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SOME FAMILIES OF HYBRID-TYPE FRACTIONAL-ORDER KINETIC EQUATIONS BASED UPON THE HILFER-TYPE AND OTHER RELATED OPERATORS OF FRACTIONAL DERIVATIVES

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Abstract. This article is motivated essentially by the fact that, in the current literature, many different kinds of operators of fractional calculus (that is, fractional-order integrals and fractional-order derivatives) have been and continue to be successfully applied in the modeling and analysis of a considerably large number of applied scientific and real-world problems in the mathematical, physical, biological, engineering and statistical sciences, and indeed also in other scientific disciplines as well. We aim here at investigating a general family of hybrid-type fractional-order kinetic equations, which is associated with the Hilfer-type fractional derivative. The main results, which we have presented herein, are sufficiently general in character and are shown to be capable of furnishing solutions of a remarkably large number relatively of simpler fractional-order kinetic equations.

1. Introduction and motivation

The recent as well as the current literature has witnessed the fact that the subject of fractional calculus, as a calculus of integrals and derivatives of any real or complex order, has gained considerable popularity and importance, which is due mainly to its successfully-demonstrated applications in the modeling and analysis of applied mathematical problems and real-world situations occurring in many seemingly diverse and widespread fields of science and engineering. It does indeed also provide several potentially useful tools and techniques for solving differential and integral equations, and various other problems involving special functions of mathematical physics and applied mathematics as well as their extensions and generalizations in one and more variables.

The most commonly-used fractional-order integrals and fractional-order derivatives are defined by the right-sided Riemann-Liouville (RL) fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a+}$ and the left-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a-}$, and the corresponding Riemann-Liouville fractional derivative operators ${}^{\mathrm{RL}}D^{\mu}_{a+}$ and

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${}^{\mathrm{RL}}D^{\mu}_{a-}$, as follows (see, for example, [4, Chapter 13], [14, pp. 69–70] and [21]):

$$\left({}^{\mathrm{RL}}I^{\mu}_{a+}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\, f(t)\,\mathrm{d}t \qquad \big(x>a;\ \Re(\mu)>0\big), \tag{1.1}$$

$$\left({}^{\mathrm{RL}}I^{\mu}_{a-}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_x^a (t-x)^{\mu-1}\, f(t)\,\mathrm{d}t \qquad \big(x<a;\ \Re(\mu)>0\big) \tag{1.2}$$

and

$$\left({}^{\mathrm{RL}}D^{\mu}_{a\pm}f\right)(x) = \left(\pm\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I^{n-\mu}_{a\pm}f\right)(x) \qquad \big(\Re(\mu)\geqq 0;\ n=[\Re(\mu)]+1\big), \tag{1.3}$$

where the function f is locally integrable, $\Re(\mu)$ abbreviates the real part of the complex number $\mu\in\mathbb{C}$ and $[\Re(\mu)]$ denotes the greatest integer in $\Re(\mu)$, and $\Gamma(z)$ is the classical (Euler's) Gamma function of argument z, defined by

$$\Gamma(z) := \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-t}\, t^{z-1}\,\mathrm{d}t & \big(\Re(z)>0\big) \\[2ex] \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1}(z+j)} & \big(z\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ n\in\mathbb{N}\big), \end{cases} \tag{1.4}$$

which happens to be one of the most fundamental and the most useful special functions of mathematical analysis, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}_0^-$ being the sets of *positive*, *non-negative* and *non-positive* integers, respectively.

Such Eulerian integrals as in the definition (1.4) occur also in defining the familiar operator $\mathcal{L}$ of the Laplace transform as follows:

$$\mathcal{L}\{f(\tau):\mathfrak{s}\} := \int_0^\infty \mathrm{e}^{-\mathfrak{s}\tau}\, f(\tau)\,\mathrm{d}\tau =: F(\mathfrak{s}) \qquad \big(\Re(\mathfrak{s})>0\big), \tag{1.5}$$

where the function $f(\tau)$ is so constrained that the Eulerian integral in (1.5) exists.

In the case of the right-sided Riemann-Liouville fractional derivative operator ${}^{RL}D^{\mu}_{0+}$ of order μ in the definition (1.3), it is readily seen that (see, for example, [20, p. 105, Eq. (2.248)])

$$\mathcal{L}\left\{\left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(t):\mathfrak{s}\right\} = \mathfrak{s}^{\mu}\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^k \left({}^{\mathrm{RL}}D^{\mu-k-1}_{0+}f\right)(0+) \tag{1.6}$$

$$\big(n-1<\Re(\mu)<n;\ n\in\mathbb{N}\big)$$

or, equivalently, that (see, for example, [14, p. 84, Eq. (2.2.37)])

$$\begin{aligned} \mathcal{L}\left\{\left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(t):\mathfrak{s}\right\} &= \mathfrak{s}^{\mu}\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{n-k-1}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left\{\left({}^{\mathrm{RL}}I^{n-\mu}_{0+}f\right)(t)\right\}\bigg|_{t=0} \\ &= \mathfrak{s}^{\mu}\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{k}\,\frac{\mathrm{d}^{n-k-1}}{\mathrm{d}t^{n-k-1}}\left\{\left({}^{\mathrm{RL}}I^{n-\mu}_{0+}f\right)(t)\right\}\bigg|_{t=0} \end{aligned} \tag{1.7}$$

$$\big(n-1<\Re(\mu)<n;\ n\in\mathbb{N}\big),$$

where, for convenience,

$$\left({}^{\mathrm{RL}}D_{0+}^{\mu-k-1}f\right)(0+) := \lim_{t\to 0+}\left\{\left({}^{\mathrm{RL}}D_{0+}^{\mu-k-1}f\right)(t)\right\} =: \left({}^{\mathrm{RL}}D_{0+}^{\mu-k-1}f\right)(t)\Big|_{t=0}$$

and

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\}\Big|_{t=0} &:= \lim_{t\to 0+}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(t)\right\} \\ &=: \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left\{\left({}^{\mathrm{RL}}I_{0+}^{n-\mu}f\right)(0+)\right\}\end{aligned}$$

$$(k \in \{0,1,2,\ldots,n-1\}).$$

On the other hand, for the *ordinary* derivative $f^{(n)}(t)$ of order $n \in \mathbb{N}_0$, it is known that

$$\mathcal{L}\left\{f^{(n)}(t) : \mathfrak{s}\right\} = \mathfrak{s}^n\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^k\, f^{(n-k-1)}(t)\Big|_{t=0} \qquad (n \in \mathbb{N}_0) \tag{1.8}$$

or, equivalently, that

$$\mathcal{L}\left\{f^{(n)}(t) : \mathfrak{s}\right\} = \mathfrak{s}^n\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{n-k-1}\, f^{(k)}(0+) \qquad (n \in \mathbb{N}_0), \tag{1.9}$$

where, as well as in all of such situations in this paper, an *empty* sum is to be interpreted as 0.

It should be remarked here that, upon comparing the Laplace transform formulas (1.6) and (1.8), it is observed that the initial values such as those that occur in (1.6) are usually not interpretable physically in a given initial-value problem. Besides, unfortunately, the Riemann-Liouville fractional derivative of a constant is not zero. Some of these and other situations and disadvantages are overcome at least partially by means of the Liouville-Caputo fractional derivative which was considered in an earlier work dated 1832 by Joseph Liouville (1809–1882) [16, p. 10] *and* which has arisen in several important recent works, dated 1969 onwards, by Michele Caputo (see, for details, [20, p. 78 *et seq.*]; see also [14, p. 90 *et seq.*] and [5]).

