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AUXILIARY PROBLEM PRINCIPLE EXTENDED TO EQUILIBRIUM PROBLEMS OVER THE INTERSECTION OF FIXED POINT SETS

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Abstract. In this paper, we introduce a new algorithm which is an application of the auxiliary problem principle for solving equilibrium problems defined over the intersection of fixed point sets in a real Hilbert space. Basing on inertial extrapolation, parallel and auxiliary principle techniques, a strongly convergence of iterative sequences is showed under standard assumptions. Several numerical experiments are showed to illustrate the efficiency and accuracy of the proposed algorithm.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. The equilibrium problems (EPs, for short) is first introduced by Blum and Oettli in [13]. In recently years, solving the problem EPs has been a key task since its importance in economics, physics, engineering, game theory, operations research and other applied science. The fixed point problem, saddle point problem, variational inequality problem and Nash-Counot equilibrium model in nonlinear analysis can be modeled in the formulation of the problem EPs (see, e.g., Akutsah et al. [3], Anh et al. [7, 11], Bianchi et al. [12], Blum et al. [13], Chbani et al. [16], Giannessi et al. [19], Iusem al. [21]), Kim et al. [24], Lotfikar et al. [27] and Noor [32]: Finding a point $x^* \in C$ satisfying

$$f(x^*, y) \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $f : C \times C \to \mathcal{R}$ is usually called to be a *cost bifunction*. We denote the solution set of the problem EPs (1.1) by $\mathrm{Sol}(C, f)$.

It is well known that the auxiliary problem principle is first introduced for solving optimization problems by Cohen in [18] and then extended to the strongly monotone and Lipschitz-type problem EPs (1.1) by Mastroeni in [30]. It becomes a useful tool for analyzing and developing efficient algorithms for the solution to various classes of mathematical programming and equilibrium problems. Recently, the auxiliary problem principle was employed and promoted by many authors for solving the monotone problem (1.1) such as auxiliary principle technique of Noor in [31], dual extragradient method of Quoc et al. in [33], auxiliary problem methods of Anh et al. in [8, 9] and the references therein.

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For each $i \in I := \{1, 2, \dots, n\}$, let the self-mappings $S_i : \mathcal{H} \to \mathcal{H}$ and a bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ such that $f(x, \cdot)$ is convex and $f(x, x) = 0$ for all $x \in \mathcal{H}$. Denote the fixed point set of S_i by $Fix(S_i) := \{x \in \mathcal{H} : S_i(x) = x\}$. Given this data, we consider the equilibrium problem, shortly $EP(\Omega, f)$, of finding a point $x^* \in \Omega := \cap_{i \in I} Fix(S_i)$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in \Omega.$$

Note that the problem $EP(\Omega, f)$ has several important special cases as seen below.

1. **Equilibrium Problem EPs** (1.1)**.** Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Taking S_i is a positive constant (or $S_i = Pr_C$ is the projection onto C) for each $i \in I$. Then, it is clear that $\Omega = \cap_{i \in I} Fix(S_i) = C$ and the problem $EP(\Omega, f)$ is written in the form EPs (1.1).

2. **General Variational Inequality Problem.** Let C be a nonempty, closed and convex subset of $\mathcal{H}$, $F : \mathcal{H} \to \mathcal{H}$, $\varphi : \mathcal{H} \to \mathcal{R}$ and $Fix(S_i) = C$ for all $i \in I$. Setting $f(x, y) := \langle F(x), y - x\rangle + \varphi(y) - \varphi(x)$ for all $x, y \in \mathcal{H}$. Then, the problem $EP(\Omega, f)$ is equivalent to the following general variational inequality problem: Find $x^* \in C$ such that

$$\langle F(x^*), x - x^*\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall x \in C.$$

3. **Common Fixed Point Problem.** Let $f(x, y) = 0$ for all $x, y \in \mathcal{H}$. The following problem is called the common fixed point problem (CFPP):

$$\text{Find } x^* \in Fix(S_i), \quad \forall i \in I.$$

We can easily see that the problem (CFPP) becomes a case of the problem $EP(\Omega, f)$.

