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UPPER BOUNDS FOR VECTOR EQUILIBRIUM PROBLEMS ASSOCIATED WITH p-ORDER CONE ON HADAMARD MANIFOLDS

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Abstract. In this paper, we study a new class of vector equilibrium problems associated with partial order provided by p-order cone on Hadamard manifolds. We first propose a new concept of $\mathcal{K}_p^k$-convexity of a vector-valued function in the setting of Hadamard manifolds and derive some regularized gap functions of the concerning problem. Then, several upper bounds for vector equilibrium problems associated with p-order cone are established via regularized gap functions. At last, we present some examples to illustrate our main results in the paper.

1. Motivation and introduction

The target problem is a class of vector equilibrium problems with partial order provided by p-order cone, particularly in the setting of Hadamard manifolds. For notational convenience, this problem is denoted by $\mathrm{VEP}(\mathcal{D}, G, \mathcal{K}_p^k)$, which is to find $u \in \mathcal{D}$ such that

$$G(u,v) \in \mathcal{K}_p^k \tag{1.1}$$

for all $v \in \mathcal{D}$, where $\mathcal{D}$ is a nonempty subset of the Hadamard manifold $\mathcal{H}$ (see Section 2 for details), $G\colon \mathcal{D}\times\mathcal{D} \to \mathbb{R}^k$ is a vector-valued function satisfying $G(u,u) = 0_{\mathbb{R}^k}$ for all $u \in \mathcal{D}$, and $\mathcal{K}_p^k$ is the p-order cone given by

$$\mathcal{K}_p^k := \left\{ w = (w_1, ..., w_k)^\top \in \mathbb{R}^k \,\middle|\, w_1 \geq \left(\sum_{j=2}^{k} |w_j|^p \right)^{\frac{1}{p}} \right\} \quad (p > 1). \tag{1.2}$$

If we write $w = (w_1, \hat{w})^\top \in \mathbb{R}\times\mathbb{R}^{k-1}$, the p-order cone $\mathcal{K}_p^k$ is equivalently expressed as

$$\mathcal{K}_p^k := \left\{ w = (w_1, \hat{w})^\top \in \mathbb{R} \times \mathbb{R}^{k-1} \,|\, w_1 \geq \|\hat{w}\|_p \right\} \quad (p > 1).$$

The p-order cone was introduced by Xue and Ye [32] and Miao et al. [24], which is an extension of the second-oder cone, corresponding to $p = 2$ in (1.2). The p-order cone $\mathcal{K}_p^k$ is closed convex on $\mathbb{R}^k$. In addition, it has been used in complementarity problems and variational inequalities under the form of constraint sets, see [23, 24, 34] and references therein. When $\mathcal{H} = \mathbb{R}^m$, the $\mathrm{VEP}(\mathcal{D}, G, \mathcal{K}_p^k)$ becomes a vector

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equilibrium problem in the admissible space $\mathbb{R}^m$ with partial order provided by p-order cone. To the best of our knowledge, this special type of problems has not been considered.

It is well known that the vector equilibrium problems include some famous problems as special cases, like vector variational inequalities, vector complementarity problems and vector optimization problems, etc. Various equilibrium problems have significant applications in the areas of network analysis, transportation, mechanics, economics, finance and operations research, see [7, 10, 17] and references therein.

On the other hand, numerous important concepts and techniques of nonlinear analysis used in optimization have been extended from Euclidean spaces to Riemannian manifolds, see [31]. From the perspective of Riemannian geometry, such extension offers certain advantages, especially some originally nonconvex and nonsmooth problems can be regarded as convex and smooth ones under some settings, see [8, 26]. In 2012, Colao et al. [6] introduced a class of "scalar" equilibrium problems in the setting of Hadamard manifolds, where the solution existence was investigated with certain coercivity condition. Batista et al. [4] further generalized it to the "vector" equilibrium problem on Hadamard manifolds and studied a sufficient condition for its the existence of solution. Thereafter, various kinds of vector equilibrium problems have been proposed and investigated, see [13, 14, 20, 21].

The upper bound, so-called error bound, is known as an upper error estimate of the distance from an arbitrary feasible point to the solutions set of a certain problem. From the viewpoint of theoretical analysis, upper bounds have been significant to studying the convergence of iterative algorithms for solving variational inequalities, complementarity problems and equilibrium problems. In 2003, Mastroeni [22] developed upper bounds for "scalar" equilibrium problems where the used main tool lies on various gap functions, originally introduced by Auslender [3]. An advantage of using gap functions is that we can reformulate the variational inequalities and equilibrium problems into equivalent optimization problems. Due to the feature of non-differentiability of Auslender gap functions in general, Fukushima [9], and Yamashita and Fukushima [33] introduced the notion of regularized gap functions for variational inequalities. They also obtained upper bounds controlled by gap functions in forms of the Fukushima regularization and the Moreau-Yosida regularization. In light of that, a large number of works discussed regularized gap functions and upper bounds for variational inequalities, hemivariational-variational inequalities, and equilibrium problems, see [1, 5, 11, 12, 18, 28–30]. In particular, Khan and Chen [18] studied regularized gap functions and upper bounds for a class of "vector" equilibrium problems with partial order provided by the usual positive cone in finite dimensional spaces. Very recently, Anh et al. [1] and Hung et al. [12] extended the results [18] to generalized mixed vector equilibrium problems with partial order given by a infinite dimensional cone. Hung et al. [15] also characterized illustrating upper bounds using regularized gap functions for vector equilibrium problems, whose final space is partially ordered by a "polyhedral cone" generated by some matrix. In the setting of Hadamard manifolds, only limited results on this interesting topic for variational inequalities and equilibrium problems of "vector" types. Jayswal et al. [16] established a gap function for a non-smooth vector variational inequality on Hadamard manifolds. Recently, Hung et al. [14] developed regularized gap functions and upper bounds for vector equilibrium problems and vector variational inequalities on Hadamard manifolds. The partial order of the vector problems considered in [14, 16] is induced from the usual "positive cone".

