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ENDPOINT THEOREMS FOR SOME GENERALIZED MULTI-VALUED NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX GEODESIC SPACES

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This work is dedicated to the memories of
Professor Kazimierz Goebel and Professor William Art Kirk

Abstract. In this paper, we introduce a new class of multi-valued mappings in metric spaces and show that it lies between the class of nonexpansive mappings and the class of semi-nonexpansive mappings. We prove the semiclosed principle and apply it to obtain endpoint and commmon endpoint theorems for mappings in this class. We also prove Δ and strong convergence theorems of the SP-iteration for semi-nonexpansive mappings in 2-uniformly convex geodesic spaces. Our results extend and improve many results in the literature.

1. Introduction

Let C be a nonempty subset of a metric space (X, d). For $x \in X$, we set

$$\operatorname{dist}(x, C) := \inf\{d(x, y) : y \in C\}, \quad R(x, C) := \sup\{d(x, y) : y \in C\},$$

and

$$\operatorname{diam}(C) := \sup\{d(y, z) : y, z \in C\}.$$

We denote by $\mathcal{CB}(C)$ the family of all nonempty closed bounded subsets of C, and by $\mathcal{K}(C)$ the family of all nonempty compact subsets of C. The Pompeiu-Hausdorff distance on $\mathcal{CB}(C)$ is defined by

$$H(A, B) := \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\} \text{ for all } A, B \in \mathcal{CB}(C).$$

A mapping T from C into $\mathcal{CB}(C)$ is called a multi-valued mapping. In particular, if $T(x)$ is a singleton for every x in C, then T is called a single-valued mapping. Notice that every single-valued mapping can be regarded as a multi-valued mapping. A point x in C is called a fixed point of T if $x \in T(x)$. We denote by $Fix(T)$ the set of all fixed points of T. Let $k \in [0, 1)$. A mapping $T : C \to \mathcal{CB}(C)$ is said to be k-contractive if

$$H(T(x), T(y)) \leq k\, d(x, y) \text{ for all } x, y \in C. \tag{1.1}$$

If (1.1) is valid when $k = 1$, then T is said to be nonexpansive.

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Fixed point theory is an important tool for finding solutions of problems in the form of equations or inequalities. One of the fundamental and celebrated results in metric fixed point theory is the so-called Banach contraction principle [6] which stated that every single-valued k-contractive mapping on a complete metric space always has a fixed point. The principle was extended to multi-valued mappings by Nadler [26] in 1969.

The concept of endpoints (or strict fixed point) for multi-valued mappings is an important concept which is weaker than the concept of fixed points for single-valued mappings and stronger than the concept of fixed points for multi-valued mappings. In 1986, Corley [10] proved that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of a certain multi-valued mapping. In 2010, Amini-Harandi [4] proved the existence of endpoints for multi-valued k-contractive mappings in complete metric spaces. After that, Ahmad et al. [3] applied his result to guarantee the existence of solutions of the mixed Hadamard and Riemann-Liouville fractional inclusion problems. For more details and further applications of endpoint theory, the reader is referred to [16, 19, 20, 41–43].

In 2015, Panyanak [29] proved the existence of endpoints for multi-valued nonexpansive mappings in uniformly convex Banach spaces. It was quickly noted by Espínola et al. [12] that the results of Panyanak can be extended to Banach spaces with the Domínguez–Lorenzo condition. In 2016, Saejung [39] obtained endpoint theorems for some generalized multi-valued nonexpansive mappings in uniformly convex Banach spaces as well as Banach spaces which satisfy the Opial's condition. Moreover, he also obtained the analogous results in complete CAT(κ) spaces. Since then endpoint theorems for several classes of generalized multi-valued nonexpansive mappings in metric and Banach spaces have been developed and many papers have appeared (see, e.g., [8, 9, 17, 22–24, 28, 31, 33, 35, 44]).

In 2011, Garcia-Falset et al. [14] generalized the concept of single-valued nonexpansive mappings in the following way: a mapping $f : C \to C$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in C$ with $\lambda\, d(x, f(x)) \leq d(x, y)$. This concept has been extended to multi-valued cases in many directions (see, e.g., [2, 13, 18, 25, 38]). Among other things, Kaewcharoen and Panyanak [18] defined a multi-valued mappings in the following manner: a mapping $T : C \to \mathcal{CB}(C)$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ if $H(T(x), T(y)) \leq d(x, y)$ for all $x, y \in C$ with $\lambda\, \mathrm{dist}(x, T(x)) \leq d(x, y)$. Recently, Panyanak [33] proved the existence of endpoints for multi-valued mappings satisfying condition (C_λ) in complete uniformly convex hyperbolic spaces.