In the cases $\mathcal{H} := \mathcal{R}^n$, for each $i \in I$, $S_i = S : \mathcal{R}^n \to \mathcal{R}^n$ is nonexpansive and so $\Omega = Fix(S)$, some methods have been proposed to solve the problem $EP(Fix(S), f)$. In [20], Iiduka and Yamada introduced a subgradient-type method as follows:

$$\begin{cases} y^k \in K_k := \{x \in \mathcal{R}^n : \|x\| \leq \rho_k + 1\}, \\ f(x^k, y^k) \geq 0, \max\{f(y, x^k) : y \in K_k\} \leq f(y^k, x^k) + \epsilon_k, \\ \xi^k \in \partial_2 f(y^k, x^k), x^{k+1} = S[x^k - \lambda_k f(y^k, x^k)\xi^k], \rho_{k+1} = \max\{\|x^{k+1}\|, \rho_k\}. \end{cases}$$

To prove the convergence of iterative sequences $\{x^k\}$ and $\{y^k\}$, the authors assumed that parameter sequence $\{\xi_k\}$ is bounded by $M > 0, \epsilon_k > 0, \lim_{k \to \infty} \epsilon_k = 0$ and

$$\{\lambda_k\} \subset [a, b] \subset \left(0, \frac{2}{M^2}\right), \{x^* \in Fix(S) : f(y^k, x^*) \leq 0, \quad \forall k \geq 1\} \neq \emptyset.$$

However, these conditions are difficult to verify.

Very recently, by using the properties of the approximation projection, the fixed point method, parallel and subgradient techniques, Anh and Hong in [10] proposed a new projection method for solving the problem $EP(\Omega, f)$ with demicontractive mappings $S_i(i \in I)$. At each iteration k, the iterate x^{k+1} is defined as follows:

$$\begin{cases} x^0 \in C, \\ y_i^k = (1-\alpha_{k,i})x^k + \alpha_{k,i}S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \text{ where } i_0 = \text{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(y^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(y^k, y^k). \end{cases}$$

Under the main assumptions that f is strongly monotone and its approximation subdifferential is Lipschitz continuous, the authors showed that the sequence $\{x^k\}$ strongly converges to a unique solution in the space $\mathcal{H}$. This scheme requires to compute an approximation diagonal subgradient at each iteration.

Motivated and inspired by the above solution methods, as well as the auxiliary problem methods in [8, 18, 30] for the problem EPs (1.1), the parallel techniques in [5, 14, 22, 23] and the inertial proximal approachs in [1, 2, 4, 15, 26] for the variational inequality problem, for solving the problem $EP(\Omega, f)$ with demicontractive mappings $S_i(i \in I)$, the purpose of this paper is three-fold. First, inertial technique makes use of two previous iterates (i.e. x^k, x^{k-1}) to update the iterate w^k. A self-adaptive updating rule is applied for the stepsize and the inertial parameters. Second, for each $i \in I$, computing the intermediate approximations u_i^k can be found in parallel via the fixed point of S_i. Then, among all $u_i^k(i \in I)$, the farest element from w^k, denoted by t^k, is chosen. Third, the next iterate x^{k+1} is based on auxiliary problem principle and Mann iteration method via the point t^k.

The outline of the paper is as follows. In Section 2 we recall some useful definitions and lemmas. The new algorithm and its analysis are presented in Section 3. In Section 4, several numerical simulation experiments are provided to illustrate the efficiency and accuracy of our proposed algorithm.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space and $C \subseteq \mathcal{H}$ is nonempty, closed and convex.

Definition 2.1. Given a mapping $T : \mathcal{H} \to \mathcal{H}$.

(1) T is called *quasi-nonexpansive* on $\mathcal{H}$, if

$$\|T(x) - \hat{x}\| \le \|x - \hat{x}\|, \quad \forall (x, \hat{x}) \in \mathcal{H} \times \text{Fix}(T).$$

(2) T is called *firmly nonexpansive*, that is, for all $x, y \in H$,

$$\langle T(x) - T(y), x - y\rangle \ge \|T(x) - T(y)\|^2.$$

(3) T is called $\tau-$*strictly pseudocontractive* on $\mathcal{H}$, where $\tau \in [0, 1)$, if

$$\|T(x) - T(y)\|^2 \le \|x - y\|^2 + \tau\|(x - y) - [T(x) - T(y)]\|^2, \quad \forall x, y \in \mathcal{H}.$$

(4) T is called $\beta-$*demicontractive* on $\mathcal{H}$ where $\beta \in [0, 1)$, if

$$\|T(x) - \hat{x}\|^2 \le \|x - \hat{x}\|^2 + \beta\|x - T(x)\|^2, \quad \forall (x, \hat{x}) \in \mathcal{H} \times \text{Fix}(T).$$

(5) T is called *demiclosed* at zero, if $\{x^k\}$ weakly converges to $\bar{x}$ and $\{(I - T)(x^k)\}$ strongly converges to 0, then $\bar{x} \in \text{Fix}(T)$.