In many recent works, especially in the theory of visco-elasticity and in hereditary solid mechanics, the following type of the definition dated 1832 of Liouville [16] and the definition dated 1969 of Caputo [2] is adopted for the fractional derivative of order μ $\big(\Re(\mu) \geqq 0\big)$ of a *causal* function $f(t)$, that is,

$$f(t) = 0 \qquad (t < 0),$$

given by

$$\begin{aligned}\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}\{f(x)\} &= \left({}^{\mathrm{LC}}D_{0+}^{\mu}f\right)(x) \\ &:= \begin{cases} f^{(n)}(x) & (\mu = n \in \mathbb{N}_0) \\[2mm] \dfrac{1}{\Gamma(n-\mu)}\displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu-n+1}}\,\mathrm{d}t & \big(n-1 < \Re(\mu) < n;\ n \in \mathbb{N}\big),\end{cases}\end{aligned} \tag{1.10}$$

where

$$n = \begin{cases} [\Re(\mu)] + 1 & (\mu \neq \mathbb{N}_0) \\ \mu & (\mu \in \mathbb{N}_0), \end{cases} \tag{1.11}$$

$f^{(n)}(t)$ denotes, as before, the usual (ordinary) derivative of $f(t)$ of order n and Γ is the familiar (Euler's) Gamma function defined by (1.4).

Unlike the Laplace transform formula (1.6) for the Riemann-Liouville fractional derivative $\left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(t)$, the following analogous formula holds true for the Liouville-Caputo fractional derivative $\left({}^{\mathrm{LC}}D^{\mu}_{0+}f\right)(t)$ defined by (1.10) (see, for example, [20, p. 80, Eq. (2.140)]; see also [14, p. 98, Eq. (2.4.62)]):

$$\begin{aligned} \mathcal{L}\left\{\left({}^{\mathrm{LC}}D^{\mu}_{0+}f\right)(t) : \mathfrak{s}\right\} &= \mathfrak{s}^{\mu}\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{\mu-k-1}\, \frac{\mathrm{d}^k}{\mathrm{d}t^k}\{f(t)\}\bigg|_{t=0} \\ &= \mathfrak{s}^{\mu}\, F(\mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{\mu-k-1}\, f^{(k)}(0+) \end{aligned} \tag{1.12}$$

$$\big(n-1 < \Re(\mu) \leqq n;\; n \in \mathbb{N}\big),$$

which does have the distinct advantage that, just as in the Laplace transform formula (1.8) or (1.9), the initial values at the lower terminal $t = 0$ involve the ordinary derivative $f^{(k)}(t)$ of integer order k given by

$$k \in \{0, 1, 2, \ldots, n-1\} \qquad (n \in \mathbb{N}).$$

We turn now to an interesting two-parameter family of fractional derivatives of order μ $(0 < \mu < 1)$ and type ν $(0 \leqq \nu \leqq 1)$, which was introduced and studied recently by Hilfer in the following form (see [7], [8] and [9]; see also [10] and [42]). Indeed, the right-sided Hilfer fractional derivative ${}^{\mathrm{H}}D^{\mu,\nu}_{a+}$ and the left-sided Hilfer fractional derivative ${}^{\mathrm{H}}D^{\mu,\nu}_{a-}$ of order μ $(0 < \mu < 1)$ and type ν $(0 \leqq \nu \leqq 1)$ with respect to x are defined, in terms of the Riemann-Liouville fractional integrals in (1.1) and (1.2), by

$$\left({}^{\mathrm{H}}D^{\mu,\nu}_{a\pm}f\right)(x) = \left(\pm\, {}^{\mathrm{RL}}I^{\nu(1-\mu)}_{a\pm} \frac{\mathrm{d}}{\mathrm{d}x}\left({}^{\mathrm{RL}}I^{(1-\nu)(1-\mu)}_{a\pm} f\right)\right)(x), \tag{1.13}$$

where it is tacitly assumed that the second member of (1.13) exists.

The generalization in the equation (1.13) yields the classical Riemann-Liouville fractional derivative operator when $\nu = 0$. Moreover, in the case when $\nu = 1$, it leads to the fractional derivative operator introduced by Liouville [16, p. 10], which is quite frequently attributed to Caputo [2], but which should more appropriately be referred to as the Liouville-Caputo fractional derivative, giving due credits to Joseph Liouville (1809–1882) who considered such fractional derivatives many decades earlier in 1832 (see [16]). Many authors (see, for example, [18] and [41]) called the general two-parameter operators in (1.13) the Hilfer fractional derivative operators. Several applications of the Hilfer fractional derivative operators $D^{\alpha,\beta}_{a\pm}$ can indeed be found in [9] (see also [24] and [25]).

By applying the formulas (1.1) and (1.2), together with the equation (1.3), we find for the fractional derivative operator ${}^{\mathrm{H}}D_{a\pm}^{\mu,\nu}$ that

$$\left({}^{\mathrm{H}}D_{a\pm}^{\mu,\nu}f\right)(x)=\left(\pm\,{}^{\mathrm{RL}}I_{a\pm}^{\nu(1-\mu)}\left({}^{\mathrm{RL}}I_{a\pm}^{\mu+\nu-\mu\nu}\,f\right)\right)(x) \tag{1.14}$$

$$\big(0<\mu<1;\ 0\leqq\nu\leqq 1\big).$$

The difference between fractional derivatives of different types becomes clearer from their Laplace transformations. For the right-sided Hilfer fractional derivative operator ${}^{H}D_{0+}^{\mu,\nu}$ of order μ and type ν in the definition (1.13), it is readily seen from the relationships in (1.13) and (1.14) with $0<\mu<1$ and $0\leqq\nu\leqq 1$ that

$$\mathcal{L}\left\{\left({}^{\mathrm{H}}D_{0+}^{\mu,\nu}f\right)(t):\mathfrak{s}\right\}=\mathfrak{s}^{\mu}\,\mathcal{L}\{f(t):\mathfrak{s}\}-\mathfrak{s}^{\nu(\mu-1)}\ \left({}^{\mathrm{RL}}I_{0+}^{(1-\nu)(1-\mu)}f\right)(0+) \tag{1.15}$$

$$\big(0<\mu<1;\ 0\leqq\nu\leqq 1\big),$$

where

$$\left({}^{\mathrm{RL}}I_{0+}^{(1-\nu)(1-\mu)}f\right)(0+)$$

is the Riemann-Liouville fractional integral of order $(1-\nu)(1-\mu)$, which is evaluated in the limit when $t\to 0+$ just as we have already explained above.

Here, in this article, we investigate some general families of hybrid-type fractional-order kinetic equations involving the Hilfer derivative operator ${}^{\mathrm{H}}D_{0+}^{\mu,\nu}$, which is given above by the equation (1.13), as well as including a remarkably general class of functions as a part of the non-homogeneous term. The main results, which are established here, are stated as Theorem 3.1, Theorem 3.3 and Theorem 3.4 in this article. Each of these main results is capable of producing solutions of a significantly large number of relatively simpler fractional-order kinetic equations. Some of this deductions from the main results (Theorem 3.1, Theorem 3.3 and Theorem 3.4 in this article) are presented here as corollaries and consequences.

