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SYLVESTER WEIGHTED POWER SUMS ASSOCIATED WITH APOSTOL-BERNOULLI POLYNOMIALS

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Abstract. In the classical and famous Diophantine problem of Frobenius, the central topic is the largest integer (Frobenius number) or the number (Sylvester number) of non-negative integers that cannot be represented by a linear combination. An extension of these notions is the sum of powers of non-negative integers that cannot be represented, given an explicit formula using the elements of the Apéry set and the Bernoulli numbers. In this paper, we give an explicit formula for a further generalized weighted sum of powers using the elements of the Apéry set and the Apostol-Bernoulli polynomials, a generalization of the Bernoulli numbers. Moreover, we generalize the set of unrepresentable non-negative integers to the set of non-negative integers that can be represented in at most p ways. That is, when $p = 0$, the generalization is reduced to the original problem. We also give an expression of the p-Volonoï type sums, included in Volonoï type congruences.

1. Introduction

Given the set of positive integers $A = \{a_1, a_2, \dots, a_k\}$ with $\gcd(a_1, a_2, \dots, a_k) = 1$. Denote by $d(N; a_1, a_2, \dots, a_k)$ the number of non-negative solutions (representations) of the linear equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = N$. The quantity $d(N; a_1, a_2, \dots, a_k)$ is actually the number of partitions of N whose summands are taken (repetitions allowed) from the sequence $a_1, a_2, \dots, a_k$. For a non-negative integer p, define

$$S_p(A) = \{N \in \mathbb{N}_0 | d(N; a_1, \dots, a_k) > p\}\,, \tag{1.1}$$

$$G_p(A) = \{N \in \mathbb{N}_0 | d(N; a_1, \dots, a_k) \le p\}\,, \tag{1.2}$$

satisfying $S_p(A) \cup G_p(A) = \mathbb{N}_0$, which is the set of non-negative integers. $S_p(A)$ is called *p-numerical semigroup* and $G_p(A)$ is the set of *p-gaps*. When $p = 0$, $S = S_0(A)$ is the classical numerical semigroup as $S = \langle A \rangle$, which is generated by A. Then, the *p-Hilbert series* is given by

$$H_p(A; x) = \sum_{N \in S_p(A)} x^N \tag{1.3}$$

and the function of p-gaps is given by

$$C_p(A; x) = \frac{1}{1-x} - H_p(A; x)\,. \tag{1.4}$$

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If $A=\{a,b\}$ with $\gcd(a,b)=1$, then

$$H_p(A;x)=\frac{x^{abp}(1-x^{ab})}{(1-x^a)(1-x^b)}=\frac{x^{abp}}{1-x}\Phi_{ab}(x)$$

([6, 18]) and

$$C_p(A;x)=\frac{1}{1-x}-\frac{x^{abp}(1-x^{ab})}{(1-x^a)(1-x^b)}\,,$$

where $\Phi_n(x)$ is the n-th cyclotomic polynomial determined by

$$\Phi_n(x)=\prod_{d|n}(x^d-1)^{\mu(n/d)}$$

with the Möbius function $\mu(n)$.

For a given set $A=\{a_1,a_2,\dots,a_k\}$ with $\gcd(a_1,a_2,\dots,a_k)=1$, the *p-Frobenius number* $g_p(A)$ and the *p-Sylvester power sum* $s_p^{(\mu)}(A)$ with a non-negative integer μ are defined by

$$g_p(A):=\max_{n\in G_p(A)} n\,, \tag{1.5}$$

$$s_p^{(\mu)}(A):=\sum_{n\in G_p(A)} n^\mu\,, \tag{1.6}$$

respectively ([12]). When $p=0$, $g_0(A)$ is the classical and famous *Frobenius number*, that is, the largest positive integer that cannot be expressed as a linear combination of non-negative integers in terms of $a_1,a_2,\dots,a_k$. And when $p=0$, $s_0^{(0)}$ and $s_0^{(1)}$ are called the *Sylvester number* (or *genus*) and the *Sylvester sum*, respectively, which have been studied by many researchers in various ways for a long time.