Let $g : \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ be a convex function. The function g is called *proper* if its effective domain $D(g) := \{x \in \mathcal{H} : g(x) < +\infty\} \neq \emptyset$ and $g(x) > -\infty$ for all $x \in \mathcal{H}$. The g is *lower semicontinuous* at $x_0 \in D(g)$ if

$$g(x_0) \leq \liminf_{x \to x_0} g(x).$$

It is called lower semicontinuous if it is lower semicontinuous at every $x_0 \in D(g)$. The *subdifferential* ∂g of a proper convex function g at $x \in \mathcal{H}$ is defined by

$$\partial g(x) = \{z \in \mathcal{H} : g(x) + \langle z, y - x\rangle \leq g(y), \quad \forall y \in \mathcal{H}\}.$$

The following lemmas are useful for our algorithm's analysis.

Lemma 2.2 ([35, Lemma 2.5]). *Let $\{a_k\}$ be a positive sequence and $\{p_k\}$ a sequence of real numbers. Let $\{\alpha_k\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$. Assume that*

$$a_{k+1} \leq (1 - \alpha_k)a_k + b_k, \quad k = 1, 2, \ldots.$$

If $\limsup_{k\to\infty} \frac{b_k}{\alpha_k} \leq 0$ or $\sum_{k=1}^{\infty} b_k < +\infty$, then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.3 ([28, Remark 4.2]). *Let $S : \mathcal{H} \to \mathcal{H}$ be a β-demicontractive mapping with $Fix(S) \neq \emptyset$ and set $S_\omega = (1 - \omega)Id + \omega S$ for $\omega \in (0, 1]$. Then, the S_ω is quasi-nonexpansive if $\omega \in [0, 1 - \beta]$ and*

$$\|S_\omega(x) - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 - \omega(1 - \beta - \omega)\|S(x) - x\|^2, \quad \forall \bar{x} \in \mathrm{Fix}(S), x \in \mathcal{H}.$$

Lemma 2.4 ([28, Remark 4.4]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let k_0 be an integer such that $a_{k_0} \leq a_{k_0+1}$ and define, for all integer $k \geq k_0$,*

$$\tau(k) = \max\{i \in \mathcal{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}.$$

Then, $0 \leq a_k \leq a_{\tau(k)+1}$ for all $k \geq k_0$. Furthermore, the sequence $\{\tau(k)\}_{k\geq k_0}$ is nondecreasing and tends to $+\infty$ as $k \to \infty$.

3. Convergent results

In this section, we introduce a new iteration algorithm for approximating a solution of the problem $EP(\Omega, f)$ and prove its strong convergence. The algorithm uses a parallel technique, auxiliary problem principle and combines the inertial iteration method with an explicit self-adaptive stepsize rule.

The parameters setup for the algorithm is as follows.

$$\begin{cases} \tau \in (0, \beta - c_1), \{\lambda_k\} \subset [\bar{a}, \hat{a}] \subset (0, 1), \lambda_k^2 + \frac{\tau - 4(\beta - c_1)}{2\tau^2(\beta - c_1)}\lambda_k + \frac{\beta - c_1 - \tau}{\tau^2(\beta - c_1)} \geq 0, \\ \zeta_k \in (0, \frac{1}{\tau\bar{a}}), \sum_{k=1}^{\infty} \zeta_k = +\infty, \tau_k > 0, \sum_{k=1}^{\infty} \tau_k < +\infty, \\ \mu_k > 0, \gamma_{k,i} \in (\bar{b}, \hat{b}) \subset (0, 1 - \max\{\beta_i : i \in I\}), \quad \forall i \in I. \end{cases} \tag{3.1}$$

The Parallel Inertial Auxiliary Principle Algorithm (PIAPA) is presented next.