Motivated by the aforementioned works, this paper focuses on a class of vector equilibrium problems VEP($\mathcal{D}, G, \mathcal{K}_p^k$), which may be viewed as a follow-up of [14,15]. The novelty of setting to VEP($\mathcal{D}, G, \mathcal{K}_p^k$) is that the space is with a partial order induced from "p-order cone" and the admissible space is the Hadamard manifolds. As below, we highlight the contributions of the paper:

- In contrast to the literature, we study several gap regularized functions of the Fukushima and Moreau-Yosida types to the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$). Our main tool in using of the gap regularized functions is based on a new concept of $\mathcal{K}_p^k$-convexity of a vector-valued function on Hadamard manifolds.
- The novelty of this paper is to derive upper bounds for problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$) controlled by these regularized gap functions and depend on the data of the component functions of the vector-valued cost function.

The paper is structured as follows. Section 2 provides several definitions, notations and properties of manifolds. In Section 3, we introduce a new concept of $\mathcal{K}_p^k$-convexity of a vector-valued function on Hadamard manifolds. Then, we derive some regularized gap functions and upper bounds for VEP($\mathcal{D}, G, \mathcal{K}_p^k$) under suitable conditions. Furthermore, some examples for illustrating our main results are provided. Finally, we give some conclusions of this paper in Section 4.

2. Preliminaries and notations

In this section, we review some basic definitions, concepts, and properties of manifolds that will be used in subsequent contents. They can be found in any standard book on Riemannian geometry [27, 31].

Let $\mathcal{H}$ be an m-dimensional differentiable manifold. The *tangent space* of $\mathcal{H}$ at z is denoted by $T_z\mathcal{H}$ and the *tangent bundle* of $\mathcal{H}$ is denoted by $T\mathcal{H} = \bigcup_{z\in\mathcal{H}} T_z\mathcal{H}$. An inner product $\langle\cdot,\cdot\rangle_{\mathcal{R}_z}$ on $T_z\mathcal{H}$ is called a *Riemannian metric* on $T_z\mathcal{H}$. A tensor field $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ is said to be a Riemannian metric on $\mathcal{H}$ if for every $z \in \mathcal{H}$, the tensor $\langle\cdot,\cdot\rangle_{\mathcal{R}_z}$ is a Riemannian metric on $T_z\mathcal{H}$, where the subscript z can be omitted if no confusion occurs.

A *Riemannian manifold*, denoted by $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{R}})$, is a differentiable manifold $\mathcal{H}$ endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{R}}$. Given a piecewise smooth curve $\delta : [a, b] \to \mathcal{H}$ joining z to w, that is, $\delta(a) = z$ and $\delta(b) = w$, the length of δ is defined by $l_{\mathcal{R}}(\delta) := \int_a^b \|\delta'(s)\|ds$. For any $z, w \in \mathcal{H}$, the Riemannian distance $d_{\mathcal{R}}(z, w)$, which induces the original topology on $\mathcal{H}$, is defined by minimizing this length over the set of all such curves joining z to w.

The Levi-Civita connection ∇ associated with the Riemannian metric and δ be a smooth curve in $\mathcal{H}$. A vector field Z is said to be *parallel* along δ if $\nabla_{\delta'} Z = \mathbf{0}$, where $\mathbf{0}$ denotes the zero tangent vector. If δ' itself is parallel along δ, we say that δ is a geodesic. A geodesic joining z to w in $\mathcal{H}$ is said to be *minimal* if its length equals $d_{\mathcal{R}}(z,w)$. A Riemannian manifold is *complete* if its geodesics $\delta(s)$ are defined for all $s \in \mathbb{R}$. It follows from Hopf-Rinow Theorem that if $\mathcal{H}$ is complete, then any point in $\mathcal{H}$ can be joined by a minimal geodesic. Furthermore, $(\mathcal{H}, d_{\mathcal{R}})$ is a complete metric space, and hence bounded closed subsets in $\mathcal{H}$ are compact. A *Hadamard manifold* is a complete simply connected Riemannian manifold of nonpositive sectional curvature.

Assume that $\mathcal{H}$ is a Hadamard manifold. The *exponential map* $\exp_z \colon T_z\mathcal{H} \to \mathcal{H}$ at $z \in \mathcal{H}$ is defined by $\exp_z(v) = \delta_v(1,z)$ for each $v \in T_z\mathcal{H}$, where $\delta(\cdot) = \delta_v(\cdot, z)$ is the geodesic starting at z with velocity v, that is, $\delta(0) = z$ and $\delta'(0) = v$. It is easy to see that $\exp_z(sv) = \delta_v(z)$ for each real number s. Moreover, the exponential map $\exp_z : T_z\mathcal{H} \to \mathcal{H}$ is a diffeomorphism for all $z \in \mathcal{H}$. For $z \in \mathcal{H}$, $\exp_z^{-1} : \mathcal{H} \to T_z\mathcal{H}$ is the inverse of the exponential map. For any $z, w \in \mathcal{H}$, we have

$$\| \exp_z^{-1}(w)\|_{\mathcal{R}} = d_{\mathcal{R}}(z,w). \tag{2.1}$$

For any two distinct points $z, w \in \mathcal{H}$, there exists a unique normalized geodesic δ joining z to w such that $\delta(s) = \exp_z(s \exp_z^{-1} w)$ for all $s \in [0,1]$. In particular, the exponential map and its inverse are continuous on a Hadamard manifold.

Unless otherwise stated, $\mathcal{H}$ is a Hadamard manifold in the rest of the section.

Definition 2.1. (see [31]) A set $\mathcal{D} \subset \mathcal{H}$ is said to be *geodesic convex* if for any two distinct points z and w in $\mathcal{D}$, the geodesic joining z to w is contained in $\mathcal{D}$, that is, if $\delta : [0,1] \to \mathcal{H}$ is a geodesic such that $z = \delta(0)$ and $w = \delta(1)$, then $\delta(\lambda) = \exp_z(\lambda \exp_z^{-1} w) \in \mathcal{D}$ for all $\lambda \in [0,1]$.

