recall that $\eta_n \to 0$. Also, as $\{x_n\}$ is bounded and $u_n \to u$, it holds that $\langle u_n - u,\ x_{n+1} - \overline{u}\rangle \to 0$. As it is assumed that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \eta_n < \infty$, from Lemma 2.1, our aim is to prove that

$$\overline{\lim_{n \to \infty}} \langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle \leq 0.$$

From (3.5), it follows that

$$\overline{\lim_{n \to \infty}} \langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle = \overline{\lim_{n \to \infty}} \langle u - \overline{u},\ A_n - \overline{u}\rangle. \tag{3.6}$$

There exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that

$$\overline{\lim_{n \to \infty}} \langle u - \overline{u},\ A_n - \overline{u}\rangle = \lim_{i \to \infty} \langle u - \overline{u},\ A_{n_i} - \overline{u}\rangle. \tag{3.7}$$

As $\{A_{n_i}\}$ is bounded, there exist a subsequence $\{A_{n_j}\}$ of $\{A_{n_i}\}$ and $p \in H$ such that $A_{n_j} \rightharpoonup p$ as $j \to \infty$. As $\{y_n\}$ is bounded, from Lemma 2.4, we obtain $p \in A(S) \cap A(T)$. Hence, from (2.2),

$$\begin{aligned} \lim_{i \to \infty} \langle u - \overline{u},\ A_{n_i} - \overline{u}\rangle &= \lim_{j \to \infty} \langle u - \overline{u},\ A_{n_j} - \overline{u}\rangle \\ &= \langle u - \overline{u},\ p - \overline{u}\rangle \leq 0 \end{aligned} \tag{3.8}$$

as $\overline{u} = P_{A(S)\cap A(T)}u$. From (3.6)-(3.8), it follows that

$$\overline{\lim_{n\to\infty}} \langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle = \langle u - \overline{u},\ p - \overline{u}\rangle \leq 0.$$

According to Lemma 2.1, we have $X_n \to 0$, which implies that $x_n \to \overline{u} = P_{A(S)\cap A(T)}u$.

Assume, in addition to the other assumptions, that C is closed in H. As a final step of this proof, we demonstrate that $x_n \to \widehat{u} = P_{F(S)\cap F(T)}u$. As $x_n \to \overline{u} \equiv P_{A(S)\cap A(T)}u$ and C is closed, we have that $\overline{u} \in C \cap A(S) \cap A(T)$. From Lemma 2.3-(b), it holds true that $\overline{u} \in F(S) \cap F(T)$. Consequently, $F(S) \cap F(T) \neq \emptyset$. From Lemma 2.2, S and T are quasi-nonexpansive. Hence, $F(S) \cap F(T)$ is a closed and convex subset of H. The metric projection $P_{F(S)\cap F(T)}$ from H onto $F(S) \cap F(T)$ is defined. We aim to show that

$$(\widehat{u} \equiv)\, P_{F(S)\cap F(T)}u = \overline{u} \left(\equiv P_{A(S)\cap A(T)}u\right).$$

As $\overline{u} \in F(S) \cap F(T)$, it is sufficient to prove that $\|u - \overline{u}\| \leq \|u - q\|$ for all $q \in F(S) \cap F(T)$. Choose $q \in F(S) \cap F(T)$ arbitrarily and fix it. As S and T are quasi-nonexpansive, according to Lemma 2.3-(c), it follows that $F(S) \cap F(T) \subset A(S) \cap A(T)$. Therefore,

$$\begin{aligned}\|u - \overline{u}\| &= \inf\{\|u - h\| : h \in A(S) \cap A(T)\}\\ &\leq \inf\{\|u - h\| : h \in F(S) \cap F(T)\}\\ &\leq \|u - q\|.\end{aligned}$$

This result implies that $\overline{u} = P_{F(S)\cap F(T)}u\,(\equiv \widehat{u})$. This completes the proof. □

4. Corollaries

In this section, we present some corollaries deduced from Theorem 3.1. Theorem 1.1 and some other existing results are derived as special cases of Theorem 3.1. A multi-step iterative method is also presented as Corollary 4.2. First, we consider a result without errors. Setting $\eta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.1 yields the following:

Corollary 4.1. *Let C be a nonempty and convex subset of a real Hilbert space H. Let S and T be commutative normally 2-generalized hybrid mappings from C into itself that satisfy $A(S) \cap A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u\,(\in H)$. Let $\{y_n\}$ be a sequence in C. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C \text{ is given,}$$

$$x_{n+1} = \lambda_n u_n + (1 - \lambda_n)\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l y_n$$

for all $n \in \mathbb{N}$. Assume that

$$\|y_n - q\| \leq \|x_n - q\| \tag{4.1}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\overline{u}$ of $A(S) \cap A(T)$, where $\overline{u} = P_{A(S)\cap A(T)}u$. Additionally, if C is closed in H, then $x_n \to \widehat{u} = P_{F(S)\cap F(T)}u$.

From this, Theorem 1.1 is derived. A proof is as follows:

Proof. Set $y_n = x_n$ for all $n \in \mathbb{N}$. Then, the condition (4.1) is fulfilled. From Corollary 4.1, a sequence $\{x_n\}$ defined by (1.8) converges strongly to a point $\overline{u} = P_{A(S)\cap A(T)}u$ of $A(S) \cap A(T)$.