In this paper, motivated by the above results, we introduce a new class of multi-valued mappings and show that it is more general than the class of mappings satisfying condition (C_λ). We also prove endpoint and commmon endpoint theorems for mappings in this class. Finally, we prove Δ and strong convergence theorems for the SP-iteration in several classes of generalized nonexpansive mappings. Our results extend and improve the results of Panyanak [33], Chuadchawna et al. [9] and many others.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let C be a nonempty subset of a metric space (X, d) and $T : C \to \mathcal{CB}(C)$ be a multi-valued mapping. A point x in C is called an endpoint of T if $T(x) = \{x\}$. We denote by $End(T)$ the set of all endpoints of T. Notice that the following statements hold:

- If x is an endpoint of T, then x is a fixed point of T.
- $x \in Fix(T)$ if and only if $\operatorname{dist}(x, T(x)) = 0$.
- $x \in End(T)$ if and only if $R(x, T(x)) = 0$.

A sequence $\{x_n\}$ in C is called an approximate endpoint sequence of T [4] if

$$\lim_{n\to\infty} R(x_n, T(x_n)) = 0.$$

Moreover, if $\{T_\alpha : \alpha \in \Omega\}$ is a family of multi-valued mappings from C into $\mathcal{CB}(C)$, then $\{x_n\}$ is called an approximate common endpoint sequence of $\{T_\alpha : \alpha \in \Omega\}$ [1] if $\lim_{n\to\infty} R(x_n, T_\alpha(x_n)) = 0$ for all $\alpha \in \Omega$.

Definition 2.1. A multi-valued mapping $T : C \to \mathcal{CB}(C)$ is said to be

(i) upper semicontinuous if for any sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \in C$ and $y_n \in T(x_n)$, the conditions $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $y \in T(x)$;

(ii) lower semicontinuous if for any sequence $\{x_n\}$ in C with $\lim_{n\to\infty} x_n = x$ and $y \in T(x)$, there exists a sequence $\{y_n\}$ such that $y_n \in T(x_n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y$;

(iii) continuous if T is both upper and lower semicontinuous;

(iv) semi-nonexpansive if $End(T) \neq \emptyset$ and

$$H(T(x), T(p)) \leq d(x, p) \text{ for all } x \in C \text{ and } p \in End(T).$$

Definition 2.2. Let $\lambda \in (0, 1)$. A multi-valued mapping $T : C \to \mathcal{CB}(C)$ is said to satisfy generalized condition (C_λ) if for each $x, y \in C$, the following implication holds:

$$\lambda\, R(x, T(x)) \leq d(x, y) \implies H(T(x), T(y)) \leq d(x, y).$$

Notice that if $0 < \lambda_1 < \lambda_2 < 1$, then the condition generalized (C_{λ_1}) implies the condition generalized (C_{λ_2}). The following proposition is easy to established.

Proposition 2.3. *The following statements hold:*

(i) *If T satisfies condition (C_λ), then T satisfies generalized condition (C_λ).*

(ii) *If T satisfies generalized condition (C_λ) and $End(T) \neq \emptyset$, then T is semi-nonexpansive.*

The following examples show that the converses of (i) and (ii) in Proposition 2.3 are not true.

Example 2.4. Let $X = \mathbb{R}$, $C = [0, 3]$ and $T : C \to \mathcal{CB}(C)$ be defined by

$$T(x) = \begin{cases} [0, x] & \text{if } x \neq 3; \\ [0.5, 1] & \text{if } x = 3. \end{cases}$$

If we choose $x = 2.1$ and $y = 3$, then

$$\frac{1}{2} \operatorname{dist}(x, T(x)) = 0 \leq d(x, y).$$

However,

$$H(T(x),T(y)) = H([0,2.1],[0.5,1]) = 1.1 > 0.9 = d(x,y).$$

This implies that T does not satisfy condition $(C_{\frac{1}{2}})$. Next, we show that T satisfies generalized condition $(C_{\frac{1}{2}})$. Let $x, y \in C$.

Case 1. $x = 3$.

If $y = 3$, then $H(T(x),T(y)) = 0$, and hence the conclusion holds.

If $y \in (2,3)$, then

$$\frac{1}{2}R(x,T(x)) > 1 > d(x,y) \text{ and } \frac{1}{2}R(y,T(y)) = \frac{y}{2} > 1 > d(x,y).$$

If $y \in [0,2]$, then

$$H(T(x),T(y)) = H([0.5,1],[0,y]) \leq 1 \leq d(x,y).$$

Case 2. $x \in (2,3)$.