Lemma 2.2. (see [2], p.3) *For $z, w \in \mathcal{H}$, $r \in (0,1)$ and a point $u_r = \delta(r) = \exp_z(r \exp_z^{-1} w)$ on the geodesic $\delta : [0,1] \to \mathcal{H}$ joining z to w, we have $\exp_z^{-1}(u_r) = r \exp_z^{-1} w$.*

Definition 2.3. (see [31]) A real-valued function $\rho \colon \mathcal{H} \to \mathbb{R}$ is said to be *geodesic convex* if, for any $z, w \in \mathcal{H}$ and $\lambda \in [0,1]$, there holds

$$\rho\left(\exp_z\left(\lambda \exp_z^{-1} w\right)\right) \le (1-\lambda)\rho(z) + \lambda\rho(w).$$

Lemma 2.4. (see [19]) *Let $z_0 \in \mathcal{H}$ and $\{z_n\}$ be a sequence in $\mathcal{H}$ such that $z_n \to z_0$. Then, the following assertions hold.*

(i) *For any $w \in \mathcal{H}$, $\exp_{z_n}^{-1} w \to \exp_{z_0}^{-1} w$ and $\exp_w^{-1} z_n \to \exp_w^{-1} z_0$;*
(ii) *If $\{u_n\}$ is a sequence such that $u_n \in T_{z_n}\mathcal{H}$ and $u_n \to u_0$, then $u_0 \in T_{z_0}\mathcal{H}$;*
(iii) *Given sequences $\{u_n\}$ and $\{t_n\}$ satisfying $u_n, t_n \in T_{z_n}\mathcal{H}$, if $u_n \to u_0$ and $t_n \to t_0$, then $\langle u_n, t_n\rangle_{\mathcal{R}} \to \langle u_0, t_0\rangle_{\mathcal{R}}$.*

3. Main results

In this section, we introduce the notion of the $\mathcal{K}_p^k$-convexity of a vector-valued function on Hadamard manifolds associated with p-order cone and the exact definition of gap functions for VEP$(\mathcal{D}, G, \mathcal{K}_p^k)$. Then regularized gap functions and upper bounds for VEP$(\mathcal{D}, G, \mathcal{K}_p^k)$ will be considered under some suitable assumptions on the data of the problem. Furthermore, some examples are given for illustrating our main results.

Throughout the paper, unless other specified, let the function $G\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}^k$ be defined by

$$G(u,v) = (G_1(u,v), ..., G_k(u,v))^\top \in \mathbb{R}^k,$$

where $G_j\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ for all $j \in \{1, ..., k\}$ and $u, v \in \mathcal{D}$. Denote the solution set of the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$) by $\mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$. Since the existence of solutions for vector equilibrium problems on Hadamard manifolds with partial order provided by closed convex cones have been well investigated, see [4, 13] and the references therein, we always assume that $\mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$ is a nonempty set.

Definition 3.1. For each $j \in \{1, ..., k\}$, let $F_j\colon \mathcal{H} \to \mathbb{R}$ be a function. A function $F := (F_1, ..., F_k)^\top$ defined by $F(z) = (F_1(z), ..., F_k(z))^\top \in \mathbb{R}^k$ is said to be $\mathcal{K}_p^k$-*convex* if, for all $z, v \in \mathcal{H}$ and $\lambda \in [0,1]$,

$$(1-\lambda)F(z) + \lambda F(v) - F\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right) \in \mathcal{K}_p^k. \tag{3.1}$$

Remark 3.2. We point out couple things regarding this new concept.

(i): Let $q > 1$ satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$. By applying Hölder's inequality, the condition (3.1) leads to

$$\begin{aligned}
&(1-\lambda)F_1(z) + \lambda F_1(v) - F_1\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right)\\
\geq &\left(\sum_{j=2}^{k} \left|(1-\lambda)F_j(z) + \lambda F_j(v) - F_j\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right)\right|^p\right)^{\frac{1}{p}}\\
\geq &(k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} \left[(1-\lambda)F_j(z) + \lambda F_j(v) - F_j\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right)\right],
\end{aligned}$$

that is,

$$\begin{aligned}
&(k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} F_j\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right) - F_1\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right)\\
\geq &(k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} [(1-\lambda)F_j(z) + \lambda F_j(v)] - [(1-\lambda)F_1(z) + \lambda F_1(v)] \qquad (3.2)
\end{aligned}$$

for all $z, v \in \mathcal{H}$ and $\lambda \in [0,1]$.

(ii): If $F_j \equiv 0$ for all $j \in \{2, ..., k\}$, then we can take $F \equiv F_1$ and it follows from (3.2) that

$$F\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right) \leq (1-\lambda)F(z) + \lambda F(v),$$

for all $z, v \in \mathcal{H}$ and $\lambda \in [0,1]$. Thus, the $\mathcal{K}_p^k$-convexity of F reduces the geodesic convexity in Definition 2.3.

Proposition 3.3. *Suppose that the function $F := (F_1, ..., F_k)^\top$ is given in Definition 3.1. If F_1 is a geodesic convex function and F_j is a linear affine function (i.e., F_j and $-F_j$ are geodesic convex) for all $j \in \{2, ..., k\}$, then the function F is $\mathcal{K}_p^k$-convex.*

Proof. Let $z, v \in \mathcal{H}$ and $\lambda \in [0, 1]$. For each $j \in \{2, ..., k\}$, since the function F_j is linear affine, we have

$$(1-\lambda)F_j(z) + \lambda F_j(v) - F_j\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right) = 0.$$

Moreover, F_1 is a geodesic convex function, i.e., we obtain

$$(1-\lambda)F_1(z) + \lambda F_1(v) - F_1\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right) \geq 0.$$

Therefore, the following inequality

$$\begin{aligned}&(1-\lambda)F_1(z) + \lambda F_1(v) - F_1\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right)\\ \geq &\left(\sum_{j=2}^{k} \left|(1-\lambda)F_j(z) + \lambda F_j(v) - F_j\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right)\right|^p\right)^{\frac{1}{p}}\end{aligned}$$

holds for all $p > 1$. This implies that

$$(1-\lambda)F(z) + \lambda F(v) - F\left(\exp_z\left(\lambda \exp_z^{-1} v\right)\right) \in \mathcal{K}_p^k$$

that is, F is $\mathcal{K}_p^k$-convex. □

The following example illustrates the $\mathcal{K}_p^k$-convexity of a vector-valued function on Hadamard manifolds.