Next, assume that C is closed in H. Then, from Corollary 4.1, we obtain $x_n \to \widehat{u} = P_{F(S)\cap F(T)}u$. This ends the proof. □

In the next corollary, we assume that C is closed in H, S and T are nonexpansive mappings, and $u_n = x_1$ for all $n \in \mathbb{N}$, for simplicity. Corollary 4.1 (Theorem 3.1) also produces the following multi-step iterative method for finding a common fixed point:

Corollary 4.2. *Let C be a nonempty, closed, and convex subset of H. Let S and T be commutative nonexpansive mappings from C into itself that satisfy $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{\mu_n\}$, $\{\nu_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ w_n &= a_n x_n + b_n S x_n + c_n T x_n, \\ z_n &= \nu_n w_n + (1 - \nu_n) S w_n, \\ y_n &= \mu_n z_n + (1 - \mu_n) T z_n, \\ x_{n+1} &= \lambda_n x + (1 - \lambda_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n \end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S)\cap F(T)}x$.

Proof. It is sufficient to demonstrate that the condition (4.1) is fulfilled. First, observe that $\|w_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. In fact, as S and T are quasi-nonexpansive, it follows that

$$\begin{aligned} \|w_n - q\| &= \|a_n x_n + b_n S x_n + c_n T x_n - q\| \\ &\leq \|a_n (x_n - q) + b_n (S x_n - q) + c_n (T x_n - q)\| \\ &\leq a_n \|x_n - q\| + b_n \|S x_n - q\| + c_n \|T x_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\| . \end{aligned}$$

Similarly, $\|z_n - q\| \leq \|w_n - q\|$ and $\|y_n - q\| \leq \|z_n - q\|$ can be proved. Consequently, we have that

$$\|y_n - q\| \leq \|z_n - q\| \leq \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Thus, the condition (4.1) is met. From Corollary 4.1, we have that $x_n \to \widehat{x} = P_{F(S)\cap F(T)}x$. This ends the proof. □

Setting $b_n = c_n = 0$ in (4.2) yields the following three-step iterative method:

$$
\begin{aligned}
z_n &= \nu_n x_n + (1 - \nu_n) S x_n, \\
y_n &= \mu_n z_n + (1 - \mu_n) T z_n, \\
x_{n+1} &= \lambda_n x + (1 - \lambda_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l y_n
\end{aligned}
\tag{4.3}
$$

where $x_1 = x \in C$ is given arbitrarily. Similarly, a two-step iteration is derived by setting $\nu_n = 1$ in (4.3). From Corollary 4.1 (Theorem 3.1), various types of iterative methods can be produced for finding common attractive and fixed points. For some variations, see Kondo [25, 27].

Corollary 4.1 (Theorem 3.1) is also effective to yield convergence theorems for a single mapping. The following is a result in Kondo and Takahashi [30]:

Corollary 4.3 ([30]). *Let C be a nonempty and convex subset of H. Let T be a normally 2-generalized hybrid mapping from C into itself that satisfies $A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Define a sequence $\{x_n\}$ in C as follows:*

$$
\begin{aligned}
&x_1, u \in C \text{ are given,} \\
&x_{n+1} = \lambda_n u + (1 - \lambda_n) \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\overline{u}$ of $A(T)$, where $\overline{u} = P_{A(T)} u$.

Proof. Using Corollary 4.1 with $S = I$, $u_n = u$, and $y_n = x_n$ for all $n \in \mathbb{N}$, we obtain that the sequence $\{x_n\}$ converges strongly to a point $\overline{u} = P_{A(T)} u$ of $A(T)$, where I is the identity mapping defined on C. This indicates that the desired result holds true. □

From Corollary 4.3, the following result by Hojo and Takahashi [14] is derived:

Corollary 4.4 ([14]). *Let C be a nonempty, closed, and convex subset of H. Let T be a generalized hybrid mapping from C into itself that satisfies $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Define a sequence $\{x_n\}$ in C as follows:*

$$
\begin{aligned}
&x_1, u \in C \text{ are given,} \\
&x_{n+1} = \lambda_n u + (1 - \lambda_n) \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(T)$, where $\widehat{u} = P_{F(T)} u$.

Proof. Recall that a generalized hybrid mapping, which is characterized by the condition (1.6), is normally 2-generalized hybrid (1.7). From Lemma 2.2, T is quasi-nonexpansive. Hence, $F(T)$ is closed and convex. As $F(T) \neq \emptyset$ is assumed, the metric projection $P_{F(T)}$ from H onto $F(T)$ exists. Furthermore, according to

Lemma 2.3-(c), $F(T) \subset A(T)$. Thus, we obtain $A(T) \neq \emptyset$ and therefore, the metric projection $P_{A(T)}$ from H onto $A(T)$ exists.

From Corollary 4.3, it follows that $x_n \to \overline{u} = P_{A(T)}u$. Our aim is to prove that $(\widehat{u} \equiv) P_{F(T)}u = \overline{u} \left(\equiv P_{A(T)}u\right)$. As $\overline{u} \in A(T) \cap C$, it follows from Lemma 2.3-(b) that $\overline{u} \in F(T)$. Therefore, it is sufficient to show that $\|u - \overline{u}\| \leq \|u - q\|$ for all $q \in F(T)$. Choose $q \in F(T)$ arbitrarily. As $F(T) \subset A(T)$, the following holds:

$$\begin{aligned} \|u - \overline{u}\| &= \inf\{\|u - h\| : h \in A(T)\} \\ &\leq \inf\{\|u - h\| : h \in F(T)\} \\ &\leq \|u - q\|, \end{aligned}$$

which implies that $\overline{u} = P_{F(T)}u \, (\equiv \widehat{u})$. This completes the proof. □

From Corollary 4.4, Theorem 4.1 in Kurokawa and Takahashi [34] is derived.

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