If $y = 3$, then

$$\frac{1}{2}R(x,T(x)) = \frac{x}{2} > 1 > d(x,y) \text{ and } \frac{1}{2}R(y,T(y)) > 1 > d(x,y).$$

If $y \in [0,3)$, then

$$H(T(x),T(y)) = H([0,x],[0,y]) = d(x,y).$$

Case 3. $x \in [0,2]$.

If $y = 3$, then

$$H(T(x),T(y)) = H([0,x],[0.5,1]) \leq 1 \leq d(x,y).$$

If $y \in [0,3)$, then

$$H(T(x),T(y)) = H([0,x],[0,y]) = d(x,y).$$

Example 2.5. Let $X = \mathbb{R}$, $C = [0,3]$ and $T : C \to \mathcal{CB}(C)$ be defined by

$$T(x) = \begin{cases} \{0\} & \text{if } x \neq 3; \\ [2,2.5] & \text{if } x = 3. \end{cases}$$

It is easy to see that $End(T) = \{0\}$. Let $x \in C$. If $x = 3$, then $T(x) = [2,2.5]$, which implies that

$$H(T(x),T(0)) = H([2,2.5],\{0\}) = 2.5 < 3 = d(x,0).$$

If $x \neq 3$, then $T(x) = \{0\}$, which implies that

$$H(T(x),T(0)) = 0 \leq d(x,0).$$

Hence, T is a semi-nonexpansive mapping.

For any $\lambda \in (0,1)$, we have $\lambda R(3,T(3)) = \lambda < 1 = d(3,2)$. However,

$$H(T(3),T(2)) = H([2,2.5],\{0\}) = 2.5 > 1 = d(3,2).$$

This shows that T does not satisfy generalized condition (C_λ).

Let (X,d) be a metric space and $x,y \in X$. A continuous mapping $\phi : [0,1] \to X$ is called a geodesic joining x and y if $\phi(0) = x, \phi(1) = y$ and

$$d(\phi(t), \phi(t')) = |t - t'| d(x,y) \text{ for all } t, t' \in [0,1].$$

A metric space (X,d) is said to be a geodesic space if for any two points in X there exists a geodesic joining them. Moreover, if any two points in X are joined by a unique geodesic, then we say that X is a uniquely geodesic space. In particular, if ϕ is the unique geodesic joining x and y, then we use the notation $(1-t)x \oplus ty$ for $\phi(t)$. A subset C of X is said to be convex if $(1-t)x \oplus ty \in C$ for all $x, y \in C$ and $t \in [0,1]$.

A geodesic space (X,d) is called 2-uniformly convex [27] if there exists a constant $c_X \in (0,1]$ such that for any $x,y,z \in X$ and for any geodesic $\phi : [0,1] \to X$ joining x and y, the following inequality holds:

$$d^2(\phi(t), z) \le (1-t)d^2(x,z) + td^2(y,z) - c_X t(1-t)d^2(x,y) \text{ for all } t \in [0,1]. \tag{2.1}$$

It is known from [37] that every 2-uniformly convex geodesic space is uniquely geodesic.

Example 2.6. (1) Every uniformly convex Banach space is a 2-uniformly convex geodesic space (see [34]).

(2) If X is a CAT(0) space, then it is a 2-uniformly convex geodesic space (see [11]).

(3) If $\kappa > 0$ and X is a CAT(κ) space with $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, then by Lemma 2.3 of [30] we can conclude that

$$d^2(\phi(t), z) \le (1-t)d^2(x,z) + td^2(y,z) - \frac{R}{2}t(1-t)d^2(x,y) \quad \text{for all } x,y \in X \text{ and } t \in [0,1],$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. This clearly implies that X is a 2-uniformly convex geodesic space.

From now on, X stands for a complete 2-uniformly convex geodesic space. Let C be a nonempty subset of X and $\{x_n\}$ be a bounded sequence in X. The asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C, \{x_n\}) := \inf \big\{ \limsup_{n\to\infty} d(x_n, x) : x \in C \big\}.$$

The asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) := \big\{ x \in C : \limsup_{n\to\infty} d(x_n, x) = r(C, \{x_n\}) \big\}.$$

It is known from [5] that if C is a nonempty closed convex subset of X, then $A(C, \{x_n\})$ consists of exactly one point.

Definition 2.7. Let C be a nonempty closed convex subset of X and $x \in C$. Let $\{x_n\}$ be a bounded sequence in X. We say that $\{x_n\}$ Δ-converges to x if $A(C, \{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $x_n \xrightarrow{\Delta} x$.