One of the central research topics in Frobenius problem is to find an explicit formula for each quantity. In 1850s, Sylvester found that for two variable sets $A=\{a,b\}$,

$$g_0(a,b)=(a-1)(b-1)-1 \quad \text{and} \quad s_0^{(0)}=\frac{(a-1)(b-1)}{2}\,.$$

Rödseth [20] found the formula of $s_0^{(\mu)}(a,b)$ by using Bernoulli numbers as an extension for $s_0^{(1)}(a,b)$ by Brown and Shiue [5]. For three or more variables, however, no explicit form has been found, but the Frobenius number cannot be given by any set of closed formulas, which can be reduced to a finite set of certain polynomials ([7]). Nevertheless, with the help of the elements of Apéry set, we can give an explicit formula of $s_0^{(\mu)}(A)$ ([9]) and $s_p^{(\mu)}(A)$ ([12]). In the special case of the set A consisting of triangular triplets [10], repunits [11] and Fibonacci triplets [13], we have successfully found the explicit forms even when $p>0$.

A further generalization of power sums is the so-called weighted power sum. For a real λ, the *p-Sylvester weighted power sum* $s_{p,\lambda}^{(\mu)}(A)$ is defined by

$$s_{p,\lambda}^{(\mu)}(A):=\sum_{n\in G_p(A)}\lambda^n n^\mu\,. \tag{1.7}$$

An explicit formula of $s_{0,\lambda}^{(1)}(a,b)$ is given in [14] when $\mu=1$ (see the identity (3.1) below). By using the Apostol-Bernoulli numbers, which is a generalization of the Bernoulli numbers, an explicit formula of $s_{0,\lambda}^{(1)}(a,b)$ is also given in [14]. For a more general set A, an explicit formula of $s_{p,1}^{(\mu)}(A)$ is given in [12] (see Proposition 3.4 below) by using Apostol-Bernoulli polynomials. More generally, an explicit formula of $s_{p,\lambda}^{(\mu)}(A)$ is given in [12] by using the elements of Apéry set and Eulerian numbers $\left\langle {n \atop l} \right\rangle$, counting the number of permutations of $1,2,\dots,n$ in which exactly l elements are greater than the previous element (see Proposition 3.1 below). In [15], the special case $p=0$ is discussed to find an explicit form of $s_{0,\lambda}^{(\mu)}(A)$. In this paper, we give an explicit form of $s_{p,\lambda}^{(\mu)}(A)$ by using Apostol-Bernoulli polynomials instead of Eulerian numbers (see Theorem 3.5 below).

2. Apéry set

Without loss of generality, set $a_1=\min(A)$. The *p-Apéry set* is given by

$$\operatorname{Ap}_p(A)=\{m_0^{(p)},m_1^{(p)},\dots,m_{a_1-1}^{(p)}\}\,, \tag{2.1}$$

where $m_j^{(p)}\equiv j \pmod{a_1}$, $m_j^{(p)}\in S_p(A)$ and $m_j^{(p)}-a_1\in G_p(A)$ $(0\le j\le a_1-1)$. Then, the p-Hilbert series can also be written as

$$\begin{aligned} H_p(A;x)&=\frac{1}{1-x^{a_1}}\sum_{N\in\operatorname{Ap}_p(A)}x^N\\ &=\frac{1}{1-x^{a_1}}\sum_{j=0}^{a_1-1}x^{m_j^{(p)}}\,. \end{aligned}$$

If $A=\{a,b\}$ with $\gcd(a,b)=1$ and $a<b$, then $\operatorname{Ap}_p(A)=\{abp+ib|0\le i\le a-1\}$. Note that the order of $m_j^{(p)}$ may be different from that of j. For example, if $A=\{5,7\}$, then $m_0^{(4)}=4\cdot5\cdot7$, $m_1^{(4)}=4\cdot5\cdot7+\underline{3}\cdot7$, $m_2^{(4)}=4\cdot5\cdot7+\underline{1}\cdot7$, $m_3^{(4)}=4\cdot5\cdot7+\underline{4}\cdot7$ and $m_4^{(4)}=4\cdot5\cdot7+\underline{2}\cdot7$. Therefore, by permutation $\pi(j)$, we get $m_{\pi(j)}^{(p)}=pab+jb$ $(0\le j\le a-1)$.