Example 3.4. The set $\mathcal{H} = \mathbb{R}_{++} = \{u \in \mathbb{R} \mid u > 0\}$ endowed with the Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ defined by $\langle z, v \rangle_{\mathcal{R}} = g(u)zv$ for all $z, v \in T_u\mathcal{H}$ (where $g(u) = \frac{1}{u^2}$) is a Riemannian manifold, $\phi\colon \mathbb{R} \to \mathcal{H}$, $\phi(u) = e^u$ is an isometry, and the sectional curvature of $\mathcal{H}$ is 0. Moreover, the tangent plane at $u \in \mathcal{H}$ is $T_u\mathcal{H} = \mathbb{R}$. For any $u, w \in \mathcal{H}$ the Riemannian distance is $d_{\mathcal{R}}\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$,

$$d_{\mathcal{R}}(u, w) = |\phi^{-1}(u) - \phi^{-1}(w)| = \left|\ln\left(\frac{u}{w}\right)\right|.$$

Thus, $\mathcal{H}$ is a Hadamard manifold (see e.g., [25]). The geodesic curve $\delta\colon \mathbb{R} \to \mathcal{H}$ starting form u ($\delta(0) = u$) will have the equation $\delta(s) = ue^{(\frac{w}{u})s}$, with $w = \delta'(0) \in T_u\mathcal{H}$ as the tangent unit vector of δ in the starting point. Then, the exponential map $\exp_u$ and the inverse exponential map $\exp_u^{-1}$ are defined by

$$\exp_u(sw) = ue^{(\frac{w}{u})s} \text{ and } \exp_u^{-1} w = u \ln\left(\frac{w}{u}\right).$$

Let the functions $F_1\colon \mathcal{H} \to \mathbb{R}$ and $F_j\colon \mathcal{H} \to \mathbb{R}$ ($j \in \{2, ..., k\}$) be defined by

$$F_1(u) = 3\ln^2 u + 1; \quad F_j(u) = \frac{5}{2}\ln u + 3, \quad \forall u \in \mathcal{H}.$$

Then, for any $u, w \in \mathcal{H}$, $\lambda \in [0, 1]$, we obtain

$$\begin{aligned}F_1\left(\exp_u\left(\lambda \exp_u^{-1} w\right)\right) &= 3\ln^2(u^{1-\lambda}w^\lambda) + 1\\ &= 3[(1-\lambda)\ln u + \lambda \ln w]^2 + 1\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\lambda)\left(3\ln^2 u + 1\right) + \lambda\left(3\ln^2 w + 1\right) \\
&= (1-\lambda)F_1(u) + \lambda F_1(w). \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
F_j\left(\exp_u\left(\lambda \exp_u^{-1} w\right)\right) &= \frac{5}{2}\ln(u^{1-\lambda}w^{\lambda}) + 3 \\
&= (1-\lambda)\left(\frac{5}{2}\ln u + 3\right) + \lambda\left(\frac{5}{2}\ln w + 3\right) \\
&= (1-\lambda)F_j(u) + \lambda F_j(w). \qquad (3.4)
\end{aligned}
$$

From (3.3) and (3.4), we get that the function F_1 is geodesic convex and the function F_j is linear affine for all $j \in \{2, ..., k\}$. However, it is easy to see that the function F_j is non-convex in the usual sense for all $j \in \{2, ..., k\}$.

Let us consider the function $F := (F_1, ..., F_k)^\top$ defined by $F(u) = (F_1(u), ..., F_k(u))^\top$ for all $u \in \mathcal{H}$. Using Proposition 3.3, we conclude that the function F is $\mathcal{K}_p^k$-convex.

We introduce the exact definition of gap functions for VEP($\mathcal{D}, G, \mathcal{K}_p^k$) as below.

Definition 3.5. A real function $\mathbf{m}\colon \mathcal{D} \to \mathbb{R}$ is said to be a *gap function* for VEP($\mathcal{D}, G, \mathcal{K}_p^k$) if the following properties hold:

(a): $\mathbf{m}(u) \geq 0$, for all $u \in \mathcal{D}$;
(b): for any $u^* \in \mathcal{D}$, $\mathbf{m}(u^*) = 0$ if and only if u^* is a solution of VEP($\mathcal{D}, G, \mathcal{K}_p^k$).

We now propose a remark to convert the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$) into a scalar equilibrium problem.

Remark 3.6. Suppose that the function $G\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}^k$ is defined by $G = (G_1, ..., G_k)^\top$ where $G_j\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ for all $j \in \{1, ..., k\}$. Then, thanks to the definition of the p-order cone $\mathcal{K}_p^k$ in (1.2), we can verify that (1.1) is equivalent to

$$
G_1(u,v) - \left(\sum_{j=2}^{k} |G_j(u,v)|^p\right)^{\frac{1}{p}} \geq 0.
$$

For each fixed constant $\xi > 0$, let us consider the following function $\Phi_\xi : \mathcal{D} \to \mathbb{R}$ defined by

$$
\Phi_\xi(u) = \sup_{v\in\mathcal{D}} \left[\left(\sum_{j=2}^{k} |G_j(u,v)|^p\right)^{\frac{1}{p}} - G_1(u,v) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u,v)\right], \qquad (3.5)
$$

for all $u \in \mathcal{D}$.

We are ready to assert that Φ_ξ is a gap function of VEP($\mathcal{D}, G, \mathcal{K}_p^k$).

Theorem 3.7. *Suppose that $\mathcal{D}$ is a closed and geodesic convex set and G is $\mathcal{K}_p^k$-convex in the second component. Then the function Φ_ξ defined by* (3.5) *for any $\xi > 0$ is a gap function of* VEP($\mathcal{D}, G, \mathcal{K}_p^k$).

Proof. For any fixed parameter $\xi > 0$, we shall prove that Φ_ξ satisfies the conditions of Definition 3.5.