It is known from [15] and [21] that every bounded sequence in X has a Δ-convergent subsequence.

Definition 2.8. Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{CB}(C)$. Let I be the identity mapping on C. We say that $I - T$ is semiclosed if for any sequence $\{x_n\}$ in C such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, T(x_n)) \to 0$, one has $T(x) = \{x\}$.

The following results can be found in [11] and [34], respectively.

Lemma 2.9. *Let C be a nonempty closed convex subset of X and $\{x_n\}$ be a bounded sequence in X. If $A(C, \{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(C, \{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.10. *Let C be a nonempty subset of X and $T : C \to \mathcal{CB}(C)$. Then the following statements hold:*
(i) If C is convex and T is semi-nonexpansive, then $End(T)$ is convex.
(ii) If C is closed and convex and $I - T$ is semiclosed, then $End(T)$ is closed.

3. Endpoint theorems

We start this section by proving the semiclosed principle for multi-valued mappings satisfying generalized condition (C_λ). Notice that this is an extension of Lemma 3.1 in [9].

Theorem 3.1. *Let C be a nonempty closed convex subset of X, and I the identity mapping on C, and $T : C \to \mathcal{CB}(C)$ a mappings satisfying generalized condition (C_λ) for some $\lambda \in (0, 1)$. If T is lower semicontinuous, then $I - T$ is semiclosed.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, T(x_n)) \to 0$. We show that $T(x) = \{x\}$.

Case 1. For each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\lambda R(x_m, T(x_m)) > d(x_m, x)$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lambda R(x_{n_k}, T(x_{n_k})) > d(x_{n_k}, x) \text{ for all } k \in \mathbb{N}.$$

This implies that $\lim_{k\to\infty} x_{n_k} = x$. Let y be an arbitrary point in $T(x)$. Since T is lower semicontinuous, for each $k \in \mathbb{N}$ there exists $y_{n_k} \in T(x_{n_k})$ such that $\lim_{k\to\infty} y_{n_k} = y$. Thus,

$$\begin{aligned} d(x, y) &\leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y) \\ &\leq d(x, x_{n_k}) + R(x_{n_k}, T(x_{n_k})) + d(y_{n_k}, y) \to 0 \text{ as } k \to \infty. \end{aligned}$$

This implies that $x = y$, and hence $T(x) = \{x\}$.

Case 2. There exists $n_0 \in \mathbb{N}$ such that $\lambda R(x_n, T(x_n)) \leq d(x_n, x)$ for all $n \geq n_0$. This implies that $H(T(x_n), T(x)) \leq d(x_n, x)$. Let v be an arbitrary point in $T(x)$. For each $n \in \mathbb{N}$, there exists $v_n \in T(x_n)$ such that $d(v, v_n) \leq \operatorname{dist}(v, T(x_n)) + \frac{1}{n}$. Thus,

$$\begin{aligned} d(x_n, v) &\leq d(x_n, v_n) + d(v_n, v) \\ &\leq R(x_n, T(x_n)) + H(T(x_n), T(x)) + \frac{1}{n} \end{aligned}$$

$$\leq \quad R(x_n, T(x_n)) + d(x_n, x) + \frac{1}{n}.$$

This implies that $\limsup_{n\to\infty} d(x_n, v) \leq \limsup_{n\to\infty} d(x_n, x)$. Since $x_n \xrightarrow{\Delta} x$, we have $v = x$. Therefore, $T(x) = \{x\}$. □

By applying Theorem 3.1, we prove a common endpoint theorem for a family of mappings satisfying generalized condition (C_λ).

Theorem 3.2. *Let C be a nonempty closed convex subset of X and $\{T_\alpha : \alpha \in \Omega\}$ a family of lower semicontinuous mappings from C into $\mathcal{CB}(C)$. Suppose that for each $\alpha \in \Omega$, there exists $\lambda \in (0, 1)$ such that T_α satisfies generalized condition (C_λ). If $\{T_\alpha : \alpha \in \Omega\}$ has a bounded approximate common endpoint sequence in C, then it has a common endpoint in C.*

Proof. Let $\{x_n\}$ be a bounded approximate common endpoint sequence of $\{T_\alpha : \alpha \in \Omega\}$. As we have observed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} x$. It follows from Theorem 3.1 that $T_\alpha(x) = \{x\}$ for all $\alpha \in \Omega$. Therefore x is a common endpoint of $\{T_\alpha : \alpha \in \Omega\}$. □

As a consequence of Theorem 3.2, we can obtain the following result. Notice that it is an extension of Theorem 3.5 in [33].