By using the elements in the p-Apéry set, we have

$$\begin{aligned} C_p(x)&=C_p(A;x)\\ &=\sum_{j=0}^{a_1-1}\left(x^{m_j^{(p)}-a_1}+x^{m_j^{(p)}-2a_1}+\cdots+x^{m_j^{(p)}-\lfloor m_j^{(p)}/a_1\rfloor a_1}\right)\\ &=\sum_{j=0}^{a_1-1}\frac{x^{m_j^{(p)}}\left(1-x^{-a_1\lfloor m_j^{(p)}/a_1\rfloor}\right)}{1-x^{-a_1}}\\ &=\sum_{j=0}^{a_1-1}\frac{x^j\left(x^{a_1\lfloor m_j^{(p)}/a_1\rfloor}-1\right)}{x^{a_1}-1}=\sum_{j=0}^{a_1-1}\frac{x^{m_j^{(p)}}-x^j}{x^{a_1}-1}\,. \end{aligned}$$

By the multisection formula (see [17, (2)], [19, §4.3], [21]), we have for $0 \le j \le a_1 - 1$

$$\frac{x^j\left(x^{a_1\left\lfloor m_j^{(p)}/a_1\right\rfloor}-1\right)}{x^{a_1}-1}=\frac{1}{a_1}\sum_{i=0}^{a_1-1}\zeta_{a_1}^{-ij}C_p(\zeta_{a_1}^i x)\,,$$

where $\zeta_{a_1}=\exp(2\pi\sqrt{-1}/a_1)$, the a_1-th root of unity. Then we obtain

$$x^{a_1\left\lfloor m_j^{(p)}/a_1\right\rfloor}=1+\frac{x^{a_1}-1}{a_1 x^j}\sum_{i=0}^{a_1-1}\zeta_{a_1}^{-ij}C_p(\zeta_{a_1}^i x)\,.$$

Taking the limit at $x\to 1$, we have

$$\left\lfloor\frac{m_j^{(p)}}{a_1}\right\rfloor=\frac{1}{a_1}\sum_{i=0}^{a_1-1}\zeta_{a_1}^{-ij}C_p(\zeta_{a_1}^i)\,.$$

In particular, if $A=\{a,b\}$ with $\gcd(a,b)=1$ and $a<b$, then we get

$$\begin{aligned}x^{a(bp+\lfloor jb/a\rfloor)}&=1+\frac{x^a-1}{ax^{\pi(j)}}\sum_{i=0}^{a-1}\zeta_a^{-i\pi(j)}C_p(\zeta_a^i x)\\&=1+\frac{x^a-1}{ax^{\pi(j)}}\sum_{i=0}^{a-1}\zeta_a^{-ijb}C_p(\zeta_a^i x)\qquad(2.2)\end{aligned}$$

and

$$\begin{aligned}bp+\left\lfloor\frac{jb}{a}\right\rfloor&=\frac{1}{a}\sum_{i=0}^{a-1}\zeta_a^{-i\pi(j)}C_p(\zeta_a^i)\\&=\frac{1}{a}\sum_{i=0}^{a-1}\zeta_a^{-ijb}C_p(\zeta_a^i)\,,\qquad(2.3)\end{aligned}$$

respectively, where $\pi(j)$ denotes the permutation, satisfying $\pi(j)\equiv jb \pmod a$. In other words, $\pi(j)$ is given by

$$\pi(j)=jb-a\left\lfloor\frac{jb}{a}\right\rfloor\,.$$

3. Apostol-Bernoulli polynomials

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$ are defined by the generating function ([3, p.165, (3.1)]):

$$\frac{te^{xt}}{\lambda e^t-1}=\sum_{n=0}^\infty\mathcal{B}_n(x,\lambda)\frac{t^n}{n!}\quad(|t+\log\lambda|<2\pi)\,.$$