Algorithm 3.1. Choose starting points $x^0, x^1 \in \mathcal{H}$.

Step 1. (Inertial technique) Given the iterates x^{k-1} and x^k, compute

$$w^k = x^k + \alpha_k(x^k - x^{k-1}), \tag{3.2}$$

where

$$\alpha_k = \begin{cases} \min\left\{\frac{\tau_k}{\|x^k - x^{k-1}\|}, \mu_k\right\}, & if \;\; \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & \text{otherwise.} \end{cases} \tag{3.3}$$

Step 2. (Parallel technique) Take

$$u_i^k = (1-\gamma_{k,i})w^k + \gamma_{k,i}S_i(w^k).$$

Set $t^k := u_{i_0}^k$, where $i_0 \in \text{argmax}\{\|u_i^k - w^k\| : i \in I\}$.

Step 3. (Auxiliary problem principle) Compute

$$y^k = \arg\min\left\{\lambda_k f(t^k, x) + \frac{1}{2}\|x - t^k\|^2 : x \in C\right\},$$

$$x^{k+1} = (1-\zeta_k)t^k + \zeta_k y^k.$$

Let $k := k+1$ and go to Step 1.

Note that, computing w^k is used by inertial technique and t^k is by parallel technique. Then, the iteration point x^{k+1} is based on the Mann iteration method and the auxiliary problem principle. We recall that a point x^k generated by Algorithm 3.1 is an $\epsilon-$solution of the problem $EP(\Omega, f)$, if $\|x^{k+1} - x^k\| \leq \epsilon$.

For the convergence of the algorithm we assume the following.

Assumption 3.2. The mapping $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is $\beta-$strongly monotone and Lipschitz continuous with positive constants c_1, c_2 such that $\beta > c_1$.

Assumption 3.3. For all $i \in I$ the mappings $S_i : \mathcal{H} \to \mathcal{H}$ are β_i-demicontractive and demiclosed at zero and the set $\Omega := \cap_{i\in I} Fix(S_i)$ is nonempty.

Theorem 3.4. *Assume that Assumptions 3.2 and 3.3 hold. Under Conditions (3.1), the sequence $\{x^k\}$ generated by Algorithm 3.1 strongly converges to a unique solution x^* of the problem $EP(\Omega, f)$.*

Proof. Let x^* be a unique solution of the problem $EP(\Omega, f)$. Since y^k is the unique solution of the strongly convex problem

$$y^k = \arg\min\left\{\lambda_k f(t^k, x) + \frac{1}{2}\|x - t^k\|^2 : x \in C\right\},$$

we have

$$0 \in \lambda_k \partial_2 f(t^k, y^k) + y^k - t^k + N_C(y^k).$$

It means that

$$t^k - y^k - \lambda_k w^k \in N_C(y^k),$$

where $w^k \in \partial_2 f(t^k, y^k)$. Using the definition of the normal cone N_C and $x^* \in C$ yields

$$\langle t^k - y^k, x^* - y^k\rangle \leq \lambda_k\langle w^k, x^* - y^k\rangle. \tag{3.4}$$

On the other hand, from $w^k \in \partial_2 f(t^k, y^k)$ it follows that

$$\lambda_k[f(t^k, x^*) - f(t^k, y^k)] \geq \lambda_k\langle w^k, x^* - y^k\rangle.$$

Combining this and (3.4), we get

$$\langle t^k - y^k, x^* - y^k\rangle \leq \lambda_k[f(t^k, x^*) - f(t^k, y^k)]. \tag{3.5}$$

Since $\lambda_k > 0$ and f is Lipschitz-type with constants $c_1 > 0$ and $c_2 > 0$, we have

$$\lambda_k[f(t^k, x^*) - f(t^k, y^k)] \leq \lambda_k f(y^k, x^*) + \lambda_k c_1\|y^k - x^*\|^2 + \lambda_k c_2\|t^k - y^k\|^2. \tag{3.6}$$