Corollary 3.3. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{CB}(C)$ a lower semicontinuous mapping. Suppose that T satisfies generalized condition (C_λ) for some $\lambda \in (0, 1)$. Then T has an endpoint if and only if T has a bounded approximate endpoint sequence in C.*

Now, we prove another common endpoint theorem which can be viewed as an extension of Theorem 3.2 in [23].

Theorem 3.4. *Let C be a nonempty closed convex subset of X and $\{T_\alpha : \alpha \in \Omega\}$ a family of lower semicontinuous mappings from C into $\mathcal{CB}(C)$. Suppose that for each $\alpha \in \Omega$, there exists $\lambda \in (0, 1)$ such that T_α satisfies generalized condition (C_λ). If there exist two disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of Ω such that $\mathcal{A} \cup \mathcal{B} = \Omega$, and for each $\alpha \in \mathcal{A}$, the mapping T_α has a bounded approximate endpoint sequence in $\cap_{\beta\in\mathcal{B}} End(T_\beta)$, then $\{T_\alpha : \alpha \in \Omega\}$ has a common endpoint in C.*

Proof. Fix $\alpha \in \mathcal{A}$ and let $\{x_n\}$ be a bounded approximate endpoint sequence of T_α in $\cap_{\beta\in\mathcal{B}} End(T_\beta)$. Without loss of generality, we may assume that $x_n \xrightarrow{\Delta} x$. According to Theorem 3.1, $x \in E(T_\alpha)$. Fix $\beta \in \mathcal{B}$ and let $y \in T_\beta(x)$. Since $\lambda R(x_n, T_\beta(x_n)) = 0 \leq d(x_n, x)$, $H(T_\beta(x_n), T_\beta(x)) \leq d(x_n, x)$. This implies that

$$\begin{aligned} d(y, x_n) &= \operatorname{dist}(y, T_\beta(x_n)) \\ &\leq H(T_\beta(x), T_\beta(x_n)) \\ &\leq d(x, x_n). \end{aligned}$$

Thus, $\limsup_{n\to\infty} d(y, x_n) \leq \limsup_{n\to\infty} d(x, x_n)$. Since $x_n \xrightarrow{\Delta} x$, we have $y = x$ for all $y \in T_\beta(x)$, and hence $T_\beta(x) = \{x\}$. This shows that x is a common endpoint of $\{T_\alpha : \alpha \in \Omega\}$. □

As a consequence of Theorem 3.4, we also obtain the following result.

Corollary 3.5. *Let C be a nonempty closed convex subset of X and $T, S : C \to \mathcal{CB}(C)$ be lower semicontinuous mappings. Suppose that T and S satisfy generalized condition (C_λ) for some $\lambda \in (0,1)$. If T has a bounded approximate endpoint sequence in $End(S)$, then T and S has a common endpoint in C.*

4. Convergence theorems

In 2011, Phuengrattana and Suantai [36] introduced an iteration process and called it the SP-iteration. They compared its convergence speed with the well-known Mann, Ishikawa and Noor iterations by showing that the SP-iteration converges faster than the others for the class of nondecreasing continuous functions on an interval. In this section, we prove Δ and strong convergence theorems of the SP-iteration for semi-nonexpansive mappings.

Let C be a nonempty convex subset of X, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$, and $T : C \to \mathcal{K}(C)$ be a multi-valued mapping. The sequence of SP-iteration [36] is defined by $x_1 \in C$,

$$z_n = \gamma_n x_n \oplus (1-\gamma_n)u_n, \;\; n \in \mathbb{N},$$

where $u_n \in T(x_n)$ such that $d(x_n, u_n) = R(x_n, T(x_n))$, and

$$y_n = \beta_n z_n \oplus (1-\beta_n)v_n, \;\; n \in \mathbb{N},$$

where $v_n \in T(z_n)$ such that $d(z_n, v_n) = R(z_n, T(z_n))$, and

$$x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)w_n, \;\; n \in \mathbb{N}, \tag{4.1}$$

where $w_n \in T(y_n)$ such that $d(y_n, w_n) = R(y_n, T(y_n))$.

A sequence $\{x_n\}$ in X is said to be Fejér monotone with respect to C [7] if

$$d(x_{n+1}, p) \le d(x_n, p) \;\text{ for all } p \in C \text{ and } n \in \mathbb{N}.$$

The following lemma shows that the sequence of SP-iteration defined by (4.1) is Fejér monotone with respect to the endpoint set of a semi-nonexpansive mapping.