When $x=0$, $\mathcal{B}_n(\lambda)=\mathcal{B}_n(0,\lambda)$ are Apostol-Bernoulli numbers. When $\lambda\to 1$, $B_n(x)=\lim_{\lambda\to 1}\mathcal{B}_n(x,\lambda)$ are Bernoulli polynomials[1] defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^\infty\mathcal{B}_n(x)\frac{t^n}{n!}\,.$$

[1]Note that $B_n(x)\ne\mathcal{B}_n(x,1)$.

Furthermore, when $x = 1$, $B_n = \lim_{\lambda\to 1}\mathcal{B}_n(1,\lambda)$ are Bernoulli numbers. For $\lambda \ne 1$, Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$ are expressed explicitly by

$$\mathcal{B}_n(x,\lambda) = \sum_{k=1}^n k\binom{n}{k}\sum_{j=0}^{k-1}(-\lambda)^j(\lambda-1)^{-j-1}j!\left\{ {k-1 \atop j} \right\} x^{n-k} \quad (n \ge 0)$$

([16, Remark 2.6]), where the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ are given by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^k (-1)^{k-j}\binom{k}{j} j^n\,.$$

The Sylvester weighted sum s_λ, defined by

$$s_\lambda(A) := \sum_{n\in G_0(A)} \lambda^n n\,,$$

can be given explicitly when $A = \{a,b\}$ with $\gcd(a,b) = 1$. For a real λ with $\lambda^a \ne 1$ and $\lambda^b \ne 1$,

$$\begin{aligned} &s_\lambda(a,b) \\ &= \frac{\lambda}{(\lambda-1)^2} + \frac{ab\lambda^{ab}}{(\lambda^a-1)(\lambda^b-1)} - \frac{(\lambda^{ab}-1)\big((a+b)\lambda^{a+b} - a\lambda^a - b\lambda^b\big)}{(\lambda^a-1)^2(\lambda^b-1)^2} \end{aligned} \tag{3.1}$$

([14, Theorem 1.1]). More generally, the p-Sylvester weighted power sum $s_{p,\lambda}^{(\mu)}$, defined by

$$s_{p,\lambda}^{(\mu)}(A) := \sum_{n\in G_p(A)} \lambda^n n^\mu\,,$$

can be given explicitly when $A = \{a,b\}$ with $\gcd(a,b) = 1$.

In order to obtain an explicit form of $s_{p,\lambda}^{(\mu)}(A)$, we need the formula in [12, Theorem 2]. The case $p = 0$ is discussed in [15].

Proposition 3.1. *Let k, p and μ be integers with $k \ge 2$, $p \ge 0$ and $\mu \ge 1$, and λ be a real with $\lambda^{a_1} \ne 1$. Then for $A = \{a_1, a_2, \dots, a_k\}$ with $\gcd(a_1, a_2, \dots, a_k) = 1$, we have*

$$\begin{aligned} &s_{p,\lambda}^{(\mu)}(A) \\ &= \sum_{n=0}^{\mu} \frac{(-a_1)^n}{(\lambda^{a_1}-1)^{n+1}}\binom{\mu}{n}\sum_{j=0}^n \left\langle {n \atop n-j} \right\rangle \lambda^{ja_1}\sum_{i=0}^{a_1-1}\big(m_i^{(p)}\big)^{\mu-n}\lambda^{m_i^{(p)}} \\ &\quad + \frac{(-1)^{\mu+1}}{(\lambda-1)^{\mu+1}}\sum_{j=0}^{\mu}\left\langle {\mu \atop \mu-j} \right\rangle \lambda^j\,, \end{aligned}$$

where $\left\langle {n \atop m} \right\rangle$ are Eulerian numbers, appearing in the generating function

$$\sum_{k=0}^\infty k^n x^k = \frac{1}{(1-x)^{n+1}}\sum_{m=0}^{n-1}\left\langle {n \atop m} \right\rangle x^{m+1} \quad (n \ge 1) \tag{3.2}$$

with $0^0 = 1$ and $\left\langle {0 \atop 0} \right\rangle = 1$.

