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# REMARKS ON OHLIN TYPE THEOREM FOR SET-VALUED MAPS

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*Dedicated to the memory of Professor Kazimierz Goebel and Professor W. Art Kirk*

Abstract. A version of the Ohlin theorem for set-valued maps convex with a modulus is presented. As an application a counterpart of the Hermite-Hadamard double inequality is obtained. Converse Ohlin's theorem for set-valued maps convex with a modulus as well as a probabilistic characterization of such maps are presented.

## 1. Introduction

Over fifty years ago J. Ohlin [10] proved the following result:

**Lemma 1.1** ([10]). *Let $X_1$, $X_2$ be two real valued random variables with the same expectations $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If there exists a number $t_0 \in \mathbb{R}$ such that the distribution functions $F_X, F_Y$ satisfy*

$$F_{X_1}(t) \le F_{X_2}(t) \quad for\ t < t_0 \quad and \quad F_{X_1}(t) \ge F_{X_2}(t) \quad for\ t > t_0, \tag{1.1}$$

*then*

$$\mathbb{E}[f(X_1)] \le \mathbb{E}[f(X_2)],$$

*for every convex function $f : \mathbb{R} \to \mathbb{R}$.*

This Ohlin's theorem plays a fundamental role in the theory of optimal reinsurance structures. It gives sufficient conditions under which the random variable $X$ is dominated by $Y$ in the convex stochastic ordering sense. It is also a very useful tool in the theory of functional inequalities. In [12], T. Rajba used the Ohlin theorem to get a very simple proof of the Hermite-Hadamard inequalities, as well as to obtain some new inequalities related to convex functions (see also [7, 11, 13, 15] for other results of this type).

A version of Ohlin's theorem for strongly convex functions was proved in [7] and converse Ohlin's theorems for convex and strongly convex functions were obtained in [1]. Counterparts of Ohlin's theorem for set-valued maps as well as their applications were investigated in [8] and [4]. The aim of this note is to present some versions of Ohlin's theorem and converse Ohlin's theorem for set-valued maps convex with a modulus $c \in \mathbb{R}$. As an application we obtain a counterpart of the Hermite-Hadamard double inequality for set-valued maps convex with a modulus and give a probabilistic characterization of such maps.

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## 2. Main results

Let $(Y, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}$ be an interval. Denote by $B$ be the closed unit ball in $Y$ and by $n(Y)$ the family of all nonempty subsets of $Y$. Assume that $c \in \mathbb{R}$. We say that a set-valued map $G : I \to n(Y)$ is *convex with modulus* $c$ if

$$tG(x_1) + (1-t)G(x_2) + ct(1-t)(x_1 - x_2)^2 B \subset G(tx_1 + (1-t)x_2) \tag{2.1}$$

for all $x_1, x_2 \in I$ and $t \in [0, 1]$. Clearly, the usual notion of convex set-valued maps corresponds to relation (2.1) with $c = 0$. If $c > 0$, then condition (2.1) defines strongly convex set-valued maps with modulus $c$ introduced by Huang [3] (see also [5]). Since $B = -B$, condition (2.1) for $c$ and $-c$ is the same. So, for $c < 0$ we get also the class of strongly convex set-valued maps (with modulus $\mid c \mid$). It is worth noting that this situation (i.e. for set-valued maps), is slightly different from that for single-valued ones. A function $f : I \to \mathbb{R}$ is called convex with modulus $c$ (cf. Gilányi at al. [2]) if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 \tag{2.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. Obviously, for $c = 0$ this is the definition of convex functions. If $f$ satisfies (2.2) with $c > 0$, it is strongly convex with modulus $c$, while, in the case when $c < 0$, we obtain a kind of approximate convexity (semi-convexity).