Lemma 4.1. *Let C be a nonempty convex subset of X and $T : C \to \mathcal{K}(C)$ a semi-nonexpansive mapping. Let $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). Then $\{x_n\}$ is Fejér monotone with respect to $End(T)$. Hence, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in End(T)$.*

Proof. Let $p \in End(T)$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(z_n, p) &\le \gamma_n d(x_n, p) + (1-\gamma_n) d(u_n, p) \\ &\le \gamma_n d(x_n, p) + (1-\gamma_n) H(T(x_n), T(p)) \\ &\le d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(y_n, p) &\le \beta_n d(z_n, p) + (1-\beta_n) d(v_n, p) \\ &\le d(z_n, p) \le d(x_n, p). \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+1}, p) &\le \alpha_n d(y_n, p) + (1-\alpha_n) d(w_n, p) \\ &\le d(y_n, p) \le d(x_n, p). \end{aligned}$$

Thus, $\{x_n\}$ is Fejér monotone with respect to $End(T)$. □

The following fact can be found in [40].

Lemma 4.2. *Let C be a nonempty closed subset of X and $\{x_n\}$ a Fejér monotone sequence with respect to C. Then $\{x_n\}$ converges strongly to an element of C if and only if* $\lim_{n\to\infty} dist(x_n, C) = 0$.

The following fact is also needed.

Lemma 4.3. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ a mapping such that $I - T$ is semiclosed. If $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} R(x_n, T(x_n)) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in End(T)$, then $\omega_w(x_n) \subseteq End(T)$. Here $\omega_w(x_n) := \bigcup A(C, \{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(C, \{u_n\}) = \{u\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $v_n \xrightarrow{\Delta} v$. It follows from Lemma 2.9 and the semiclosedness of $I - T$ that $u = v \in End(T)$, which implies $\omega_w(x_n) \subseteq End(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(C, \{u_n\}) = \{u\}$ and let $A(C, \{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq End(T)$, $\{d(x_n, u)\}$ converges. By Lemma 2.9, $x = u$. This completes the proof. □

Now, we prove a Δ-convergence theorem.

Theorem 4.4. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ a semi-nonexpansive mapping such that $I - T$ is semiclosed. Let $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). Then $\{x_n\}$ Δ-converges to an endpoint of T.*

Proof. Let $p \in End(T)$. It follows from (2.1) that

$$\begin{aligned} d^2(z_n, p) &\leq \gamma_n d^2(x_n, p) + (1 - \gamma_n) d^2(u_n, p) - c_X \gamma_n (1 - \gamma_n) d^2(x_n, u_n) \\ &\leq d^2(x_n, p) - c_X \gamma_n (1 - \gamma_n) d^2(x_n, u_n) \end{aligned}$$

and

$$\begin{aligned} d^2(y_n, p) &\leq \beta_n d^2(z_n, p) + (1 - \beta_n) d^2(v_n, p) - c_X \beta_n (1 - \beta_n) d^2(z_n, v_n) \\ &\leq d^2(z_n, p) - c_X \beta_n (1 - \beta_n) d^2(z_n, v_n). \end{aligned}$$

This implies that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \alpha_n d^2(y_n, p) + (1 - \alpha_n) d^2(w_n, p) - c_X \alpha_n (1 - \alpha_n) d^2(y_n, w_n) \\ &\leq d^2(y_n, p) - c_X \alpha_n (1 - \alpha_n) d^2(y_n, w_n) \\ &\leq d^2(x_n, p) - c_X \gamma_n (1 - \gamma_n) d^2(x_n, u_n). \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} c_X a^2 (1 - b) d^2(x_n, u_n) < \infty.$$

Also, $\lim_{n\to\infty} d^2(x_n, u_n) = 0$, and hence

$$\lim_{n\to\infty} R(x_n, T(x_n)) = \lim_{n\to\infty} d(x_n, u_n) = 0. \tag{4.2}$$

By Lemma 4.1, $\{d(x_n, p)\}$ converges for all $p \in End(T)$. By Lemma 4.3, $\omega_w(x_n)$ consists of exactly one point and is contained in $End(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $End(T)$. □

As a consequence of Proposition 2.3 and Theorem 4.4, we can obtain the following result.

Corollary 4.5. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ a lower semicontinuous mapping with $End(T) \neq \emptyset$. Suppose that T satisfies generalized condition (C_λ) for some $\lambda \in (0, 1)$. Let $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). Then $\{x_n\}$ Δ-converges to an endpoint of T.*

Next, we prove strong convergence theorems. Recall that a mapping $T : C \to \mathcal{K}(C)$ is said to satisfy condition (J) [32] if $End(T) \neq \emptyset$ and there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0, \infty)$ such that

$$R(x, T(x)) \geq g(\operatorname{dist}(x, End(T))) \text{ for all } x \in C.$$

The mapping T is said to be semicompact if for any sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} R(x_n, T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = q \in C$.