(a) Let $u \in \mathcal{D}$ be arbitrary. Since $G_j(u,u) = 0$ for all $j \in \{1, ..., k\}$ and $u \in \mathcal{D}$, it follows from the definition of Φ_ξ that

$$\Phi_\xi(u) \geq \left(\sum_{j=2}^{k} |G_j(u,u)|^p\right)^{\frac{1}{p}} - G_1(u,u) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u,u) = 0.$$

(b) Suppose $u^* \in \mathcal{D}$ such that $\Phi_\xi(u^*) = 0$, namely,

$$\sup_{v \in \mathcal{D}} \left[\left(\sum_{j=2}^{k} |G_j(u^*,v)|^p\right)^{\frac{1}{p}} - G_1(u^*,v) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u^*,v)\right] = 0.$$

This implies

$$\left(\sum_{j=2}^{k} |G_j(u^*,v)|^p\right)^{\frac{1}{p}} - G_1(u^*,v) \leq \frac{1}{2\xi} d_{\mathcal{R}}^2(u^*,v),$$

for all $v \in \mathcal{D}$. For any $u \in \mathcal{D}$ and $\lambda \in (0,1)$, we set

$$v_\lambda := \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right).$$

As $\mathcal{D}$ is geodesic convex, $v_\lambda \in \mathcal{D}$, and hence

$$\begin{aligned}
&\left(\sum_{j=2}^{k} |G_j\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right)|^p\right)^{\frac{1}{p}} - G_1\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right) \\
&\leq \frac{1}{2\xi} d^2\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right) \\
&= \frac{1}{2\xi} \left\|\exp_{u^*}^{-1}\left[\exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right]\right\|_{\mathcal{R}}^2 \\
&= \frac{1}{2\xi} \left\|\lambda \exp_{u^*}^{-1} u\right\|_{\mathcal{R}}^2 = \frac{\lambda^2}{2\xi} d_{\mathcal{R}}^2(u^*,u). \qquad (3.6)
\end{aligned}$$

Applying the Hölder's inequality, the $\mathcal{K}_p^k$-convexity of $v \mapsto G(u^*,v)$ and (3.2) with the condition $G_j(u^*,u^*) = 0$ for all $j \in \{1, ..., k\}$, we have

$$\begin{aligned}
&\left(\sum_{j=2}^{k} |G_j\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right)|^p\right)^{\frac{1}{p}} - G_1\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right) \\
&\geq (k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} G_j\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} v\right)\right) - G_1\left(u^*, \exp_{u^*}\left(\lambda \exp_{u^*}^{-1} u\right)\right) \\
&\geq (k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} [(1-\lambda) G_j(u^*,u^*) + \lambda G_j(u^*,u)] - [(1-\lambda) G_1(u^*,u^*) + \lambda G_1(u^*,u)]
\end{aligned}$$

(3.7)

$$= \lambda \left[(k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} G_j(u^*, u) - G_1(u^*, u) \right],$$

where $q > 1$ satisfies $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Combining relations (3.6) and (3.7), it gives

$$(k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} G_j(u^*, u) - G_1(u^*, u) \leq \frac{\lambda}{2\xi} d_{\mathcal{R}}^2(u^*, u),$$

for all $u \in \mathcal{D}$. In the inequality above, letting $\lambda \to 0^+$ yields

$$\begin{aligned} 0 &\geq (k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} G_j(u^*, u) - G_1(u^*, u) \\ &\geq \left(\sum_{j=2}^{k} |G_j(u^*, u)|^p \right)^{\frac{1}{p}} - G_1(u^*, u) \end{aligned}$$

for all $u \in \mathcal{D}$, that is,

$$G(u^*, u) \in \mathcal{K}_p^k, \forall u \in \mathcal{D}.$$

Thus, u^* is a solution of VEP($\mathcal{D}, G, \mathcal{K}_p^k$).

Conversely, suppose that u^* is a solution of VEP($\mathcal{D}, G, \mathcal{K}_p^k$), that is, $u^* \in \mathcal{D}$ and $G(u^*, v) \in \mathcal{K}_p^k, \forall v \in \mathcal{D}$. Then, for every $v \in \mathcal{D}$, we have

$$\left(\sum_{j=2}^{k} |G_j(u^*, v)|^p \right)^{\frac{1}{p}} - G_1(u^*, v) \leq 0.$$

Therefore, for any $\xi > 0$, we obtain

$$\Phi_\xi(u^*) = \sup_{v \in \mathcal{D}} \left[\left(\sum_{j=2}^{k} |G_j(u^*, v)|^p \right)^{\frac{1}{p}} - G_1(u^*, v) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u^*, v) \right] \leq 0.$$

Since $\Phi_\xi(u) \geq 0$ for all $u \in \mathcal{D}$, one has $\Phi_\xi(u^*) = 0$. Then, the proof is complete. □

Lemma 3.8. *Assume that all hypotheses of Theorem 3.7 hold and, further, that the set $\mathcal{D}$ is compact, G_j is continuous for all $j \in \{1, \cdots, k\}$. Then, for each $\xi > 0$, the gap function Φ_ξ is continuous on $\mathcal{D}$.*

Proof. Since the function G_j is continuous for each $j \in \{1, ..., k\}$, we know that the function $P \colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ defined by

$$P(u, v) = \left(\sum_{j=2}^{k} |G_j(u, v)|^p \right)^{\frac{1}{p}} - G_1(u, v) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u, v)$$

is continuous on $\mathcal{D} \times \mathcal{D}$. Moreover, $\mathcal{D}$ is a compact set, so the function Φ_ξ defined by

$$\Phi_\xi(u) = \sup_{v \in \mathcal{D}} P(u, v)$$

is continuous on $\mathcal{D}$. This completes the proof. □

Next, we introduce the following gap function in the form of the Moreau-Yosida regularization of the function Φ_ξ:

$$\mathbf{R}_{\Phi_\xi,\zeta}(u) = \inf_{z \in \mathcal{D}} \left[\Phi_\xi(z) + \zeta d^2_{\mathcal{R}}(u, z)\right], \tag{3.8}$$

for all $u \in \mathcal{D}$. Accordingly, we can rewrite $\mathbf{R}_{\Phi_\xi,\zeta}$ defined by (3.8) as follows:

$$\mathbf{R}_{\Phi_\xi,\zeta}(u) = \inf_{z \in \mathcal{D}} \left\{ \sup_{v \in \mathcal{D}} \left[\left(\sum_{j=2}^{k} |G_j(z, v)|^p \right)^{\frac{1}{p}} - G_1(z, v) - \frac{1}{2\xi} d^2_{\mathcal{R}}(z, v) \right] + \zeta d^2_{\mathcal{R}}(u, z)) \right\}.$$

Theorem 3.9. *Suppose that all assumptions of Lemma 3.8 hold. Then, the function* $\mathbf{R}_{\Phi_\xi,\zeta}$ *defined by* (3.8) *for any* $\xi, \zeta > 0$ *is a gap function for* $\mathrm{VEP}(\mathcal{D}, G, \mathcal{K}^k_p)$.