When $A=\{a,b\}$ with $\gcd(a,b)=1$ and $a<b$, by applying Proposition 3.1 as $m_{\pi(i)}^{(p)}=abp+ib$, we obtain the following. Note that $\left\langle {1 \atop 1}\right\rangle=0$ and $\left\langle {1 \atop 0}\right\rangle=1$.

Proposition 3.2. *For integers p and μ with $p\ge 0$ and $\mu\ge 1$, and a real λ with $\lambda^a\ne 1$ and $\lambda^b\ne 1$, we have*

$$s_{p,\lambda}^{(\mu)}(a,b)=\sum_{n=0}^{\mu}\frac{(-a)^n}{(\lambda^a-1)^{n+1}}\binom{\mu}{n}\sum_{j=0}^{n}\left\langle {n \atop n-j}\right\rangle\lambda^{ja}\sum_{i=0}^{a-1}(abp+ib)^{\mu-n}\lambda^{abp+ib}$$
$$+\frac{(-1)^{\mu+1}}{(\lambda-1)^{\mu+1}}\sum_{j=0}^{\mu}\left\langle {\mu \atop \mu-j}\right\rangle\lambda^j\,.$$

In particular, for the simple p-Sylvester weighted sum as $\mu=1$, we have the following. When $p=0$, the identity (3.1) is reduced.

Proposition 3.3. *For a non-negative integer p and a real λ with $\lambda^a\ne 1$ and $\lambda^b\ne 1$, we have*

$$s_{p,\lambda(a,b)}:=\sum_{n\in G_p(A)}\lambda^n n$$
$$=\frac{\lambda}{(\lambda-1)^2}+\frac{ab\big(p(\lambda^{ab}-1)+\lambda^{ab}\big)}{(\lambda^a-1)(\lambda^b-1)}-\frac{(\lambda^{ab}-1)\big((a+b)\lambda^{a+b}-a\lambda^a-b\lambda^b\big)}{(\lambda^a-1)^2(\lambda^b-1)^2}\,.$$

On the other hand, the simple p-Sylvester sum (the weight λ is equal to 1) can be explicitly given, as in [12, Theorem 1] (see also [15]).

Proposition 3.4. *Let k, p and μ be integers with $k\ge 2$, $p\ge 0$ and $\mu\ge 1$. Then for $A=\{a_1,a_2,\dots,a_k\}$ with $\gcd(a_1,a_2,\dots,a_k)=1$, we have*

$$s_p^{(\mu)}(A)=\frac{1}{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu+1}{\kappa}B_\kappa a_1^{\kappa-1}\sum_{i=0}^{a_1-1}\big(m_i^{(p)}\big)^{\mu+1-\kappa}+\frac{B_{\mu+1}}{\mu+1}(a_1^{\mu+1}-1)\,,$$

where B_n are Bernoulli numbers.

The finite part of the infinite sum

$$M_N^{(n)}(x):=\sum_{k=0}^{N}k^n x^k\,,$$

appearing in the generating function (3.2) about Eulerian numbers, is called the *Mirimanoff polynomial* and discussed in [22]. Namely, we have

$$\lim_{N\to\infty}M_N^{(n)}(x)=\frac{1}{(1-z)^{n+1}}\sum_{m=0}^{n-1}\left\langle {n \atop m}\right\rangle x^{m+1}\quad(n\ge 1)\,.$$

As Carlitz pointed out ([4]), by using Apostol-Bernoulli polynomials, the Mirimanoff polynomial can be expressed as

$$M_N^{(n)}(x)=\frac{z^N\mathcal{B}_{n+1}(N,x)-\mathcal{B}_{n+1}(0,x)}{n+1}\,.$$

Using the Mirimanoff polynomials instead of Eulerian numbers, the p-Sylvester weighted power sum can be given in a different form from Proposition 3.1 ([12, Theorem 2], [15]).