Theorem 4.6. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ a semi-nonexpansive mapping such that $I - T$ is semiclosed. Let $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). If T satisfies condition (J), then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. By Lemma 2.10, $End(T)$ is closed. Since T satisfies condition (J), by (4.2) we get that $\lim_{n\to\infty} \operatorname{dist}(x_n, End(T)) = 0$. By Lemma 4.1, $\{x_n\}$ is Fejér monotone with respect to $End(T)$. The conclusion follows from Lemma 4.2. □

Corollary 4.7. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ a lower semicontinuous mapping. Suppose that T satisfies generalized condition (C_λ) for some $\lambda \in (0, 1)$. Let $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). If T satisfies condition (J), then $\{x_n\}$ converges strongly to an endpoint of T.*

Theorem 4.8. *Let C be a nonempty convex subset of X and $T : C \to \mathcal{K}(C)$ a semi-nonexpansive mapping. Let $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). If T is semicompact and lower semicontinuous, then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. By (4.2) and the semicompactness of T, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in C$. Let v be an arbitrary point in $T(q)$. Since T is lower semicontinuous, for each $k \in \mathbb{N}$ there exists v_{n_k} in $T(x_{n_k})$ such that $\lim_{k\to\infty} v_{n_k} = v$. This implies that

$$\begin{aligned} d(q, v) &\leq d(q, x_{n_k}) + d(x_{n_k}, v_{n_k}) + d(v_{n_k}, v) \\ &\leq d(q, x_{n_k}) + R(x_{n_k}, T(x_{n_k})) + d(v_{n_k}, v) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus, $v = q$ for all $v \in T(q)$. Therefore $q \in End(T)$. By Lemma 4.1, $\lim_{n\to\infty} d(x_n, q)$ exists and hence q is the strong limit of $\{x_n\}$. □

As a consequence of Proposition 2.3 and Theorem 4.8, we can obtain the following result.

Corollary 4.9. *Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ a lower semicontinuous mapping with $End(T) \neq \emptyset$. Suppose that T satisfies generalized condition (C_λ) for some $\lambda \in (0, 1)$. Let $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of SP-iteration defined by (4.1). If T is semicompact, then $\{x_n\}$ converges strongly to an endpoint of T.*

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References

[1] M. Abbas and F. Khojasteh, *Common f-endpoint for hybrid generalized multi-valued contraction mappings*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **108** (2014), 369–375.

[2] A. Abkar and M. Eslamian, *Common fixed point results in CAT(0) spaces*, Nonlinear Anal. **74** (2011), 1835–1840.

[3] B. Ahmad, S. K. Ntouyas and J. Tariboon, *A study of mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions via endpoint theory*, Appl. Math. Lett. **52** (2016), 9–14.

[4] A. Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal. **72** (2010), 132–134.

[5] D. Ariza-Ruiz, G. López-Acedo and A. Nicolae, *The asymptotic behavior of the composition of firmly nonexpansive mappings*, J Optim. Theory Appl. **167** (2015), 409–429.

[6] S. Banach, *Sur les operations dans les ensembles abstraits et leurs applications*, Fund. Math. **3** (1922), 133–181.

[7] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[8] L. Chen, L. Gao and D. Chen, *Fixed point theorems of mean nonexpansive set-valued mappings in Banach spaces*, J. Fixed Point Theory Appl. **19** (2017), 2129–2143.

[9] P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen, *Convergence theorems and approximating endpoints for multivalued Suzuki mappings in hyperbolic spaces*, J. Comp. Anal. Appl. **28** (2020), 903–916.

[10] H. W. Corley, *Some hybrid fixed point theorems related to optimization*, J. Math. Anal. Appl. **120** (1986), 528–532.

[11] S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56** (2008), 2572–2579.

[12] R. Espínola, M. Hosseini and K. Nourouzi, *On stationary points of nonexpansive set-valued mappings*, Fixed Point Theory Appl. **2015** (2015): :236.

[13] R. Espínola, P. Lorenzo and A. Nicolae, *Fixed points, selections and common fixed points for nonexpansive-type mappings*, J. Math. Anal. Appl. **382** (2011), 503–515.

[14] J. García-Falset, E. Llorens-Fuster and T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl. **375** (2011), 185–195.

[15] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[16] G. Haddad, *Monotone viable trajectories for functional-differential inclusions*, J. Differential Equ. **42** (1981), 1–24.