Using the $\beta-$strongly monotone assumption of f yields

$$f(y^k, x^*) \leq -f(x^*, y^k) - \beta\|y^k - x^*\|^2.$$

By the definition of x^* and $y^k \in C$, it follows $f(x^*, y^k) \geq 0$. Then, we deduce

$$\lambda_k f(y^k, x^*) \leq -\lambda_k\beta\|y^k - x^*\|^2.$$

Combining this, (3.5) and (3.6), we obtain

$$\begin{aligned}\langle t^k - y^k, x^* - y^k\rangle =&\frac{1}{2}(\|y^k - x^*\|^2 + \|t^k - y^k\|^2 - \|t^k - x^*\|^2)\\ \leq&\lambda_k[f(t^k, x^*) - f(t^k, y^k)]\\ \leq&\lambda_k f(y^k, x^*) + \lambda_k c_1\|y^k - x^*\|^2 + \lambda_k c_2\|t^k - y^k\|^2\\ \leq&-\lambda_k\beta\|y^k - x^*\|^2 + \lambda_k c_1\|y^k - x^*\|^2 + \lambda_k c_2\|t^k - y^k\|^2.\end{aligned}$$

Note that the first equality is followed from the relation

$$\langle a, b\rangle = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|a - b\|^2) \quad \forall a, b \in \mathcal{H}.$$

Consequently, we have

$$(1 + 2\lambda_k\beta - 2\lambda_k c_1)\|y^k - x^*\|^2 \leq \|t^k - x^*\|^2 - (1 - 2\lambda_k c_2)\|y^k - t^k\|^2. \tag{3.7}$$

From the condition in (3.1) that

$$\tau \in (0, \beta - c_1), \quad \{\lambda_k\} \subset [\bar{a}, \hat{a}] \subset (0, 1), \quad \lambda_k^2 + \frac{\tau - 4(\beta - c_1)}{2\tau^2(\beta - c_1)}\lambda_k + \frac{\beta - c_1 - \tau}{\tau^2(\beta - c_1)} \geq 0,$$

it follows

$$0 < \frac{1}{1 + 2\lambda_k\beta - 2\lambda_k c_1} \leq (1 - \tau\lambda_k)^2.$$

Using (3.7), we have

$$\|y^k - x^*\|^2 \leq (1 - \tau\lambda_k)^2\|t^k - x^*\|^2 - \frac{1 - 2\lambda_k c_2}{1 + 2\lambda_k\beta - 2\lambda_k c_1}\|y^k - t^k\|^2. \tag{3.8}$$

Otherwise, since (3.2) and (3.3), we have

$$\begin{aligned}\|w^k - x\| =&\|x^k - \alpha_k(x^k - x^{k-1}) - x\|\\ \leq&\|x^k - x\| + \alpha_k\|x^k - x^{k-1}\|\\ \leq&\|x^k - x\| + \tau_k \quad \forall x \in \mathcal{H}.\end{aligned} \tag{3.9}$$

For each $\bar{x} \in \Omega$, it follows from Step 2 and Lemma 2.3 that

$$\begin{aligned}\|t^k - \bar{x}\|^2 =&\|u^k_{i_0} - \bar{x}\|^2\\ =&\left\|(1 - \gamma_{k,i_0})w^k + \gamma_{k,i_0}S_{i_0}(w^k) - \bar{x}\right\|^2\end{aligned}$$

$$\leq \|w^k - \bar{x}\|^2 - \gamma_{k,i_0}(1 - \gamma_{k,i_0} - \beta_{i_0})\|S_{i_0}(w^k) - w^k\|^2. \tag{3.10}$$

Combining Step 3, (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned}
\|x^{k+1} - x^*\| = &\left\|(1-\zeta_k)t^k + \zeta_k y^k - x^*\right\| \\
\leq &(1-\zeta_k)\|t^k - x^*\| + \zeta_k\|y^k - x^*\| \\
\leq &(1-\zeta_k)\|t^k - x^*\| + \zeta_k(1-\tau\lambda_k)\|t^k - x^*\| \\
= &(1-\tau\zeta_k\lambda_k)\|t^k - x^*\| \\
\leq &(1-\tau\zeta_k\lambda_k)\|w^k - x^*\| \\
\leq &(1-\tau\zeta_k\lambda_k)(\|x^k - x^*\| + \tau_k) \\
\leq &(1-\tau\zeta_k\lambda_k)\|x^k - x^*\| + \tau_k \\
\leq &(1-\tau\bar{a}\zeta_k)\|x^k - x^*\| + \tau_k.
\end{aligned}$$