2. Conventions, definitions and preliminary results

In this section, we choose first to remark that most (if not all) of the various claimed one-variable and multi-parameter (or multi-index) "generalizations" of the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter extension $E_{\alpha,\beta}(z)$, which are defined as follows:

$$E_\alpha(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\qquad\text{and}\qquad E_{\alpha,\beta}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{2.1}$$

$$\big(z,\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\big),$$

are no more than fairly obvious specialized or limit cases of the substantially much more general Fox-Wright function ${}_p\Psi_q$ $(p,q\in\mathbb{N}_0)$ or ${}_p\Psi_q^*$ $(p,q\in\mathbb{N}_0)$. As a matter of fact, the familiar and widely-investigated Fox-Wright function ${}_p\Psi_q$ $(p,q\in\mathbb{N}_0)$ or ${}_p\Psi_q^*$ $(p,q\in\mathbb{N}_0)$ happens to be the Fox-Wright generalization of the relatively more familiar hypergeometric function ${}_pF_q$ $(p,q\in\mathbb{N}_0)$, with p numerator parameters $a_1,\ldots,a_p$ and q denominator parameters $b_1,\ldots,b_q$ such that

$$a_j\in\mathbb{C}\quad(j=1,\ldots,p)\qquad\text{and}\qquad b_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\quad(j=1,\ldots,q).$$

These general Fox-Wright functions ${}_p\Psi_q$ $(p,q \in \mathbb{N}_0)$ and ${}_p\Psi_q^*$ $(p,q \in \mathbb{N}_0)$ are indeed defined by (see, for details, [3, p. 183] and [39, p. 21]; see also [13, p. 65], [14, p. 56] and [23])

$$
\begin{aligned}
&{}_p\Psi_q^*\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p);\\ \\ (b_1,B_1),\ldots,(b_q,B_q);\end{array} z\right] \\
&\qquad := \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n}\cdots(a_p)_{A_p n}}{(b_1)_{B_1 n}\cdots(b_q)_{B_q n}}\,\frac{z^n}{n!} \\
&\qquad = \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\;{}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p);\\ \\ (b_1,B_1),\ldots,(b_q,B_q);\end{array} z\right]
\end{aligned}
\tag{2.2}
$$

$$
\Bigg(\Re(A_j)>0\;\;(j=1,\ldots,p);\;\Re(B_j)>0\;\;(j=1,\ldots,q);
$$

$$
1+\Re\left(\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\right)\geqq 0\Bigg),
$$

where, and elsewhere in this article, $(\lambda)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial, since

$$
(1)_n = n! \qquad (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\;\mathbb{N}:=\{1,2,3,\ldots\}),
$$

which is defined (for $\lambda,\nu\in\mathbb{C}$ and in terms of familiar Gamma function in the equation (1.4)) by

$$
(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\lambda\in\mathbb{C}\setminus\{0\}) \\ \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu=n\in\mathbb{N};\lambda\in\mathbb{C}), \end{cases}
\tag{2.3}
$$

in which we have assumed *conventionally* that $(0)_0 := 1$ and understood *tacitly* that the Γ-quotient exists. In general, we suppose that

$$
a_j, A_j\in\mathbb{C}\quad(j=1,\ldots,p)\qquad\text{and}\qquad b_j,B_j\in\mathbb{C}\quad(j=1,\ldots,q)
$$

and that the equality in the convergence condition in the definition (2.2) holds true only for suitably-bounded values of $|z|$ given by

$$
|z|<\nabla:=\left(\prod_{j=1}^{p}A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^{q}B_j^{B_j}\right).
$$

The above-mentioned generalized hypergeoemtric function ${}_pF_q$ $(p,q\in\mathbb{N}_0)$, with p numerator parameters $a_1,\ldots,a_p$ and q denominator parameters $b_1,\ldots,b_q$, happens to be a widely- and extensively-investigated and potentially useful special case of the general Fox-Wright function ${}_p\Psi_q$ $(p,q\in\mathbb{N}_0)$ when

$$
A_j=1\quad(j=1,\ldots,p)\qquad\text{and}\qquad B_j=1\quad(j=1,\ldots,q).
$$

We find it to be important to turn now to a series of monumental works (see, for example, [43], [44] and [45]) by Sir Edward Maitland Wright (1906–2005). Fortunately (for me, of course), during my visit to the University of Aberdeen in the year 1976, I had the privilege to have met and discussed with Sir Wright researches emerging from his publications on hypergeometric and related functions. In fact, as long ago as 1940, Sir Wright introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [43, p. 424]):

$$\mathfrak{E}_{\alpha,\beta}(\phi;z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n+\beta)}\, z^n \qquad \big(\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\big), \tag{2.4}$$

where $\phi(t)$ is a function satisfying suitable conditions. Remarkably, it was my proud privilege to have also met many times and discussed mathematical researches, especially on various families of higher transcendental functions and related topics, with my Canadian colleague, Charles Fox (1897–1977) of birth and education in England, both at McGill University and Sir George Williams University (*now* Concordia University) in Montréal, mainly during the 1970s (see, for details, [23]).

The above-cited contributions by Sir Wright were motivated essentially by the earlier developments reported for simpler cases by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1905, Anders Wiman (1865–1959) in 1905, Ernest William Barnes (1874–1953) in 1906, Godfrey Harold Hardy (1877–1947) in 1905, George Neville Watson (1886–1965) in 1913, Charles Fox (1897–1977) in 1928, and other authors. In particular, the aforementioned work [1] by *Bishop* Ernest William Barnes (1874–1953) of the Church of England in Birmingham considered the asymptotic expansions of functions in the class which is defined below:

$$E^{(\kappa)}_{\alpha,\beta}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s\,\Gamma(\alpha n+\beta)} \qquad \big(\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\big) \tag{2.5}$$

for suitably-restricted parameters κ and s. Clearly, we have the following relationship:

$$\lim_{\alpha\to 0}\left\{E^{(\kappa)}_{\alpha,\beta}(s;z)\right\} = \frac{1}{\Gamma(\beta)}\,\Phi(z,s,\kappa)$$

with the classical Lerch transcendent (or the Hurwitz-Lerch zeta function) $\Phi(z,s,\kappa)$ defined by (see, for example, [3, p. 27, Eq. 1.11 (1)]; see also [37] and [38])

$$\Phi(z,s,\kappa) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s} \tag{2.6}$$

$$\big(\kappa\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C}\quad\text{when}\quad |z|<1;\ \Re(s)>1\quad\text{when}\quad |z|=1\big).$$

The Hurwitz-Lerch zeta function $\Phi(z,s,\kappa)$ defined by (2.6) contains, as its *special* cases, not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,\kappa)$:

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = \Phi(1,s,1) \qquad\text{and}\qquad \zeta(s,\kappa) := \sum_{n=0}^{\infty}\frac{1}{(n+\kappa)^s} = \Phi(1,s,\kappa), \tag{2.7}$$

and the Lerch zeta function $\ell_s(\xi)$ defined by (see, for details, [3, Chapter I] and [37, Chapter 2])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2n\pi \mathrm{i}\xi}}{n^s} = \mathrm{e}^{2\pi \mathrm{i}\xi}\ \Phi\left(\mathrm{e}^{2\pi \mathrm{i}\xi}, s, 1\right) \tag{2.8}$$

$$\left(\mathrm{i} = \sqrt{-1};\ \xi \in \mathbb{R};\ \Re(s) > 1\right),$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or the *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\ \Phi(z, s, 1) \tag{2.9}$$

$$\left(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right)$$

and the Lipschitz-Lerch zeta function (see [37, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi, \kappa, s) := \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2n\pi \mathrm{i}\xi}}{(n+\kappa)^s} = \Phi\left(\mathrm{e}^{2\pi \mathrm{i}\xi}, s, \kappa\right) =: L\left(\xi, s, \kappa\right) \tag{2.10}$$

$$\left(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\right),$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see, for details, [27] and [28]).