Proof. (a) For any $\xi, \zeta > 0$ fixed, since $\Phi_\xi(u) \geq 0$ for all $u \in \mathcal{D}$, it is easy to verify that $\mathbf{R}_{\Phi_\xi,\zeta}(u) \geq 0$ for all $u \in \mathcal{D}$.

(b) Assume that $u^* \in \mathbf{S}(\mathcal{D}, G, \mathcal{K}^k_p)$. Thanks to the proof of part(a) and Theorem 3.7, we have $\Phi_\xi(u^*) = 0$ and

$$0 \leq \mathbf{R}_{\Phi_\xi,\zeta}(u^*) = \inf_{z \in \mathcal{D}} \left\{\Phi_\xi(z) + \zeta d^2_{\mathcal{R}}(z, u^*)\right\} \leq \Phi_\xi(u^*) + \zeta d^2_{\mathcal{R}}(u^*, u^*) = 0.$$

Thus, we obtain $\mathbf{R}_{\Phi_\xi,\zeta}(u^*) = 0$.

Conversely, if $u^* \in \mathcal{D}$ and $\mathbf{R}_{\Phi_\xi,\zeta}(u^*) = 0$, namely,

$$\inf_{z \in \mathcal{D}} \{\Phi_\xi(z) + \zeta d^2_{\mathcal{R}}(z, u^*)\} = 0,$$

which implies that for each n, there exists a sequence $\{z_n\} \subset \mathcal{D}$ such that

$$0 \leq \Phi_\xi(z_n) + \zeta d^2_{\mathcal{R}}(z_n, u^*) < \frac{1}{n}.$$

Thus, $\Phi_\xi(z_n) \to 0$ and $d_{\mathcal{R}}(z_n, u^*) \to 0$, as $n \to +\infty$, that is, $\Phi_\xi(z_n) \to 0$ and $z_n \to u^*$, as $n \to +\infty$. Using the continuity of Φ_ξ in Lemma 3.8, we achieve

$$\Phi_\xi(u^*) = \lim_{n \to +\infty} \Phi_\xi(z_n) = 0.$$

Then, applying Theorem 3.7 gives $u^* \in \mathbf{S}(\mathcal{D}, G, \mathcal{K}^k_p)$. This completes the proof. □

Remark 3.10. We point out some comments regarding Theorem 3.7 and Theorem 3.9.

- **(i):** The gap functions Φ_ξ and $\mathbf{R}_{\Phi_\xi,\zeta}$ transform $\mathrm{VEP}(\mathcal{D}, G, \mathcal{K}^k_p)$ into minimization problems. Accordingly, one can apply optimization methods to solve the problem $\mathrm{VEP}(\mathcal{D}, G, \mathcal{K}^k_p)$.

(ii): As mentioned in the introduction, there is no result concerning gap functions in forms of the regularization for the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$). Thus, Theorem 3.7 and Theorem 3.9 are new in considering regularized gap functions to vector equilibrium problems on Hadamard manifolds, whose final space is partially ordered by p-order cone.

In order to establish upper bounds for VEP($\mathcal{D}, G, \mathcal{K}_p^k$), it needs the following hypotheses.

Assumption ($\mathcal{A}_G^\mu$):

(i): For the function $G_1 \colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, for any $(u, v) \in \mathcal{D} \times \mathcal{D}$, there exists $\mu_1 > 0$ such that

$$G_1(u, v) + G_1(v, u) + \mu_1 d_{\mathcal{R}}^2(u, v) \leq 0.$$

(ii): For each $j \in \{2, ..., k\}$, the function $G_j \colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ is such that for any $(u, v) \in \mathcal{D} \times \mathcal{D}$, there exists $\mu_j > 0$ such that

$$G_j(u, v) + G_j(v, u) - \mu_j d_{\mathcal{R}}^2(u, v) \geq 0.$$

Proposition 3.11. *Under the assumptions ($\mathcal{A}_G^\mu$), the solution set of the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$) is singleton.*

Proof. Suppose that $u_1, u_2 \in \mathcal{D}$ are two solutions of VEP($\mathcal{D}, G, \mathcal{K}_p^k$). Then, for each $j = \{1, 2\}$, we have

$$G_1(u_i, v) - \left(\sum_{j=2}^{k} |G_j(u_i, v)|^p \right)^{\frac{1}{p}} \geq 0, \ \forall v \in \mathcal{D}.$$

This implies

$$G_1(u_i, v) - (k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} G_j(u_i, v) \geq 0, \ \forall v \in \mathcal{D}$$

for $i = \{1, 2\}$. Taking $v = u_2$ if $i = 1$ and $v = u_1$ if $i = 2$ for the inequalities above, we sum up the resulting inequalities to achieve

$$G_1(u_1, u_2) + G_1(u_2, u_1) - (k-1)^{-\frac{1}{q}} \sum_{j=2}^{k} [G_j(u_1, u_2) + G_j(u_2, u_1)] \geq 0 \tag{3.9}$$

By the assumptions ($\mathcal{A}_G^\mu$), it follow from (3.9) that

$$-\left[\mu_1 + (k-1)^{-\frac{1}{q}} \left(\sum_{j=2}^{k} \mu_j \right) \right] d_{\mathcal{R}}^2(u_1, u_2) \geq 0,$$

which says

$$\left[\mu_1 + (k-1)^{-\frac{1}{q}} \left(\sum_{j=2}^{k} \mu_j \right) \right] d_{\mathcal{R}}^2(u_1, u_2) \leq 0.$$

Since $\mu_1 + (k-1)^{-\frac{1}{q}} \left(\sum_{j=2}^{k} \mu_j\right) > 0$, the above inequality implies $d_{\mathcal{R}}(u_1, u_2) = 0$, and so $u_1 = u_2$. Therefore, VEP$(\mathcal{D}, G, \mathcal{K}_p^k)$ has a unique solution $u \in \mathcal{D}$ provided that the solution set of VEP$(\mathcal{D}, G, \mathcal{K}_p^k)$ is nonempty, i.e., $\mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$ is a singleton set. □

The following results provide upper bounds for VEP$(\mathcal{D}, G, \mathcal{K}_p^k)$ using the gap functions Φ_ξ and $\mathbf{R}_{\Phi_\xi,\zeta}$.