Applying Lemma 2.2 for $a_k := \|x^k - x^*\|^2, \alpha_k := \tau\bar{a}\zeta_k, b_k := \tau_k$, and using the condtion (3.1), we obtain the limit $\lim_{k\to\infty}\|x^k - x^*\|^2 = 0$. Which completes the proof. □

4. Numerical experiments

We start with some numerical examples in which we compare the algorithm (PIAPA) with two other: Parallel Projection Algorithm (PPA) introduced by Anh et al. [10, Scheme (3.1)] and Subgradient-type Algorithm (STA) suggested by Iiduka et al. [20, Algorithm 3.2] where $T := S_n S_{n-1} \ldots S_2 S_1$.

Example 4.1. We use the equilibrium bifunction $f : \mathcal{R}^m \times \mathcal{R}^m \to \mathcal{R}$ is first introduced in [34], later in [7, 9, 10], of the form

$$f(x,y) = \langle F(x) + Qy + q, y - x\rangle, \tag{4.1}$$

where A is an $m \times m$ matrix, B is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, $Q = AA^T + B + D$ and q is a vector in $\mathcal{R}^m$, $\xi > 1 + \|Q\|$. The mapping F is defined by

$$F(x) = (\xi x_1 + \xi x_2 + \sin(x_1), -\xi x_1 + \xi x_2 + \sin(x_2), (\xi-1)x_3, \ldots, (\xi-1)x_m)^\top.$$

By a similar way as the proof of [34, Lemma 6.1], Anh et al. [7] showed that

(1) If $\xi > 1 + \|Q\|$ then f is strongly monotone with constant $\beta = \xi - 1 - \|Q\|$;
(2) F is $L-$Lipschitz continuous, where $L = \sqrt{2(2\xi^2 + 2\xi + 1)}$. By [34, Lemma 6.2], f has Lipschitz-type constants c_1 and c_2 satisfying $2\sqrt{c_1c_2} \geq L + \|Q\|$.

Next, we consider the feasible set C and mappings S_1, S_2, S_3 given in [10] as follows:

$$C = \left\{x \in \mathcal{R}^m : 0 \leq x, e^\top x \leq g\right\}, e \in \mathcal{R}^m, g \in \mathcal{R},$$

$$S_1(x) = x \quad \forall x \in \mathcal{R}^m,$$

$$S_2(x) = Pr_G(x), G = C \cap \left\{x = (x_1, x_2, \ldots, x_m)^\top \in \mathcal{R}^m : x_i \leq 3 \quad \forall i = 1, 2, \ldots, m\right\},$$

$$S_3(x) = (\sin^2 x_1, 1 + x_2, x_3, \ldots, x_m)^\top,$$

where Pr_G is the metric projection onto G. Then, for each $i \in I = \{1,2,3\}$, the mapping $S_i : \mathcal{R}^m \to C$ is nonexpansive.

Test 1. First, let us consider the algorithm (PIAPA) in $\mathcal{R}^5$. The matrices A, B, D, the vectors q, e and real number g are randomly chosen as follows:

$$A = \begin{bmatrix} 0 & 1 & 0.5 & 2 & 1 \\ -1 & 1 & -0.5 & 0 & -2 \\ -0.5 & 0.5 & -0.8 & 5 & 1 \\ 3 & 4 & -5 & 4 & 7 \\ -6 & 0.5 & 8 & 2 & 9 \end{bmatrix}, B = \begin{bmatrix} 1.5 & -1 & 0.5 & 0 & 0 \\ 1 & 3 & -1.25 & -1 & 0 \\ -0.5 & 1.25 & 5 & 0 & -4 \\ 0 & -1 & 0 & 7 & 0 \\ 0 & 0 & 4 & 0 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, q = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 8 \\ 22 \end{bmatrix}, e = \begin{bmatrix} 3 \\ -5 \\ 10 \\ 3 \\ 7 \end{bmatrix}, g = 15.$$

It is easy to evaluate that

$$eig(Q) = \{197.5373,\ 135.0908,\ 30.3720,\ 7.4079,\ 3.9820\}, \|Q\| = 197.7064,$$

and hence

$$L := \max\{t : t \in eig(Q)\} = 197.5373 \text{ and } \beta := \min\{t : t \in eig(Q)\} = 3.9820.$$