Theorem 3.5. *Let k, p and μ be integers with $k \ge 2$, $p \ge 0$ and $\mu \ge 1$, and λ be a real with $\lambda^{a_1} \ne 1$. Then for $A = \{a_1, a_2, \dots, a_k\}$ with $\gcd(a_1, a_2, \dots, a_k) = 1$, we have*

$$
\begin{aligned}
s_{p,\lambda}^{(\mu)}(A) &= \sum_{i=0}^{a_1-1} \lambda^{m_i^{(p)}} \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \bigl(m_i^{(p)}\bigr)^{\mu-\kappa} (-a_1)^\kappa M_{\ell_i}^{(\kappa)}(\lambda^{-a_1}) \\
&= \sum_{i=0}^{a_1-1} \lambda^{m_i^{(p)}} \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \frac{\bigl(m_i^{(p)}\bigr)^{\mu-\kappa}(-a_1)^\kappa}{\kappa+1} \\
&\quad \times \bigl(\lambda^{-a_i \ell_i} \mathcal{B}_{\kappa+1}(\ell_i, \lambda^{-a_i}) - \mathcal{B}_{\kappa+1}(0, \lambda^{-a_i})\bigr) ,
\end{aligned}
$$

where

$$
\ell_i = \frac{m_i^{(p)} - i}{a_1} = \left\lfloor \frac{m_i^{(p)}}{a_1} \right\rfloor \quad (0 \le i \le a_1 - 1) .
$$

Proof. We have

$$
\begin{aligned}
s_{p,\lambda}^{(\mu)}(A) &= \sum_{n \in G_p(A)} \lambda^n n^\mu \\
&= \sum_{i=0}^{a_1-1} \sum_{j=1}^{\ell_i} \lambda^{m_i^{(p)} - j a_1} (m_i^{(p)} - j a_1)^\mu \\
&= \sum_{i=0}^{a_1-1} \lambda^{m_i^{(p)}} \sum_{j=1}^{\ell_i} (\lambda^{-a_1})^j \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \bigl(m_i^{(p)}\bigr)^{\mu-\kappa} (-j a_1)^\kappa \\
&= \sum_{i=0}^{a_1-1} \lambda^{m_i^{(p)}} \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \bigl(m_i^{(p)}\bigr)^{\mu-\kappa} (-a_1)^\kappa M_{\ell_i}^{(\kappa)}(\lambda^{-a_1}) .
\end{aligned}
$$

□

Remark 3.6. Apostol-Bernoulli polynomials are also related to Hurwitz-Lerch Zeta functions, defined by the infinite series

$$
\Phi(z, \alpha, s) = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha + k)^s} ,
$$

where $\alpha, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $|z| \le 1$, $z \ne 0$, and $\Re(s) > 1$ ($s \in \mathbb{C}$). When $|z| = 1$, for a non-negative integer n,

$$
\Phi(z, \alpha, -n) = -\frac{\mathcal{B}_{n+1}(\alpha, z)}{n+1} .
$$

Namely, when $\lambda = e^{2\pi\sqrt{-1}z}$, since

$$
\phi(z, \alpha, -n) = -\frac{\mathcal{B}_{n+1}(\alpha, e^{2\pi\sqrt{-1}z})}{n+1} ,
$$

the p-Sylvester weighted sum can be expressed by using the Lipschitz-Lerch Zeta function ([3, p.161]), defined by

$$\phi(z,\alpha,s)=\sum_{k=0}^{\infty}\frac{e^{2k\pi\sqrt{-1}z}}{(\alpha+k)^s}\,.$$