Theorem 3.12. *Let u^* be a solution of VEP$(\mathcal{D}, G, \mathcal{K}_p^k)$. Suppose that all hypotheses of Theorem 3.7 are satisfied and the conditions $(\mathcal{A}_G^\mu)$ hold. Then, for each $u \in \mathcal{D}$ and ξ satisfying*

$$\frac{1}{2}\left[\mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j\right)\right]^{-1} < \xi,$$

we have

$$d_{\mathcal{R}}(u, u^*) \le \left[\mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j\right) - \frac{1}{2\xi}\right]^{-\frac{1}{2}} \left[\Phi_\xi(u)\right]^{\frac{1}{2}}. \tag{3.10}$$

Proof. Let $u^* \in \mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$. For each $u \in \mathcal{D}$, by the definition of Φ_ξ and applying Hölder's inequality, we obtain

$$\begin{aligned}
\Phi_\xi(u) &= \sup_{v\in\mathcal{D}} \left[\left(\sum_{j=2}^{k} |G_j(u,v)|^p\right)^{\frac{1}{p}} - G_1(u,v) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u,v)\right] \\
&\ge \left(\sum_{j=2}^{k} |G_j(u,u^*)|^p\right)^{\frac{1}{p}} - G_1(u,u^*) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u,u^*) \\
&\ge (k-1)^{\frac{1-p}{p}} \sum_{j=2}^{k} G_j(u,u^*) - G_1(u,u^*) - \frac{1}{2\xi} d_{\mathcal{R}}^2(u,u^*). \qquad (3.11)
\end{aligned}$$

Using the condition $(\mathcal{A}_G^\mu)$(i) yields

$$-G_1(u,u^*) \ge G_1(u^*,u) + \mu_1 d_{\mathcal{R}}^2(u,u^*). \tag{3.12}$$

Then, it follows from the condition $(\mathcal{A}_G^\mu)$(ii) that for each $j \in \{2, ..., k\}$, there holds

$$G_j(u,u^*) \ge -G_j(u^*,u) + \mu_j d_{\mathcal{R}}^2(u,u^*),$$

which implies

$$\begin{aligned}
&(k-1)^{\frac{1-p}{p}} \sum_{j=2}^{k} G_j(u,u^*) \\
&\ge -(k-1)^{\frac{1-p}{p}} \sum_{j=2}^{k} G_j(u^*,u) + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j\right) d_{\mathcal{R}}^2(u,u^*). \qquad (3.13)
\end{aligned}$$

Since $u^* \in \mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$, one has

$$\begin{aligned} 0 &\leq G_1(u^*, u) - \left(\sum_{j=2}^{k} |G_j(u^*, u)|^p \right)^{\frac{1}{p}} \\ &\leq G_1(u^*, u) - (k-1)^{\frac{1-p}{p}} \sum_{j=2}^{k} G_j(u^*, u). \end{aligned} \tag{3.14}$$

Combining (3.12)-(3.14) leads to

$$\Phi_\xi(u) \geq \left[\mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi} \right] d_{\mathcal{R}}^2(u, u^*). \tag{3.15}$$

This implies that

$$d_{\mathcal{R}}(u, u^*) \leq \left[\mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi} \right]^{-\frac{1}{2}} [\Phi_\xi(u)]^{\frac{1}{2}}$$

and the inequality (3.10) is valid. □

Theorem 3.13. *Let u^* be a solution of VEP($\mathcal{D}, G, \mathcal{K}_p^k$). Suppose that the assumptions of Theorem 3.9 and Theorem 3.12 hold. Then, for each $u \in \mathcal{D}$, we have*

$$d_{\mathcal{R}}(u, u^*) \leq \left[\frac{1}{2} \min \left\{ \mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi}, \zeta \right\} \right]^{-\frac{1}{2}} \left[\mathbf{R}_{\Phi_\xi, \zeta}(u) \right]^{\frac{1}{2}}. \tag{3.16}$$

Proof. Let $u^* \in \mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$. In view of (3.15), we obtain

$$\begin{aligned} \mathbf{R}_{\Phi_\xi, \zeta}(u) &= \inf_{z \in \mathcal{D}} \left\{ \Phi_\xi(z) + \zeta d_{\mathcal{R}}^2(u, z) \right\} \\ &\geq \inf_{z \in \mathcal{D}} \left\{ \left[\mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi} \right] d_{\mathcal{R}}^2(z, u^*) + \zeta d_{\mathcal{R}}^2(u, z) \right\} \\ &\geq \min \left\{ \mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi}, \zeta \right\} \inf_{z \in \mathcal{D}} \left\{ d_{\mathcal{R}}^2(z, u^*) + d_{\mathcal{R}}^2(u, z) \right\} \\ &\geq \min \left\{ \mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi}, \zeta \right\} \frac{[d_{\mathcal{R}}(z, u^*) + d_{\mathcal{R}}(u, z)]^2}{2} \\ &\geq \frac{1}{2} \min \left\{ \mu_1 + (k-1)^{\frac{1-p}{p}} \left(\sum_{j=2}^{k} \mu_j \right) - \frac{1}{2\xi}, \zeta \right\} d_{\mathcal{R}}^2(u, u^*). \end{aligned}$$

This further leads to

$$d_{\mathcal{R}}(u,u^*) \leq \left[\frac{1}{2}\min\left\{\mu_1 + (k-1)^{\frac{1-p}{p}}\left(\sum_{j=2}^{k}\mu_j\right) - \frac{1}{2\xi}, \zeta\right\}\right]^{-\frac{1}{2}} \left[\mathbf{R}_{\Phi_\xi,\zeta}(u)\right]^{\frac{1}{2}}.$$

Then, the proof is complete. □

Next, we present an example to illustrate our main results on regularized gap functions and upper bounds for $\mathrm{VEP}(\mathcal{D},G,\mathcal{K}_p^k)$.

Example 3.14. Let $(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{R}})$ be the Hadamard manifold considered in Example 3.4.