In what follows $(Y, \|\cdot\|)$ is a separable Banach space and $B$ is the closed unit ball in $Y$. We denote by $cl(Y)$ the family of all closed nonempty subsets of $Y$. Assume that $(\Omega, \mathcal{A}, P)$ is a probability space with a nonatomic measure $P$ and $I \subset \mathbb{R}$ is an open interval. We denote by $\mathbb{E}[X]$ and $\mathbb{D}^2[X]$ the expectation and variance of a random variable $X$, respectively. For a given set-valued map $G : \Omega \to n(Y)$ the integral $\int_\Omega G(\omega)dP$ is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable (in the sense of Bochner) selections of the map $G$. A set-valued map $G : \Omega \to n(Y)$ is called *integrable bounded* if there exists a nonnegative integrable function $k : \Omega \to \mathbb{R}$ such that $G(\omega) \subset k(\omega)B$, for all $\omega \in \Omega$.

Recently counterparts of Ohlin's theorem for convex and strongly convex set-valued maps were proved in [8] and [4], respectively. We can reformulate both these results as:

**Theorem 2.1.** *Let $c \in \mathbb{R}$. Assume that $X_1, X_2 : \Omega \to I$ are square integrable random variables such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If for some $t_0 \in \mathbb{R}$ condition* (1.1) *holds, then*

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP. \tag{2.3}$$

*for every integrable bounded set-valued map $G : I \to cl(Y)$ convex with modulus $c$.*

As an example of possible application of the above Ohlin type theorem we present a counterpart of the classical Hermite-Hadamard double inequality for set-valued maps convex with a modulus $c \in \mathbb{R}$. It is an extension of the result obtained in [8] for $c = 0$ (cf. also [6]). Other examples of this type one can find in [8] and [4].

**Theorem 2.2.** *Let $G : I \to cl(Y)$ be an integrable bounded set-valued map convex with a modulus $c \in \mathbb{R}$. Assume that $[a, b] \subset I$ and $\mu$ is a Borel measure on $[a, b]$ with $\mu([a, b]) > 0$. Denote by $s_\mu = \frac{1}{\mu([a,b])} \int_a^b x\, d\mu(x)$ the barycenter of $\mu$ on $[a, b]$. Then*

$$\frac{1}{\mu([a,b])} \int_a^b G(x)\, d\mu(x) + c\Big(\frac{1}{\mu([a,b])} \int_a^b x^2\, d\mu(x) - s_\mu^2\Big)B \subset G(s_\mu) \tag{2.4}$$

*and*

$$\begin{aligned}\frac{b - s_\mu}{b - a} G(a) + \frac{s_\mu - a}{b - a} G(b) \\ + c\Big((b - s_\mu)(s_\mu - a) + s_\mu^2 - \frac{1}{\mu([a,b])} \int_a^b x^2\, d\mu(x)\Big)B \\ \subset \frac{1}{\mu([a,b])} \int_a^b G(x)\, d\mu(x).\end{aligned} \tag{2.5}$$

*Proof.* By the mean value theorem $s_\mu \in [a, b]$. Assume that $(\Omega, \mathcal{A}, P)$ is a probability space with a nonatomic measure $P$ and take random variables $X_1, X_2, X_3 : \Omega \to [a, b]$ with the distributions

$$\mu_{X_1} = \delta_{x_\mu}, \quad \mu_{X_2} = \frac{b - s_\mu}{b - a}\delta_a + \frac{s_\mu - a}{b - a}\delta_b, \quad \mu_{X_3} = \frac{1}{\mu([a,b])}\, \mu.$$

Then the distribution functions $F_{X_1}, F_{X_3}$ and $F_{X_3}, F_{X_2}$ satisfy condition (1.1) as well as $\mathbb{E}[X_1] = s_\mu = \mathbb{E}[X_3]$ and $\mathbb{E}[X_3] = s_\mu = \mathbb{E}[X_2]$ . We have also

$$\int_\Omega G\big(X_1(\omega)\big) dP = G(s_\mu), \quad \int_\Omega G\big(X_2(\omega)\big) dP = \frac{b - s_\mu}{b - a} G(a) + \frac{s_\mu - a}{b - a} G(b)$$

and

$$\int_\Omega G\big(X_3(\omega)\big) dP = \frac{1}{\mu([a,b])} \int_a^b G(x)\, d\mu(x).$$