Choosing $\xi = 250$ and $c_1 = 50$. From $2\sqrt{c_1 c_2} \geq L + \|Q\|$ and $\beta = \xi - 1 - \|Q\|$, it follows

$$\beta = 51.2936, c_2 \geq \frac{(L + \|Q\|)^2}{4c_1} = 781.0879.$$

The parameters satisfying (3.1) are set as follows:

$$\begin{cases} \tau = 0.001 \in (0, \beta - c_1) = (0, 1.2936), \bar{a} = 0.001, \hat{a} = 0.8668, \\ \lambda_k = 0.01 + \frac{1}{10k+9} \in (0, 0.4997), \\ \zeta_k = \frac{1}{5k+1} \in (0, \frac{1}{\tau \bar{a}}), \mu_k = \frac{k}{10k+1} > 0, \tau_k = \frac{1}{20k^2+7}, \bar{b} = 0.001, \hat{b} = 0.9882, \\ \gamma_{k,i} = 0.01 + \frac{1}{30k+100}, \end{cases}$$

where $a := \frac{\tau - 4(\beta - c_1)}{2\tau^2(\beta - c_1)}, b := \frac{\beta - c_1 - \tau}{\tau^2(\beta - c_1)}$ and $\frac{-a - \sqrt{a^2 - 4b}}{2} = 0.4997$. We take $x^0 = (1,2,0,0,1)^\top, x^1 = (1,2,1,3,0)^\top$ and the tolerance $\epsilon = 10^{-3}$. The numerical results are showed in Figure 1 and Table 1.

Test 2. Second, we compare the $(PIAPA)$ with two algorithm: The (PPA) and the (STA). The stopping criterion of the algorithms is $\|x^{k+1} - x^k\| \leq \epsilon$. Let $e = (3, -5, 10, 3, 7)^\top, g = 15$, all entries B, D, E and vector q be randomly generated by using the commands in Matlab $A = 3 * rand(5,5); B = skewdec(5,1); D = 3 * diag(1:5)$ giving $D = (e_{ij})_{5\times 5}$ where $e_{ij} = 0$ for all $i \neq j$ and $e_{ii} = 3i$ for all $i \in \{1, \ldots, 5\}$; $q = rand(5,1)$. The termination criterion is $\|x^{k+1} - x^k\| \leq \epsilon = 10^{-3}$. Data of the algorithms are given as follows:

(a) The algorithm (PIAPA): The starting points $x^0 = (1,2,0,0,1)^\top$, $x^1 = (1,2,1,3,0)^\top, \tau_k = \frac{1}{(10k+6)^2}$, $\mu_k = 0.1 + \frac{1}{20k+1}$, $\gamma_{k,i} = 0.0001 + \frac{2}{100k+9}$, $\lambda_k = 0.02 + \frac{1}{10k+21}$ and $\zeta_k = \frac{1}{15k+6}$.

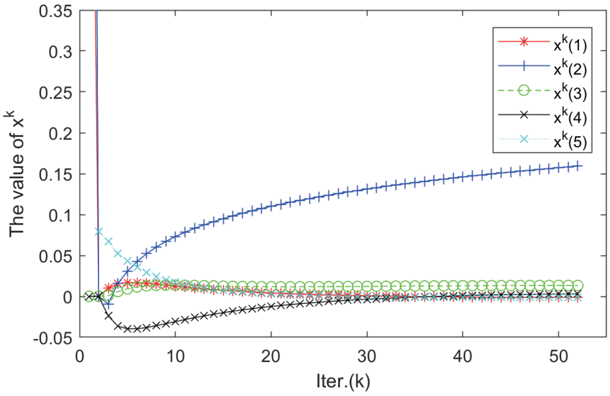

FIGURE 1. Performance of the seuquence $\{x^k\}$ in the algorithm (PI-APA) with the tolerence $\|x^{k+1} - x^k\| \leq \epsilon = 10^{-3}$. The approximate solution is $x^{51} = (0.0000, 0.1593, 0.0133, 0.0032, 0.0000)^\top$.

TABLE 1. Iterations (Iter.) and CPU times (Times) with randomly different parameters.