Let $k=3$, $\mathcal{D}=\{u\in\mathbb{R}\mid u=e^{\frac{5}{2}r+1}, r\in[0,1]\}\subset\mathcal{H}$, for each $j\in\{1,2,3\}$, the functions $G_j\colon\mathcal{D}\times\mathcal{D}\to\mathbb{R}$ be defined by

$$\begin{aligned} G_1(u,v) &= 2\ln u\ln^2 v - \ln^3 u,\\ G_2(u,v) &= \frac{3}{2}\ln^2 u - \frac{3}{2}\ln u\ln v,\\ G_3(u,v) &= 2\ln^2 u - 2\ln u\ln v, \end{aligned}$$

for all $u,v\in\mathcal{D}$. Moreover, let $G\colon\mathcal{D}\times\mathcal{D}\to\mathbb{R}^3$ be defined by

$$G(u,v) = (G_1(u,v),G_2(u,v),G_3(u,v))^\top, \text{ for all } u,v\in\mathcal{D}.$$

Consider the problem $\mathrm{VEP}(\mathcal{D},G,\mathcal{K}_p^k)$ with $p=q=2$, i.e., finding $u\in\mathcal{D}$ such that

$$\left(\left(\frac{3}{2}\ln^2 u - \frac{3}{2}\ln u\ln v\right)^2 + \left(2\ln^2 u - 2\ln u\ln v\right)^2\right)^{\frac{1}{2}} \leq \ln u\ln^2 v - \ln^3 u,\ \forall v\in\mathcal{D},$$

which is equivalent to solving

$$\frac{5}{2}\sqrt{\ln^2 u(\ln v-\ln u)^2} \leq \ln u(\ln v-\ln u)(\ln u+\ln v),\ \forall v\in\mathcal{D}.$$

Following from a direct computation, it is easy to verify that $\mathbf{S}(\mathcal{D},G,\mathcal{K}_p^k)=\{e\}$.

In addition, it is not hard to show that $\mathcal{D}$ is a nonempty, compact and geodesic convex set, the function G_j is continuous and $G_j(u,u)=0$ for all $j\in\{1,2,3\}$ and so $G(u,u)=0_{\mathbb{R}^3}$ for all $u\in\mathcal{D}$. Furthermore, thanks to Example 3.4, we know that the function $v\mapsto G_1(u,v)$ is geodesic convex and the functions $v\mapsto G_2(u,v)$ and $v\mapsto G_3(u,v)$ are linear affine for all $u\in\mathcal{D}$. Applying Proposition 3.3, G is the $\mathcal{K}_p^k$-convex function. Thus, all assumptions of Theorem 3.7 and Theorem 3.9 are satisfied, which says, for any $\xi,\zeta>0$, Φ_ξ and $\mathbf{R}_{\Phi_\xi,\zeta}$ are gap functions of $\mathrm{VEP}(\mathcal{D},G,\mathcal{K}_p^k)$.

For any $u,v\in\mathcal{D}$, we have

$$\begin{aligned} G_1(u,v)+G_1(v,u) &= \ln u\ln^2 v - \ln^3 u + \ln v\ln^2 u - \ln^3 v\\ &= -(\ln u+\ln v)(\ln v-\ln u)^2\\ &\leq -2d_{\mathcal{R}}^2(u,v);\\ G_2(u,v)+G_2(v,u) &= \frac{3}{2}(\ln v-\ln u)^2 = \frac{3}{2}d_{\mathcal{R}}^2(u,v); \end{aligned}$$

$$G_3(u,v) + G_3(v,u) = 2(\ln v - \ln u)^2 = 2d_{\mathcal{R}}^2(u,v).$$

Hence, the conditions ($\mathcal{A}_G^{\mu}$) are satisfied with $\mu_1 = \mu_3 = 2, \mu_2 = \frac{3}{2}$. Therefore, for any

$$\xi > \frac{1}{2}\left[\mu_1 + (k-1)^{-\frac{1}{q}}\left(\sum_{j=2}^{k}\mu_j\right)\right]^{-1} \approx 0.1117,$$

which implies the inequalities (3.10) and (3.16) hold. In particular, taking $\xi = \frac{1}{2}$, by the definition of the function Φ_ξ in (3.5), we obtain

$$\Phi_\xi(u) = \sup_{v\in\mathcal{D}}\left[\frac{5}{2}\sqrt{\ln^2 u(\ln v - \ln u)^2} - \ln u(\ln^2 v - \ln^2 u) - (\ln v - \ln u)^2\right],$$

and with $u^* = e \in \mathbf{S}(\mathcal{D}, G, \mathcal{K}_p^k)$, we have

$$d_{\mathcal{R}}(u,e) \le \frac{2}{\sqrt{4+7\sqrt{2}}}\sqrt{\Phi_\xi(u)}.$$

for all $u \in \mathcal{D}$.

Remark 3.15. There are some comments regarding Theorem 3.12 and Theorem 3.13.

- **(i):** The upper bound results in Theorem 3.12 and Theorem 3.13 provide the upper estimates of the Riemannian distance from an arbitrary point in the admissible set $\mathcal{D}$ to the unique solution set of VEP($\mathcal{D}, G, \mathcal{K}_p^k$) on Hadamard manifolds. Computing the upper estimates in (3.10) and (3.16) is controlled by the regularized gap functions Φ_ξ and $\mathbf{R}_{\Phi_\xi,\zeta}$ and depends on the data in the component functions $G_1, \cdots, G_k$.
- **(ii):** To the best of our knowledge, up to now, deriving upper bounds for vector equilibrium problems on Hadamard manifolds, whose final space is partially ordered by p-order cone using gap regularized functions has not been studied. Therefore, our results in Theorem 3.12 and Theorem 3.13 are new to the literature.

4. Conclusions

In this article, we have investigated a new class of vector equilibrium problems associated with partial order provided by p-order cone in the setting of Hadamard manifolds. In light of the concept of $\mathcal{K}_p^k$-convexity of a vector-valued function on Hadamard manifolds, we have proposed gap regularized functions of the Fukushima and Moreau-Yosida types for tackling the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$) (Theorem 3.7 and Theorem 3.9). Especially, we have also developed several upper bounds for the problem VEP($\mathcal{D}, G, \mathcal{K}_p^k$) controlled by these regularized gap functions under suitable assumptions (Theorem 3.12 and Theorem 3.13). The main results are new to the literature even in the special case where $\mathcal{H}$ is a finite dimensional space.

Based on the obtained results, a future direction is studying the solution methods including numerical algorithms and convergence analyses to solve the problem

VEP($\mathcal{D}, G, \mathcal{K}_p^k$) via the minimization problems using the regularized gap functions Φ_ξ and $\mathbf{R}_{\Phi_\xi,\zeta}$.

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