A natural unification and generalization of the Fox-Wright function ${}_p\Psi_q^*$ defined by (2.2) as well as the Hurwitz-Lerch zeta function $\Phi(z, s, \kappa)$ defined by (2.6) was indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition (2.6). For this purpose, in addition to the symbol ∇^* defined by

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right), \tag{2.11}$$

the following notations will be employed:

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \qquad \text{and} \qquad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}. \tag{2.12}$$

Then the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}(z, s, \kappa)$$

is defined by (see [40, p. 503, Equation (6.2)]; see also [26] and [38])

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+\kappa)^s} \tag{2.13}$$

$$\Bigg(p, q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j = 1, \ldots, p);\ \kappa, \mu_j \in \mathbb{C} \setminus Z_0^-\ (j = 1, \ldots, q);$$

$$\rho_j, \sigma_k \in \mathbb{R}^+ \ (j = 1, \ldots, p;\ k = 1, \ldots, q); \Delta > -1 \ \text{ when } \ s, z \in \mathbb{C};$$
$$\Delta = -1 \ \text{ and } \ s \in \mathbb{C} \ \text{ when } \ |z| < \nabla^*;$$
$$\left.\Delta = -1 \ \text{ and } \ \Re(\Xi) > \frac{1}{2} \ \text{ when } \ |z| = \nabla^* \right).$$

An interesting and potentially useful family of the λ-generalized Hurwitz-Lerch zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z, s, \kappa)$$

defined by (2.13), was introduced and investigated systematically by Srivastava [26], who discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [26]).

We now introduce some general families of the Riemann-Liouville type fractional integrals and fractional derivatives by making use of the following interesting unification of the definitions in (2.4), (2.5) and (2.13) for a suitably-restricted function $\varphi(\tau)$ given by

$$\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\kappa)^s\, \Gamma(\alpha n + \beta)}\, z^n \tag{2.14}$$
$$\big(\alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0\big),$$

where the parameters α, β, s and κ are appropriately constrained as above. The resulting general right-sided fractional integral operator $\mathcal{I}^{\mu}_{a+}(\varphi; z, s, \kappa, \nu)$ and the general left-sided fractional integral operator $\mathcal{I}^{\mu}_{a-}(\varphi; z, s, \kappa, \nu)$, and the corresponding fractional derivative operators

$$\mathcal{D}^{\mu}_{a+}(\varphi; z, s, \kappa, \nu) \qquad \text{and} \qquad \mathcal{D}^{\mu}_{a-}(\varphi; z, s, \kappa, \nu),$$

each of the Riemann-Liouville type, are defined by (see, for details, [30], [31], [32] and [22, Chapter 1])

$$\big(\mathcal{I}^{\mu}_{a+}(\varphi; z, s, \kappa, \nu) f\big)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z(x-t)^{\nu}, s, \kappa\big)\, f(t)\, \mathrm{d}t \tag{2.15}$$
$$\big(x > a;\ \Re(\mu) > 0\big),$$

$$\big(\mathcal{I}^{\mu}_{a-}(\varphi; z, s, \kappa, \nu) f\big)(x) = \frac{1}{\Gamma(\mu)} \int_x^a (t-x)^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z(t-x)^{\nu}, s, \kappa\big)\, f(t)\, \mathrm{d}t \tag{2.16}$$
$$\big(x < a;\ \Re(\mu) > 0\big)$$

and

$$\big(\mathcal{D}^{\mu}_{a\pm}(\varphi; z, s, \kappa, \nu) f\big)(x) = \left(\pm \frac{d}{dx}\right)^n \big(\mathcal{I}^{n-\mu}_{a\pm}(\varphi; z, s, \kappa, \nu) f\big)(x) \tag{2.17}$$
$$\big(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1\big),$$

where the function f is in the space $L(\mathfrak{a},\mathfrak{b})$ of Lebesgue integrable functions on a finite closed interval $[\mathfrak{a},\mathfrak{b}]$ $(\mathfrak{b}>\mathfrak{a})$ of the real line $\mathbb{R}$ given by

$$L(\mathfrak{a},\mathfrak{b})=\left\{f:\ \ \|f\|_1=\int_{\mathfrak{a}}^{\mathfrak{b}}|f(x)|\,\mathrm{d}x<\infty\right\}, \tag{2.18}$$

it being *tacitly* assumed that, in situations such as those occurring in conjunction with the usages of the definitions in (2.15), (2.16) and (2.17), the point $\mathfrak{a}$ in all such function spaces as (for example) the function space $L(\mathfrak{a},\mathfrak{b})$ coincides precisely with the *lower* terminal a in the integrals involved in the definitions (2.15), (2.16) and (2.17).

Next, in terms of the operator $\mathcal{L}$ of the Laplace transform given by the equation (1.5), it is easily seen for the function $\mathcal{E}_{\alpha,\beta}\big(\varphi;z,s,\kappa\big)$, defined above by the equation (2.14), that

$$\mathcal{L}\left\{\tau^{\mu-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;z\tau^{\nu},s,\kappa\big):\mathfrak{s}\right\}=\frac{1}{\mathfrak{s}^{\mu}}\sum_{k=0}^{\infty}\frac{\varphi(k)\,\Gamma(\nu k+\mu)}{(k+\kappa)^{s}\,\Gamma(\alpha k+\beta)}\,\left(\frac{z}{\mathfrak{s}^{\nu}}\right)^{k} \tag{2.19}$$

$$\big(\min\left\{\Re(\mathfrak{s}),\Re(\mu),\Re(\nu),\Re(\alpha)\right\}>0\big),$$

provided that each member of (2.19) exists. Obviously, upon setting $\mu=\beta$ and $\nu=\alpha$, the Laplace transform formula (2.19) simplifies to the following form:

$$\mathcal{L}\left\{\tau^{\beta-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;z\tau^{\alpha},s,\kappa\big):\mathfrak{s}\right\}=\frac{1}{\mathfrak{s}^{\mu}}\sum_{k=0}^{\infty}\frac{\varphi(k)}{(k+\kappa)^{s}}\,\left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^{k} \tag{2.20}$$

$$\big(\min\left\{\Re(\mathfrak{s}),\Re(\alpha),\Re(\beta)\right\}>0\big).$$

In case we apply the limit formula given by

$$\mathfrak{E}_{\alpha,\beta}(\phi;z)=\lim_{s\to 0}\left\{\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa)\right\}\Big|_{\varphi\equiv\phi} \tag{2.21}$$

or, alternatively, if we make use of the definitions in (2.4) and (1.5), we find for Wright's function $\mathfrak{E}_{\alpha,\beta}\big(\phi;z\big)$ that

$$\mathcal{L}\left\{\tau^{\mu-1}\,\mathfrak{E}_{\alpha,\beta}\big(\phi;z\tau^{\nu}\big):\mathfrak{s}\right\}=\frac{1}{\mathfrak{s}^{\mu}}\sum_{k=0}^{\infty}\frac{\phi(k)\,\Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)}\,\left(\frac{z}{\mathfrak{s}^{\nu}}\right)^{k} \tag{2.22}$$

$$\big(\min\left\{\Re(\mathfrak{s}),\Re(\mu),\Re(\nu),\Re(\alpha)\right\}>0\big),$$

which, in the special case when $\nu=\alpha$ and $\mu=\beta$, yields

$$\mathcal{L}\left\{\tau^{\beta-1}\,\mathfrak{E}_{\alpha,\beta}\big(\phi;z\tau^{\alpha}\big):\mathfrak{s}\right\}=\frac{1}{\mathfrak{s}^{\beta}}\sum_{k=0}^{\infty}\phi(k)\,\left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^{k} \tag{2.23}$$

$$\big(\min\left\{\Re(\mathfrak{s}),\Re(\alpha),\Re(\beta)\right\}>0\big).$$

Moreover, in the case when the sequence $\{\varphi(n)\}_{n=0}^{\infty}$ is given by

$$\varphi(n)=\frac{\Gamma(\alpha n+\beta)\,\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\cdot\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}}\qquad(n\in\mathbb{N}_0), \tag{2.24}$$

the Laplace transformation formula (2.20) would yield the following result:

$$\begin{aligned}&\mathcal{L}\left\{\tau^{\mu-1}\,\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}\left(z\tau^{\nu},s,\kappa\right):\mathfrak{s}\right\}\\ &\qquad=\frac{\Gamma(\mu)}{\mathfrak{s}^{\mu}}\,\Phi^{(\nu,\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\mu,\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}\left(\frac{z}{\mathfrak{s}^{\nu}},s,\kappa\right)\end{aligned}\tag{2.25}$$

$$\left(\min\left\{\Re(\mathfrak{s}),\Re(\mu),\Re(\nu),\Re(\alpha)\right\}>0\right)$$

for the extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,\kappa)$$

defined by the equation (2.13).