Corollary 3.7. *Let k, p and μ be integers with $k\ge 2$, $p\ge 0$ and $\mu\ge 1$, and $|z|\le 1$ with $a_1 z\notin\mathbb{Z}$. Then we have*

$$\begin{aligned}s^{(\mu)}_{p,e^{2\pi\sqrt{-1}z}}(A)=\sum_{i=0}^{a_1-1}e^{2\pi\sqrt{-1}m_i^{(p)}z}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\bigl(m_i^{(p)}\bigr)^{\mu-\kappa}(-a_1)^{\kappa}\\ \times\bigl(-\phi(-a_1 z,\ell_i,-\kappa)+\phi(-a_1 z,0,-\kappa)\bigr)\,.\end{aligned}$$

When $A=\{a,b\}$ in Theorem 3.5, we have a different expression of Proposition 3.2.

Corollary 3.8. *Let p and μ be integers with $p\ge 0$ and $\mu\ge 1$, and λ be a real with $\lambda^{a_1}\ne 1$. Then we have*

$$\begin{aligned}s^{(\mu)}_{p,\lambda}(a,b)&=\sum_{j=0}^{a-1}\lambda^{pab+jb}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(pab+jb)^{\mu-\kappa}(-a)^{\kappa}M^{(\kappa)}_{pb+\lfloor jb/a\rfloor}(\lambda^{-a})\\ &=\sum_{j=0}^{a-1}\lambda^{pab+jb}\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(pab+jb)^{\mu-\kappa}(-a)^{\kappa}}{\kappa+1}\\ &\quad\times\bigl(\lambda^{-pab-a\lfloor jb/a\rfloor}\mathcal{B}_{\kappa+1}(pb+\lfloor jb/a\rfloor,\lambda^{-a})-\mathcal{B}_{\kappa+1}(0,\lambda^{-a})\bigr)\,,\end{aligned}$$

Proof. Since

$$\begin{aligned}\ell_{\pi(j)}&=\frac{m^{(p)}_{\pi(j)}-\pi(j)}{a}\\ &=\frac{1}{a}\left(pab+jb-jb+a\left\lfloor\frac{jb}{a}\right\rfloor\right)\\ &=pb+\left\lfloor\frac{jb}{a}\right\rfloor\,,\end{aligned}$$

by rearranging the order of the summation in Theorem 3.5, we get the desired result. □

4. Voronoï type sums

In [1] Agoh studied Voronoï type congruence, including the sum of the type

$$V_{m,n}(a,b):=\sum_{k=1}^{a-1}k^m\left(\left\lfloor\frac{kb}{a}\right\rfloor\right)^n\,.$$

We study a more general Voronoï type sum:

$$\begin{aligned} V_{m,n}(a,b,p) :&= \sum_{k=1}^{a-1} k^m \left(bp + \left\lfloor \frac{kb}{a} \right\rfloor \right)^n \\ &= \sum_{\nu=1}^{n} \binom{n}{\nu} (bp)^{n-\nu} V_{m,\nu}(a,b)\,. \end{aligned}$$

This sum can be expressed as a function of p-gaps and Mirimanoff polynomials.

Theorem 4.1. *We have*

$$V_{m,n}(a,b,p) = \frac{1}{a^n} \sum_{\substack{l_0+\cdots+l_{a-1}=n \\ l_0,\ldots,l_{a-1}\ge 0}} \binom{n}{l_0,\ldots,l_{a-1}} \prod_{i=0}^{a-1} C_p^{l_i}(\zeta_a^i) M_{a-1}^{(m)}\left(\zeta_a^{-b\sum_{i=0}^{a-1} il_i}\right),$$

where $\binom{n}{l_0,\ldots,l_{a-1}} = \frac{n!}{l_0!\ldots l_{a-1}!}$ *is the multinomial coefficient. In particular, we have*

$$V_{m,1}(a,b,p) = \frac{1}{a} \sum_{i=0}^{a-1} C_p(\zeta_a^i) M_{a-1}^{(m)}\left(\zeta_a^{-bi}\right),$$

$$V_{1,1}(a,b,p) = \sum_{i=1}^{a-1} \frac{C_p(\zeta_a^i)}{\zeta_a^{-bi}-1} + \frac{a-1}{2}\left(abp + \frac{(a-1)(b-1)}{2} \right).$$