[17] M. Hosseini, K. Nourouzi, D. O'Regan, *Stationary points of set-valued contractive and nonexpansive mappings on ultrametric spaces*, Fixed Point Theory **19** (2018), 587–594.

[18] A. Kaewcharoen and B. Panyanak, *Fixed point theorems for some generalized multivalued nonexpansive mappings*, Nonlinear Anal. **74** (2011), 5578–5584.

[19] P. Q. Khanh and V. S. T. Long, *Invariant-point theorems and existence of solutions to optimization-related problems*, J. Global Optim. **58** (2014), 545–564.

[20] H. Khatibzadeh, M. R. Piranfar and J. Rooin, *Convergence of dissipative-like dynamics and algorithms governed by set-valued nonexpansive mappings*, Bull. Malays. Math. Sci. Soc. **44** (2021), 1101–1121.

[21] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.

[22] A. Kudtha and B. Panyanak, *Common endpoints for Suzuki mappings in uniformly convex hyperbolic spaces*, Thai J. Math. **2018**(2018), 159–168.

[23] T. Laokul and B. Panyanak, *Common endpoints for non-commutative Suzuki mappings*, Thai J. Math. **17** (2019), 821–828.

[24] T. Laokul and B. Panyanak, *A generalization of the (CN) inequality and its applications*, Carpathian J. Math. **36** (2020), 81–90.

[25] W. Laowang and B. Panyanak, *Common fixed points for some generalized multivalued nonexpansive mappings in uniformly convex metric spaces*, Fixed Point Theory Appl. **2011** (2011): 20.

[26] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–487.

[27] A. Naor and L. Silberman, *Poincare inequalities, embeddings, and wild groups*, Compos. Math. **147** (2011), 1546–1572.

[28] D. M. Oyetunbi and A. R. Khan, *Approximating common endpoints of multivalued generalized nonexpansive mappings in hyperbolic spaces*, Appl. Math. Comput. **392** (2021): Paper No. 125699.

[29] B. Panyanak, *Endpoints of multivalued nonexpansive mappings in geodesic spaces*, Fixed Point Theory Appl. **2015** (2015): 147.

[30] B. Panyanak, *On an open problem of Kyung Soo Kim*, Fixed Point Theory Appl. **2015** (2015): 186.

[31] B. Panyanak, *The demiclosed principle for multi-valued nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal. **17** (2016), 2063–2070.

[32] B. Panyanak, *Approximating endpoints of multi-valued nonexpansive mappings in Banach spaces*, J. Fixed Point Theory Appl. **20** (2018): Paper No. 77.

[33] B. Panyanak, *Stationary points of lower semicontinuous multifunctions*, J. Fixed Point Theory Appl. **22**:43 (2020), 1–12.

[34] B. Panyanak, *Endpoints of generalized Berinde nonexpansive mappings in hyperbolic spaces*, Fixed Point Theory **24** (2023), 13 pp.

[35] B. Panyanak and S. Suantai, *Diametrically regular mappings and Browder's theorem without the endpoint condition*, Numer. Funct. Anal. Optim. **41** (2020), 495–505.

[36] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math. **235** (2011), 3006–3014.

[37] J. Puiwong and S. Saejung, *On convergence theorems for single-valued and multi-valued mappings in p-uniformly convex metric spaces*, Carpathian J. Math. **37** (2021), 513–527.

[38] A. Razani and H. Salahifard, *Invariant approximation for CAT(0) spaces*, Nonlinear Anal. TMA. **72** (2010), 2421–2425.

[39] S. Saejung, *Remarks on endpoints of multivalued mappings in geodesic spaces*, Fixed Point Theory Appl. **2016** (2016): 52.

[40] S. Saejung, S. Suantai and P. Yotkaew, *A note on "Common fixed point of multistep Noor iteration with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings"*, Abstr. Appl. Anal. **2009** (2009): Article ID 283461.

[41] E. Tarafdar, P. Watson and X. Z. Yuan, *Poincare's recurrence theorems for set-valued dynamical systems*, Appl. Math. Lett. **10** (1997), 37–44.

[42] E. Tarafdar and X. Z. Yuan, *The set-valued dynamic system and its applications to Pareto optima*, Acta Appl. Math. **46** (1997), 93–106.

[43] M. Turinici, *Differential Lipschitzianness tests on abstract quasimetric spaces*, Acta Math. Hungar. **41** (1983), 93–100.

[44] I. Uddin, S. Aggarwal and A. A. N. Abdou, *Approximation of endpoints for α-Reich-Suzuki nonexpansive mappings in hyperbolic metric spaces*, Mathematics **9** (2021): 1692.

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