3. General hybrid-type families of fractional-order kinetic equations

Given an arbitrary reaction, which is characterized by a time-dependent quantity $N = N(t)$, it is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\mathfrak{d}$ and the production rate $\mathfrak{p}$ of N, that is,

$$\frac{dN}{dt}=-\mathfrak{d}+\mathfrak{p}.$$

By means of feedback or other interaction mechanism, the destruction and the production depend on the quantity N itself, that is,

$$\mathfrak{d}=\mathfrak{d}(N)\qquad\text{and}\qquad\mathfrak{p}=\mathfrak{p}(N).$$

Since the destruction or the production at a time t depends not only on $N(t)$, but also on the past history $N(\eta)$ $(\eta < t)$ of the variable N, such dependence is, in general, complicated. We may formally represent this by the following equation (see [6]):

$$\frac{dN}{dt}=-\mathfrak{d}\left(N_t\right)+\mathfrak{p}\left(N_t\right),\tag{3.1}$$

where N_t denotes the function defined by

$$N_t\left(t^*\right)=N\left(t-t^*\right)\qquad(t^*>0).$$

Haubold and Mathai [6] studied a special case of the equation (3.1) in the following form:

$$\frac{dN_j}{dt}=-c_j\,N_j\left(t\right),\tag{3.2}$$

that is,

$$\frac{dN_j(t)}{N_j(t)}=-c_j\,dt,\tag{3.3}$$

with the initial condition given by

$$N_j\left(t\right)\Big|_{t=0}=N_0,$$

where $N_j(t)$ is the number density of the species j at time $t = 0$ and the constant $c_j > 0$. This is known as a standard kinetic equation. The solution of the equation (3.2) is readily seen to be given by

$$N_j(t) = N_0 \, \mathrm{e}^{-c_j t}, \tag{3.4}$$

which, upon integration, yields the following alternative form of the solution of the equation (3.2) (*without* the subscript j):

$$N(t) - N_0 = c \cdot {}_0D_t^{-1} \{N(t)\}, \tag{3.5}$$

where ${}_0D_t^{-1}$ is the standard (ordinary) integral operator and c is a constant of integration.

A fractional-order generalization of the equation (3.5) is given as in the following form (see [6]):

$$N(t) - N_0 = c^{\nu} \left({}^{\mathrm{RL}}I_{0+}^{\nu} N\right)(t) \tag{3.6}$$

in terms of the familiar right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I_{0+}^{\nu}$ of order ν defined, as in (1.1), by (see, for example, [14])

$$\left({}^{\mathrm{RL}}I_{0+}^{\nu} f\right)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} \, f(u) \, \mathrm{d}u \qquad \big(t > 0;\ \Re(\nu) > 0\big). \tag{3.7}$$

For a notably large number of extensions and further generalizations of the fractional-order kinetic equation (3.6), the interested reader should refer (for example) to [15], [29] and [30] as well as to the other relevant references which are cited in each of these earier publications. We propose here to investigate the solution of a general hybrid-type family of the fractional-order kinetic equations which are associated with the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ defined by the equation (2.14), as well as with the Hilfer-type fractional derivative operator ${}^{\mathrm{H}}D_{0+}^{\sigma,\omega}$ defined by the equation (1.13) with μ and ν replaced by σ and ω, respectively. The results presented here are sufficiently general in character and are indeed capable of being specialized appropriately to include solutions of the corresponding (known or new) fractional-order kinetic equations associated with a wide variety of simpler functions.

Theorem 3.1. *Let each of the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$, $0 < \sigma < 1$ *and* $0 \leqq \omega \leqq 1$ *hold true. Suppose also that the general function* $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, *defined by the equation* (2.14), *exists. If we set*

$$\chi_0(\sigma, \omega) := \left({}^{\mathrm{RL}}I_{0+}^{(1-\omega)(1-\sigma)} N\right)(0+) \qquad \big(0 < \sigma < 1;\ 0 \leqq \omega \leqq 1\big), \tag{3.8}$$

then the solution of the following general hybrid-type family of the fractional-order kinetic equations:

$$N(t) - N_0 \, t^{\mu-1} \, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big) = -c^{\rho} \left({}^{\mathrm{H}}D_{0+}^{\sigma,\omega} N\right)(t) \tag{3.9}$$

is given by

$$N(t) = N_0 \, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k + \mu)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)\, \Gamma\big(\nu k + (r+1)\sigma + \mu\big)}\, (zt^\nu)^k$$
$$+ \chi_0(\sigma, \omega)\, t^{\sigma + \omega(1-\sigma) - 1}$$
$$\cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma(r+1) + \omega(1-\sigma)\big)} \left(\frac{t^\sigma}{c^\rho}\right)^r \qquad (t > 0), \tag{3.10}$$

provided that the second member of the solution given by the equation (3.10) *exists.*

Proof. Under the hypotheses involving the various parametric constraints in Theorem 3.1, we first apply the operator $\mathcal{L}$ of the Laplace transform on both sides of the kinetic equation (3.9). Thus, upon setting

$$\mathcal{N}(\mathfrak{s}) := \mathcal{L}\{N(t) : \mathfrak{s}\} = \int_0^\infty \mathrm{e}^{-\mathfrak{s}t}\, N(t)\, \mathrm{d}t, \tag{3.11}$$

we make use of the formula (2.20) and the formula (1.15) in the following form:

$$\mathcal{L}\left\{\left({}^{\mathrm{H}}D_{0+}^{\sigma,\omega} N\right)(t) : \mathfrak{s}\right\} = \mathfrak{s}^\sigma\, \mathcal{L}\{N(t) : \mathfrak{s}\} - \mathfrak{s}^{\omega(\sigma-1)} \left({}^{\mathrm{RL}}I_{0+}^{(1-\omega)(1-\sigma)} N\right)(0+)$$
$$= \mathfrak{s}^\sigma\, \mathcal{N}(\mathfrak{s}) - \mathfrak{s}^{\omega(\sigma-1)}\, \chi_0(\sigma, \omega) \tag{3.12}$$
$$\big(0 < \sigma < 1;\ 0 \leqq \omega \leqq 1\big),$$

where, as in the equation (3.8),

$$\left({}^{\mathrm{RL}}I_{0+}^{(1-\omega)(1-\sigma)} N\right)(0+) =: \chi_0(\sigma, \omega)$$

is the Riemann-Liouville fractional integral of order $(1-\omega)(1-\sigma)$, which is evaluated in the limit when $t \to 0+$ just as we have already explained above. We then find that

$$\mathcal{N}(\mathfrak{s}) - \frac{N_0}{\mathfrak{s}^\mu} \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k + \mu)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)} \left(\frac{z}{\mathfrak{s}^\nu}\right)^k$$
$$= -c^\rho \left[\mathfrak{s}^\sigma\, \mathcal{N}(\mathfrak{s}) - \mathfrak{s}^{\omega(\sigma-1)}\, \chi_0(\sigma, \omega)\right]$$
$$= -c^\rho\, \mathfrak{s}^\sigma\, \mathcal{N}(\mathfrak{s}) + c^\rho\, \chi_0(\sigma, \omega)\, \mathfrak{s}^{\omega(\sigma-1)}, \tag{3.13}$$

which leads us to the following result:

$$\mathcal{N}(\mathfrak{s}) = \frac{N_0}{1 + c^\rho\, \mathfrak{s}^\sigma} \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k + \mu)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)} \frac{z^k}{\mathfrak{s}^{\nu k + \mu}} + \frac{c^\rho\, \chi_0(\sigma, \omega)}{1 + c^\rho\, \mathfrak{s}^\sigma}\, \mathfrak{s}^{\omega(\sigma-1)}. \tag{3.14}$$