Proof. We have

$$\begin{aligned} V_{m,n}(a,b,p) &= \frac{1}{a^n} \sum_{k=1}^{a-1} k^m \sum_{\substack{l_0+\cdots+l_{a-1}=n \\ l_0,\ldots,l_{a-1}\ge 0}} \binom{n}{l_0,\ldots,l_{a-1}} \zeta_a^{-bk\sum_{i=0}^{a-1} il_i} \prod_{i=0}^{a-1} C_p^{l_i}(\zeta_a^i) \\ &= \frac{1}{a^n} \sum_{\substack{l_0+\cdots+l_{a-1}=n \\ l_0,\ldots,l_{a-1}\ge 0}} \binom{n}{l_0,\ldots,l_{a-1}} \prod_{i=0}^{a-1} C_p^{l_i}(\zeta_a^i) \sum_{k=1}^{a-1} k^m \zeta_a^{-bk\sum_{i=0}^{a-1} il_i} \\ &= \frac{1}{a^n} \sum_{\substack{l_0+\cdots+l_{a-1}=n \\ l_0,\ldots,l_{a-1}\ge 0}} \binom{n}{l_0,\ldots,l_{a-1}} \prod_{i=0}^{a-1} C_p^{l_i}(\zeta_a^i) M_{a-1}^{(m)}\left(\zeta_a^{-b\sum_{i=0}^{a-1} il_i}\right). \end{aligned}$$

In particular, when $n = 1$, we have

$$V_{m,1}(a,b,p) = \frac{1}{a} \sum_{i=0}^{a-1} C_p(\zeta_a^i) M_{a-1}^{(m)}\left(\zeta_a^{-bi}\right).$$

Furthermore, when $m = n = 1$, by

$$C_p(1) = abp + \frac{(a-1)(b-1)}{2}$$

and

$$\frac{M_{a-1}^{(1)}(\zeta_a^{-bi})}{a} = \left. \frac{ax^a(x-1) - x(x^a-1)}{a(x-1)^2} \right|_{x\to\zeta_a^{-bi}}$$

$$= \frac{1}{\zeta_a^{-bi} - 1} \quad (0 < i < a)$$

with

$$\frac{M_{a-1}^{(1)}(\zeta_a^0)}{a} = \frac{a-1}{2},$$

we have

$$\begin{aligned} V_{1,1}(a,b,p) &= \frac{1}{a}\sum_{i=0}^{a-1} C_p(\zeta_a^i) M_{a-1}^{(1)}\left(\zeta_a^{-bi}\right) \\ &= \sum_{i=1}^{a-1} \frac{C_p(\zeta_a^i)}{\zeta_a^{-bi} - 1} + \frac{a-1}{2}\left(abp + \frac{(a-1)(b-1)}{2}\right). \end{aligned}$$

□

Proposition 4.2. *We have*

$$abp + \sum_{k=0}^{a-1} \left\lfloor \frac{bk}{a} \right\rfloor x^k = \frac{x^a - 1}{a} \sum_{i=0}^{a-1} \frac{C_p(\zeta_a^i)}{\zeta_a^{-ib} x - 1}.$$

Proof. By (2.3), we have

$$\begin{aligned} \sum_{k=0}^{a-1} \left(bp + \left\lfloor \frac{bk}{a} \right\rfloor\right) x^k &= \frac{1}{a}\sum_{k=0}^{a-1} x^k \sum_{i=0}^{a-1} \zeta_a^{-ibk} C_p(\zeta_a^i) \\ &= \frac{1}{a}\sum_{i=0}^{a-1} C_p(\zeta_a^i) \sum_{k=0}^{a-1} (\zeta_a^{-ib} x)^k \\ &= \frac{x^a - 1}{a} \sum_{i=0}^{a-1} \frac{C_p(\zeta_a^i)}{\zeta_a^{-ib} x - 1}. \end{aligned}$$

□

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