Test	τ_k	μ_k	γ_{ki}	λ_k	ζ_k	Iter.	Times
1	$\frac{1}{(20k+7)^2}$	1	$0.01 + \frac{1}{30k+100}$	$0.01 + \frac{1}{10k+9}$	$\frac{1}{5k+1}$	51	6.8750
2	$\frac{1}{(30k+9)^2}$	$1 + \frac{1}{2k+1}$	$0.001 + \frac{1}{100k+8}$	$0.02 + \frac{1}{5k+7}$	$\frac{1}{7k+5}$	246	8.1719
3	$\frac{1}{(20k+7)^2}$	$1 + \frac{1}{2k+1}$	$0.001 + \frac{1}{30k+100}$	$0.01 + \frac{1}{10k+9}$	$\frac{1}{5k+1}$	230	9.9531
4	$\frac{1}{(100k+7)^2}$	$1 + \frac{1}{2k+1}$	$0.001 + \frac{1}{30k+100}$	$0.01 + \frac{1}{10k+9}$	$\frac{1}{5k+1}$	132	5.6563
5	$\frac{1}{100k^2+7}$	$1 + \frac{1}{2k+1}$	$0.0001 + \frac{1}{100k+9}$	$0.01 + \frac{1}{10k+9}$	$\frac{1}{5k+1}$	19	0.6719
6	$\frac{1}{20k^2+7}$	$1 + \frac{1}{2k+1}$	$0.001 + \frac{1}{30k+100}$	$0.05 + \frac{1}{50k+7}$	$\frac{1}{5k+1}$	64	2.5938
7	$\frac{1}{20k^2+7}$	$1 + \frac{1}{2k+1}$	$0.001 + \frac{1}{30k+100}$	$0.05 + \frac{1}{50k+7}$	$\frac{1}{15k+6}$	203	6.0781
8	$\frac{1}{20k^2+7}$	$8 + \frac{1}{10k+1}$	$0.001 + \frac{1}{30k+100}$	$0.05 + \frac{1}{50k+7}$	$\frac{1}{15k+6}$	201	6.0156
9	$\frac{5}{100k^2+9}$	$8 + \frac{1}{10k+1}$	$0.001 + \frac{1}{30k+100}$	$0.02 + \frac{1}{10k+21}$	$\frac{1}{15k+6}$	399	11.7031
10	$\frac{1}{10k^2+6}$	$2 + \frac{1}{20k+1}$	$0.0005 + \frac{1}{30k+100}$	$0.02 + \frac{1}{10k+21}$	$\frac{1}{15k+6}$	236	7.0625

(a) (PPA): $\alpha_{k,i} := 0.001 + \frac{1}{k+100}$ for all $i \in I, \epsilon_k = 0, \tau_k = 0, \gamma_k = \frac{1}{7k+10}$, for all $k \in \mathcal{N}$, the starting point $x^0 = (1, 2, 0, 0, 1)^\top$.

(b) (STA): $\mu = 1.65\frac{\beta}{L^2} \in (0, \frac{2\beta}{L^2})$ where $\beta = \min\{m : m \in eig(Q)\}$ and $L = \max\{k : k \in eig(Q)\}$. Parameters $\lambda_k := \frac{1}{\sqrt{3k+5}}$ (k=1,2,...) satisfy the conditions

$$\lim_{k\to\infty} \lambda_k = 0, \sum_{k=1}^{\infty} \lambda_k = +\infty, \lim_{k\to\infty} \frac{\lambda_k - \lambda_{k+1}}{\lambda_{k+1}} = 0.$$

The starting point: $x^0 = (1, 2, 0, 0, 1)^\top$.

The numerical results are showed in Table 2.

TABLE 2. Comparison results.

Tests	No. Iter.			CPU times		
	(PPA)	(STA)	$(PIAPA)$	(PPA)	(STA)	$(PIAPA)$
1	153	6093	157	10.5601	297.0412	4.6406
2	253	1105	1158	22.5024	116.8923	32.5781
3	96	3205	1497	7.9941	105.9602	42.1406
4	184	9402	941	12.6981	326.8830	13.0551
5	1104	4057	138	210.4831	130.8024	7.0884
6	8302	3592	297	170.5241	90.8540	7.5015
7	130	2905	94	11.0938	143.9054	1.4431
8	342	6605	164	54.0951	373.7548	6.5201
9	361	403	43	80.5629	294.8840	0.9662
10	126	3055	702	10.0741	109.0413	15.7724

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