We now apply the series expansion given by

$$\frac{1}{1 + c^\rho\, \mathfrak{s}^\sigma} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\left(c^\rho\, \mathfrak{s}^\sigma\right)^{r+1}} \qquad \left(|c^\rho\, \mathfrak{s}^\sigma| > 1\right),$$

so that the equation (3.14) can be rewritten as follows:

$$\mathcal{N}(\mathfrak{s}) = N_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{c^{\rho(r+1)}} \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k + \mu)}{(k+\kappa)^s\, \Gamma(\alpha k + \beta)} \frac{z^k}{\mathfrak{s}^{\nu k + \sigma(r+1) + \mu}}$$

$$+\,\chi_0(\sigma,\omega)\sum_{r=0}^{\infty}\frac{(-1)^r}{c^{\rho r}\;\mathfrak{s}^{\sigma(r+1)+\omega(1-\sigma)}}. \tag{3.15}$$

Finally, if we invert the Laplace transforms occurring in (3.15) by means of the following well-known identities for the operators $\mathcal{L}$ and $\mathcal{L}^{-1}$ of the Laplace transform and the inverse Laplace transform, respectively:

$$\mathcal{L}\left\{t^{\lambda}:\mathfrak{s}\right\}=\frac{\Gamma(\lambda+1)}{\mathfrak{s}^{\lambda+1}}$$
$$\Longleftrightarrow\;\mathcal{L}^{-1}\left(\frac{1}{\mathfrak{s}^{\lambda+1}}\right)=\frac{t^{\lambda}}{\Gamma(\lambda+1)}\qquad\big(\Re(\lambda)>-1;\;\Re(\mathfrak{s})>0\big), \tag{3.16}$$

we obtain the solution (3.10) asserted by Theorem 3.1. This evidently completes the proof of Theorem 3.1. □

Remark 3.2. The use of the general function $\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa)$, defined by the equation (2.14), in the non-homogeneous term of the fractional-order kinetic equation (3.9) in Theorem 3.1 provides a distinct advantage in its generality so that solutions of other kinetic equations involving relatively simpler non-homogeneous terms can be derived by appropriately specializing the solution (3.10) which is asserted by Theorem 3.1. In what follows, we choose to record two relatively simpler versions of Theorem 3.1.

Theorem 3.3. *Let each of the parametric constraints* $c,\mu,\nu,\rho\in\mathbb{R}^+$, $0<\sigma<1$ *and* $0\leqq\omega\leqq 1$ *hold true. Suppose also that the general function* $\mathfrak{E}_{\alpha,\beta}(\phi;z)$, *defined by the equation* (2.4), *exists. If* $\chi_0(\sigma,\omega)$ *is defined by the equation* (3.8), *then the solution of the following general hybrid-type family of the fractional-order kinetic equations*:

$$N(t)-N_0\,t^{\mu-1}\,\mathfrak{E}_{\alpha,\beta}\big(\varphi;zt^{\nu}\big)=-c^{\rho}\,\big({}^{\mathrm{H}}D_{0+}^{\sigma,\omega}\,N\big)(t) \tag{3.17}$$

is given by

$$N(t)=N_0\,t^{\mu-1}\sum_{r=0}^{\infty}(-1)^r\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1}$$
$$\cdot\sum_{k=0}^{\infty}\frac{\phi(k)\,\Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)\,\Gamma\big(\nu k+(r+1)\sigma+\mu\big)}\,(zt^{\nu})^k$$
$$+\,\chi_0(\sigma,\omega)\,t^{\sigma+\omega(1-\sigma)-1}$$
$$\cdot\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma\big(\sigma(r+1)+\omega(1-\sigma)\big)}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r}\qquad(t>0), \tag{3.18}$$

it being assumed that the second member of the solution given by the equation (3.18) *exists.*

Proof. Our demonstration of Theorem 3.3 is indeed analogous to that of Theorem 3.1. Here we make use of the definition (2.4) and the Laplace transform formula (2.22). The details involved in the proof of Theorem 3.3 are being omitted here. □

Theorem 3.4. *Under the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$, $0 < \sigma < 1$ *and* $0 \leqq \omega \leqq 1$, *let the extended Hurwitz-Lerch zeta function*:

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}\big(z, s, \kappa\big),$$

defined by the equation (2.13), *exist. If* $\chi_0(\sigma,\omega)$ *is defined by the equation* (3.8), *then the solution of the following generalized hybrid-type fractional-order kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}\big(zt^{\nu}, s, \kappa\big) = -c^{\rho}\, \big({}^{\mathrm{H}}D_{0+}^{\sigma,\omega}\, N\big)(t) \tag{3.19}$$

is given by

$$\begin{aligned} N(t) = N_0\, t^{\mu-1} &\sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1} \frac{\Gamma(\mu)}{\Gamma\big(\sigma(r+1)+\mu\big)} \\ &\cdot \Phi_{\mu,\lambda_1,\ldots,\lambda_p;\sigma(r+1)+\mu,\mu_1,\ldots,\mu_q}^{(\nu,\rho_1,\ldots,\rho_p;\nu,\sigma_1,\ldots,\sigma_q)}\big(zt^{\nu}, s, \kappa\big) \\ &+ \chi_0(\sigma,\omega)\, t^{\sigma+\omega(1-\sigma)-1} \\ &\cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma(r+1)+\omega(1-\sigma)\big)} \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r} \qquad (t > 0), \end{aligned} \tag{3.20}$$

provided that the second member of the solution in the equation (3.20) *exists.*

Proof. Theorem 3.4 can be proven, along the lines which are parallel to those of our demonstrations of Theorem 3.1 and Theorem 3.3. In this case, we apply the definition (2.13) and the Laplace transform formula (2.25). We choose to omit the details involved in our proof of Theorem 3.4. □

4. Corollaries and consequences

We begin this section by presenting the following sequel to Remark 3.2 of the preceding section (Section 3).

Remark 4.1. As we observed above, since

$${}^{\mathrm{RL}}D_{0+}^{\sigma} := {}^{\mathrm{H}}D_{0+}^{\sigma,0} \qquad \text{and} \qquad {}^{\mathrm{LC}}D_{0+}^{\sigma} := {}^{\mathrm{H}}D_{0+}^{\sigma,1} \tag{4.1}$$

for the operators of the Riemann-Liouville and the Liouville-Caputo fractional-order derivatives, each of our main results (Theorems 3.1, 3.3 and 3.4) can be specialized to deduce the solution of the corresponding hybrid-type fractional-order kinetic equations involving these simpler and relatively more familiar fractional derivatives (see, for details, [30], [35] and [36]). Here, in this section, we state each of these corollaries and consequences of Theorems 3.1, 3.3 and 3.4. Corollaries 4.2, 4.3 and 4.4 below would follow when we appropriately apply the first relationship in (4.1).