Moreover,

$$\mathbb{D}^2[X_1] = 0, \quad \mathbb{D}^2[X_2] = \frac{b - s_\mu}{b - a} a^2 + \frac{s_\mu - a}{b - a} b^2 - s_\mu^2 = (b - s_\mu)(s_\mu - a)$$

and

$$\mathbb{D}^2[X_3] = \frac{1}{\mu([a,b])} \int_a^b x^2\, d\mu(x) - s_\mu^2.$$

Hence, applying Theorem 2.1 for $X_1, X_3$ and for $X_3, X_2$, we obtain (2.4) and (2.5).

□

If $\mu$ in Theorem 2.2 is the Lebesgue measure on $[a, b]$, then $\mu([a, b]) = b - a$ and $s_\mu = \frac{a+b}{2}$. Moreover, for the random variables $X_1, X_2, X_3$ appearing in the proof we have

$$\mathbb{D}^2[X_1] = 0, \quad \mathbb{D}^2[X_2] = \left(b - \frac{a + b}{2}\right)\left(\frac{a + b}{2} - a\right) = \frac{1}{4}(b - a)^2$$

and

$$\mathbb{D}^2[X_3] = \frac{1}{b-a}\int_a^b x^2\,dx - \Big(\frac{a+b}{2}\Big)^2 = \frac{1}{12}(b-a)^2.$$

Therefore, as a consequence of Theorem 2.2 we get the following Hermite-Hadamard type inclusions obtained for strongly convex set-valued maps in [9] (cf. also [4]).

**Corollary 2.3.** *If a set-valued map $G : I \to cl(Y)$ is convex with modulus $c \in \mathbb{R}$ and integrable bounded, then*

$$\frac{1}{b-a}\int_a^b G(x)dx + \frac{c}{12}(a-b)^2 B \subset G\left(\frac{a+b}{2}\right)$$

*and*

$$\frac{G(a)+G(b)}{2} + \frac{c}{6}(a-b)^2 B \subset \frac{1}{b-a}\int_a^b G(x)dx$$

*for any $a, b \in I$, $a < b$. In particular, if $c = 0$, then*

$$\frac{G(a)+G(b)}{2} \subset \frac{1}{b-a}\int_a^b G(x)dx \subset G\left(\frac{a+b}{2}\right)$$

*for any $a, b \in I$, $a < b$.*

Now we will show that the converse Ohlin's theorem for set-valued maps also holds.

**Theorem 2.4.** *Let $G : I \to cl(Y)$ be a given set-valued map and $c \in \mathbb{R}$. If for any discrete random variables $X_1, X_2 : \Omega \to I$ satisfying $\mathbb{E}[X_1] = \mathbb{E}[X_2]$ and condition (1.1) with some $t_0 \in \mathbb{R}$, we have*

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP. \tag{2.6}$$

*then $G$ is convex with modulus $c$.*

*Proof.* Let $G : I \to cl(Y)$ be a given set-valued map. Fix any $x_1, x_2 \in I$ and $t \in (0,1)$. Consider two random variables $X_1, X_2 : \Omega \to I$ with the distributions $\mu_{X_1} = \delta_{tx_1+(1-t)x_2}$ and $\mu_{X_2} = t\delta_{x_1} + (1-t)\delta x_2$, respectively. Then

$$\mathbb{E}[X_1] = tx_1 + (1-t)x_2 = \mathbb{E}[X_2]$$

and, moreover, the distribution functions $F_{X_1}, F_{X_2}$ satisfy condition (1.1) with $t_0 = \mathbb{E}[X_1]$. Therefore $G$ satisfies (2.6). Note that

$$\mathbb{D}^2[X_1] = \mathbb{E}[(X_1)^2] - \big(\mathbb{E}[X_1]\big)^2 = 0$$

and

$$\begin{aligned}\mathbb{D}^2[X_2] &= \mathbb{E}[(X_2)^2] - \big(\mathbb{E}[X_2]\big)^2 = tx_1^2 + (1-t)x_2^2 - \big(tx_1 + (1-t)x_2\big)^2 \\ &= t(1-t)(x_1-x_2)^2.\end{aligned}$$