Corollary 4.2. *Let each of the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$ *hold true. Suppose also that the general function* $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, *defined by the equation* (2.14), *exist. If we set*

$$\Upsilon_0(\sigma) := \left({}^{\mathrm{RL}}I_{0+}^{1-\sigma} N\right)(0+) \qquad \big(0 < \sigma < 1\big), \tag{4.2}$$

then the solution of the following general hybrid-type family of the fractional-order kinetic equations:

$$N(t) - N_0\, t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}, s, \kappa\big) = -c^{\rho}\, \big({}^{\mathrm{RL}}D^{\sigma}_{0+}\, N\big)(t) \tag{4.3}$$

is given by

$$\begin{aligned} N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1} & \\ \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)\, \Gamma\big(\nu k+(r+1)\sigma+\mu\big)}\, (zt^{\nu})^k & \\ + \Upsilon_0(\sigma)\, t^{\sigma-1} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma(r+1)\big)} \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r} \qquad (t>0), & \end{aligned} \tag{4.4}$$

provided that the second member of the solution given by the equation (4.4) *exists.*

Corollary 4.3. *Let each of the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$ *hold true. Suppose also that the general function* $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, *defined by the equation* (2.4), *exists. If* $\Upsilon_0(\sigma)$ *is defined by the equation* (4.2), *then the solution of the following general hybrid-type family of the fractional-order kinetic equations*:

$$N(t) - N_0\, t^{\mu-1}\, \mathfrak{E}_{\alpha,\beta}\big(\varphi; zt^{\nu}\big) = -c^{\rho}\, \big({}^{\mathrm{RL}}D^{\sigma}_{0+}\, N\big)(t) \tag{4.5}$$

is given by

$$\begin{aligned} N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1} & \\ \cdot \sum_{k=0}^{\infty} \frac{\phi(k)\, \Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)\, \Gamma\big(\nu k+(r+1)\sigma+\mu\big)}\, (zt^{\nu})^k & \\ + \Upsilon_0(\sigma)\, t^{\sigma-1} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma(r+1)\big)} \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r} \qquad (t>0), & \end{aligned} \tag{4.6}$$

it being assumed that the second member of the solution given by the equation (4.6) *exists.*

Corollary 4.4. *Under the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$, $0 < \sigma < 1$ *and* $0 \leqq \omega \leqq 1$, *let the extended Hurwitz-Lerch zeta function*:

$$\Phi^{(\rho_1,\dots,\rho_p;\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}\big(z, s, \kappa\big),$$

defined by the equation (2.13), *exist. If* $\Upsilon_0(\sigma)$ *is defined by* (4.2), *then the solution of the following generalized hybrid-type fractional-order kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \Phi^{(\rho_1,\dots,\rho_p;\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}\big(zt^{\nu}, s, \kappa\big) = -c^{\rho}\, \big({}^{\mathrm{H}}D^{\sigma}_{0+}\, N\big)(t) \tag{4.7}$$

is given by

$$\begin{aligned} N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{r+1} \frac{\Gamma(\mu)}{\Gamma\big(\sigma(r+1)+\mu\big)} & \\ \cdot \Phi^{(\nu,\rho_1,\dots,\rho_p;\nu,\sigma_1,\dots,\sigma_q)}_{\mu,\lambda_1,\dots,\lambda_p;\sigma(r+1)+\mu,\mu_1,\dots,\mu_q}\big(zt^{\nu}, s, \kappa\big) & \end{aligned}$$

$$+ \Upsilon_0(\sigma)\, t^{\sigma-1} \cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma(r+1)\big)} \left(\frac{t^\sigma}{c^\rho}\right)^r \qquad (t > 0), \tag{4.8}$$

provided that the second member of the solution in the equation (3.20) *exists.*

Remark 4.5. In the case when we apply the second relation in the equation (4.1) in conjunction with Theorems 3.1, 3.3 and 3.4, we are led to Corollaries 4.6, 4.7 and 4.8, respectively.

Corollary 4.6. *Let each of the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$ *hold true. Suppose also that the general function* $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, *defined by the equation* (2.14), *exists. If we set*

$$\Xi_0 = N(0+) := N(t)\big|_{t\to 0+}, \tag{4.9}$$

then the solution of the following general hybrid-type family of the fractional-order kinetic equations:

$$N(t) - N_0\, t^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zt^\nu, s, \kappa\big) = -c^\rho \left({}^{\mathrm{LC}}D^{\sigma}_{0+} N\right)(t) \tag{4.10}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^\sigma}{c^\rho}\right)^{r+1} \cdot \sum_{k=0}^{\infty} \frac{\varphi(k)\, \Gamma(\nu k+\mu)}{(k+\kappa)^s\, \Gamma(\alpha k+\beta)\, \Gamma\big(\nu k+(r+1)\sigma+\mu\big)}\, (zt^\nu)^k + \Xi_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma r+1\big)} \left(\frac{t^\sigma}{c^\rho}\right)^r \qquad (t > 0), \tag{4.11}$$

provided that the second member of the solution in the equation (4.11) *exists.*

Corollary 4.7. *Let each of the parametric constraints* $c, \mu, \nu, \rho \in \mathbb{R}^+$ *and* $0 < \sigma < 1$ *hold true. Suppose also that the general function* $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, *defined by the equation* (2.4), *exists. If* Ξ_0 *is defined by the equation* (4.9), *then the solution of the following general hybrid-type family of the fractional-order kinetic equations*:

$$N(t) - N_0\, t^{\mu-1}\, \mathfrak{E}_{\alpha,\beta}\big(\varphi; zt^\nu\big) = -c^\rho \left({}^{\mathrm{LC}}D^{\sigma}_{0+} N\right)(t) \tag{4.12}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r \left(\frac{t^\sigma}{c^\rho}\right)^{r+1} \cdot \sum_{k=0}^{\infty} \frac{\phi(k)\, \Gamma(\nu k+\mu)}{\Gamma(\alpha k+\beta)\, \Gamma\big(\nu k+(r+1)\sigma+\mu\big)}\, (zt^\nu)^k + \Xi_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma r+1\big)} \left(\frac{t^\sigma}{c^\rho}\right)^r \qquad (t > 0), \tag{4.13}$$

it being assumed that the second member of the solution given in the equation (4.13) *exists.*

Corollary 4.8. *Under the parametric constraints $c, \mu, \nu, \rho \in \mathbb{R}^+$ and $0 < \sigma < 1$, let the extended Hurwitz-Lerch zeta function*:

$$\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}\big(z, s, \kappa\big),$$

defined by the equation (2.13), *exist. If Ξ_0 is defined by the equation* (3.8), *then the solution of the following generalized hybrid-type fractional-order kinetic equation*:

$$N(t) - N_0\, t^{\mu-1}\, \Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}\big(zt^\nu, s, \kappa\big) = -c^\rho\, \big({}^{\mathrm{LC}}D^\sigma_{0+}\, N\big)(t) \tag{4.14}$$

is given by

$$\begin{aligned} N(t) = N_0\, t^{\mu-1} &\sum_{r=0}^{\infty} (-1)^r \left(\frac{t^\sigma}{c^\rho}\right)^{r+1} \frac{\Gamma(\mu)}{\Gamma\big(\sigma(r+1)+\mu\big)} \\ &\cdot \Phi^{(\nu,\rho_1,\ldots,\rho_p;\nu,\sigma_1,\ldots,\sigma_q)}_{\mu,\lambda_1,\ldots,\lambda_p;\sigma(r+1)+\mu,\mu_1,\ldots,\mu_q}\big(zt^\nu, s, \kappa\big) \\ &+ \Xi_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\big(\sigma r+1\big)} \left(\frac{t^\sigma}{c^\rho}\right)^r \qquad (t > 0), \end{aligned} \tag{4.15}$$

provided that the second member of the solution in the equation (4.15) *exists.*

5. Concluding remarks and observations

In our present investigation, we have established the explicit solution of some significantly general hybrid-type families of fractional-order kinetic equations involving the Hilfer-type fractional derivative operator ${}^{\mathrm{H}}D^{\mu,\nu}_{0+}$, which is given (for convenience) by (1.13) or (1.14) for $a = 0$, and also involving a remarkably general class of functions as a part of the non-homogeneous term. Our main results (Theorem 3.1, Theorem 3.3 and Theorem 3.4 in this article) include, as a part of the non-homogeneous term, such general functions as the functions $\mathcal{E}_{\alpha,\beta}\big(\varphi; z, s, \kappa\big)$, $\mathfrak{E}_{\alpha,\beta}\big(\phi; z\big)$ and

$$\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}\big(z, s, \kappa\big),$$

which are defined by the equations (2.14), (2.4) and (2.13), respectively. We have also shown as to how each of these main results is indeed capable of yielding solutions of a significantly large number of (known or new) simpler fractional-order kinetic equations.

As corollaries and consequences of our main results (Theorem 3.1, Theorem 3.3 and Theorem 3.4 in this article), we have successfully deduced the explicit solutions of the corresponding general hybrid-type families of fractional-order kinetic equations involving the Riemann-Liouville fractional derivative operator ${}^{\mathrm{RL}}D^{\mu}_{0+}$, which are stated as Corollaries 4.2, 4.3 and 4.4, and also the explicit solutions of the corresponding general hybrid-type families of fractional-order kinetic equations involving the Liouville-Caputo fractional derivative operator ${}^{\mathrm{LC}}D^{\mu}_{0+}$, which are stated as Corollaries 4.6, 4.7 and 4.8. Each of these corollaries itself is sufficiently general in character and can yield the solution of many relatively simpler fractional-order kinetic equations.

We choose to conclude this article by remarking that the current literature is being flooded unnecessarily by seemingly amateurish-type publications in which several obviously false and misleading claims to generalization are made by trivially and inconsequentially introducing some parametric and argument variations in the well-established and widely-investigated known definitions and known theories. The following two of many such false and misleading claims are concerned with the Eulerian integrals defining the classical (Euler's) Gamma function in the equation (1.4) and the classical Laplace transform in the equation (1.5).

For the first example, we can cite the so-called k-Gamma function $\Gamma_k(z)$ with a trivially forced-in redundant (or superfluous) parameter k by making the following inconsequential change of the variable of integration in the integral definition in the equation (1.4) (see, for details, [31, Section 3]):

$$t = \frac{\tau^k}{k} \quad \text{and} \quad \mathrm{d}t = \tau^{k-1}\,\mathrm{d}\tau \qquad (k > 0),$$

so that, upon trivially replacing the argument z by $\frac{z}{k}$ $(k > 0)$, we have

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\,\Gamma\left(\frac{z}{k}\right) := \int_0^\infty \tau^{z-1}\,\mathrm{e}^{-\frac{\tau^k}{k}}\,\mathrm{d}\tau \qquad \big(\Re(z) > 0;\ k > 0\big). \tag{5.1}$$

It is indeed regrettable to observe further that, by replacing the classical (Euler's) Gamma function $\Gamma(z)$ by this rather inconsequential k-Gamma function $\Gamma_k(z)$ in the standard definitions of the such operators of fractional calculus as (for example) the Riemann-Liouville fractional integral and derivative operators and the Liouville-Caputo fractional derivative operator, which we have worked with in this article, many seemingly amateurish-type authors and researchers are being fooled or misled to believe that they have produced a "generalization" of the corresponding extensively-studied fractional integral and fractional derivative operators.

Our second example pertains to the Eulerian integral defining the classical Laplace transform in the equation (1.5) as well as its following $\mathfrak{s}$-multiplied version studied by the American transmission theorist, John Renshaw Carson (1886–1940):

$$\mathcal{LC}\left\{f(\tau) : \mathfrak{s}\right\} := \mathfrak{s} \int_0^\infty \mathrm{e}^{-\mathfrak{s}\tau}\, f(\tau)\, \mathrm{d}\tau =: F_{\mathcal{LC}}(\mathfrak{s}), \tag{5.2}$$

which has one distinct advantage over the familiar Laplace transform in the equation (1.5) in the fact that the Laplace-Carson transform of a constant in the equation (5.2) is the same constant (see, for details, [19]). Remarkably, many obviously trivial and inconsequential variations have been and continue to be made in the parameter (or index) $\mathfrak{s}$ or in the integration variable τ (or in both $\mathfrak{s}$ and τ), ridiculously giving a "new" name to each of such parametric and argument variations of the classical Laplace transform in the equation (1.5) or its $\mathfrak{s}$-multiplied version in (5.2) by forcing-in some obviously redundant (or superfluous) parameters. Some of these examples can be found in [31, pp. 1508–1510] and in [33, Section 5, pp. 36–38] and, more recently, in [34, pp. 2341–2346] and [36, pp. 58–60]. Yet another somehow missed-out instance of such trivialities can be exemplified by Yang's attempt to produce what he called a "new" integral transform by replacing the parameter (or index)

$\mathfrak{s}$ in the equation (1.5) by $\frac{1}{\mathfrak{s}}$ (see, for details, [46] and [47]). Furthermore, several presumably amateurish-type researchers are misled (or "fooled") by the obviously false claims that the classical (Euler's) Gamma function as well as its various related special functions (such as, for example, the Beta function, the hypergeometric series and their numerous associated functions, and so on) can be "generalized" by making some rather trivial, redundant and inconsequential variable and/or parameter (index) changes in the defining integrals and series. Such demonstrably trivial and obviously inconsequential parametric and argument variations as those that we have recalled above continue to flood the literature merely to unnecessarily repeat or translate the already-published remarkably successfully developments using the original integrals and the original special functions themselves.

Finally, we recall a recent publication by Jafari [11] in which the following variation of the classical Laplace transform was shown to be useful for solving higher-order initial-value problems, integral equations and fractional-order integral equations in just about the same way as it has already been done widely and extensively by means of the classical Laplace transform itself (see [11, p. 134, Definition 1]):

$$\begin{aligned} \mathcal{T}\{f(t) : \mathfrak{s}\} &:= \mathfrak{p}(\mathfrak{s}) \int_0^\infty \mathrm{e}^{-\mathfrak{q}(\mathfrak{s})t}\, f(t)\, \mathrm{d}t \\ &=: \mathfrak{p}(\mathfrak{s})\, \mathcal{L}\{f(t) : \mathfrak{q}(\mathfrak{s})\} \qquad \big(\min\{\mathfrak{p}(\mathfrak{s}), \mathfrak{q}(\mathfrak{s})\} > 0\big), \end{aligned} \tag{5.3}$$

the extension of which to suitably-constrained complex-valued functions $\mathfrak{p}(\mathfrak{s})$ and $\mathfrak{q}(\mathfrak{s})$ of $\mathfrak{s} \in \mathbb{C}$ is a trivial matter. The case of the equation (5.3) without the obviously inconsequential multiplying factor $\mathfrak{p}(\mathfrak{s})$ was considered independently by Yang [48, p. 866, Definition 7.73].

Two sequels to the above-mentioned work [11] are worth mentioning here. One by Meddahi *et al.* [17] dealt essentially similarly with an analogous double-integral version of the definition (5.3). The other by Khan and Khalid [12] trivially reproduced the well-established theory of the classical Laplace transform itself by simply replacing the complex parameter $\mathfrak{s}$ $\big(\Re(\mathfrak{s}) > 0\big)$ by a positive real parameter $\mathfrak{s}^n > 0$ $(n \in \{1, 3, 5, \dots\})$ and they named it rather strangely and ridiculously as "Fareeha transform", which obviously is a special case of the so-called "Sadik transform" (with $v = \mathfrak{s}$) when $\alpha = n$ $(n \in \{1, 3, 5, \dots)\}$ and $\beta = 0$ (see, for details, [33, p. 38, Eq. (52)]). It is unfortunate to observe numerous erroneous and misleading claims and statements throughout the paper [12]. All such obviously unnecessary and demonstrably inconsequential flooding of the literature by some amateurish-type publications with the sole aim to somehow produce "new" papers with hardly any new or nontrivial content should not be encouraged by the editors and reviewers of respectable journals.

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