Moreover,

$$\int_\Omega G\big(X_1(\omega)\big)dP = G\big(tx_1 + (1-t)x_2\big)$$

and

$$\int_\Omega G\big(X_2(\omega)\big)dP = tG(x_1) + (1-t)G(x_2).$$

Therefore, by (2.6), we obtain

$$tG(x_1) + (1-t)G(x_2) + ct(1-t)(x_1 - x_2)^2 B \subset G(tx_1 + (1-t)x_2),$$

which proves that $G$ is convex with modulus $c$. $\square$

**Remark 2.5.** In fact, in Theorem 2.4 it is enough to assume that condition (2.6) holds for any random variables $X_1, X_2$ with distributions concentrated at one or two points, satisfying $\mathbb{E}[X_1] = \mathbb{E}[X_2]$ and condition (1.1).

It is known that a function $f : I \to \mathbb{R}$ is convex if and only if for every random variable $X$ taking values in $I$ the following Jensen's inequality holds:

$$f\big(\mathbb{E}[X]\big) \le \mathbb{E}[f(X)].$$

In [14] Rajba and Wąsowicz extended this classical result to strongly convex functions proving that $f : I \to \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if for every random variable $X$ taking values in $I$

$$f\big(\mathbb{E}[X]\big) \le \mathbb{E}[f(X)] - c\mathbb{D}^2[X].$$

As a consequence of Theorems 2.2, we obtain the following counterpart of these results for set-valued maps convex with modulus $c$. For $c = 0$ analogous characterization was given in [8].

**Corollary 2.6.** *An integrable bounded set-valued map $G : I \to cl(Y)$ is convex with modulus $c \in \mathbb{R}$ if and only if*

$$\int_\Omega G\big(X(\omega)\big)dP + c\mathbb{D}^2[X]B \subset G\Big(\int_\Omega X(\omega)dP\Big), \tag{2.7}$$

*for every square integrable random variable $X : \Omega \to I$.*

*Proof.* Assume first that $G : I \to cl(Y)$ is convex with modulus $c$ and fix a square integrable random variable $X : \Omega \to I$. Take a random variable $X_1 : \Omega \to I$ with the distribution $\mu_{X_1} = \delta_{\mathbb{E}[X]}$. Then $\mathbb{E}[X_1] = \mathbb{E}[X]$ and the distribution functions $F_{X_1}$ and $F_X$ satisfy condition (1.1). Therefore, by Theorem 2.2, we obtain

$$\int_\Omega G\big(X(\omega)\big)dP + c\big(\mathbb{D}^2[X] - \mathbb{D}^2[X_1]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP. \tag{2.8}$$

Since $\mathbb{D}^2[X_1] = 0$ and

$$\int_\Omega G\big(X_1(\omega)\big)dP = G\big(\mathbb{E}[X]\big) = G\Big(\int_\Omega X(\omega)dP\Big),$$

by (2.8) we obtain (2.7).

Conversely, assume now that $G$ satisfies condition (2.7). Fix any $x_1, x_2 \in I$ and $t \in (0,1)$. Consider a random variable $X : \Omega \to I$ with the distributions $\mu_X = t\delta_{x_1} + (1-t)\delta x_2$. Then $\int_\Omega X(\omega)dP = tx_1 + (1-t)x_2$ and $\int_\Omega G\big(X(\omega)\big)dP =$

$tG(x_1) + (1-t)G(x_2)$. Moreover, $\mathbb{D}^2[X] = t(1-t)(x_1 - x_2)^2$. Therefore, by (2.7), we obtain

$$tG(x_1) + (1-t)G(x_2) + ct(1-t)(x_1 - x_2)^2 B \subset G(tx_1 + (1-t)x_2),$$

which shows that $G$ is convex with modulus $c$.

□

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