Journal of Nonlinear and Convex Analysis
Volume 24, Number 12, 2023, 2573–2592

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ε-WOLFE TYPE DUALITY FOR CONVEX OPTIMIZATION PROBLEMS UNDER DATA UNCERTAINTY

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Dedicated to the Memory of Kazimierz Goebel and W. Art Kirk

Abstract. In this paper, we investigate a pretty new concept of approximate solutions, called quasi (α, ϵ)-solutions, for convex optimization problems, which involve data uncertainty in the sense of its constraints. Using the robust optimization approach (worst-case approach), approximate optimality conditions for quasi (α, ϵ)-solutions to the considered convex optimization problems, are established. Approximate duality theorems for ϵ-Wolfe type dual model are discussed. Finally, we propose some results on approximate saddle point theorems. Besides, we design some examples to illustrate the obtained results.

1. Introduction

The study of convex optimization problems that are affected by uncertain data is considered increasingly important in mathematical programming. Many research papers investigated (approximate) optimality and duality theories for vector/scalar linear or convex programming problems under data uncertainty with the worst-case approach (the robust approach); see, for example, [1, 3, 5, 15–18, 21–25, 28, 29] and the reference therein. It is worth noting that, according to [1, 15, 16], we know that the value of the robust counterpart of the primal problem is equal to the value of the optimistic counterpart of the dual problem; roughly speaking, "primal worst equals dual best".

1.1. **Problems Formulation.**

Now, we recall the models from the *standard* convex optimization problem to the *robust* one. A standard convex optimization problem [4, 7] is the one:

$$\text{(CP)} \qquad \min\ f(x) \quad \text{subject to} \quad g_i(x) \leq 0,\ i = 1, \ldots, m,$$

where $f,\ g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are convex functions.

The standard convex optimization problem (CP) in the face of data uncertainty *in the constraints* can be captured by the one:

$$\text{(UCP)} \qquad \min\ f(x) \quad \text{subject to} \quad g_i(x, v_i) \leq 0,\ i = 1, \ldots, m,$$

2020 *Mathematics Subject Classification.* 90C25, 90C46, 49N15.

Key words and phrases. Quasi (α, ϵ)-solutions, robust characteristic cone constraint qualification, approximate optimality conditions, approximate duality theorems, approximate saddle point theorems.

This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2019R1A2C1008672). The third author was supported by the National Science Foundation of China (11961073).

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where $f : \mathbb{R}^n \to \mathbb{R}$, is still a convex function, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is continuous such that $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to some set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$.

Following the robust (worst-case) approach, the robust counterpart of problem (UCP) is stated as follows:

$$\text{(RCP)} \qquad \min f(x) \quad \text{subject to} \quad g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m.$$

The feasible set F of problem (RCP) (also known as the *robust* feasible set of problem (UCP), see [37]) is defined by

$$F := \{x \in \mathbb{R}^n \colon g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\}, \tag{1.1}$$

which is assumed to be nonempty. Then $\bar{x} \in F$ is called a (*global*) *optimal* solution to problem (RCP), if $f(\bar{x}) \leq f(x)$, for all $x \in F$.

1.2. **Approximate Solution Concepts.** It is well-known that an optimal solution to a convex optimization problem may not be exact but very near to it; for instance, minimizing the convex function e^x over $(-\infty, 0]$. This fact leads to the notion of *approximate* solutions, which also play a key role in algorithmic study of optimization problems. A large number of research papers studied the characterizations of approximate solutions in mathematical optimization problems; see, for example, [2, 6, 8, 9, 12, 18, 20–24, 27, 30, 32, 33, 35, 36] and the references therein.

Below, we recall the definition of approximate solutions to problem (RCP).

Definition 1.1. Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given, then $\bar{x}$ is said to be

- an *ϵ-solution* to problem (RCP), if $f(\bar{x}) \leq f(x) + \epsilon$, for all $x \in F$;
- a *quasi α-solution* to problem (RCP), if $f(\bar{x}) \leq f(x) + \alpha\|x - \bar{x}\|$, for all $x \in F$;
- a *regular (α, ϵ)-solution* to problem (RCP), if for any $x \in F$, $\bar{x}$ is an ϵ-solution to problem (RCP) as well as a quasi α-solution;
- a *quasi (α, ϵ)-solution* to problem (RCP), if

$$f(\bar{x}) \leq f(x) + \alpha\|x - \bar{x}\| + \epsilon, \text{ for all } x \in F. \tag{1.2}$$

Remark 1.2. (i) By definition, if $\bar{x}$ is an ϵ-solution to problem (RCP), then it is a quasi (α, ϵ)-solution, but the converse is not true. Similarly, if $\bar{x}$ is a quasi α-solution to problem (RCP), then it is a quasi (α, ϵ)-solution, again the converse is not true.

(ii) It is also apparent that, if $\alpha = 0$ (resp., $\epsilon = 0$), then the notion of a quasi (α, ϵ)-solution defined above coincides with an ϵ-solution (resp., a quasi α-solution); see, for example [18, 21–24]. Moreover, if $\alpha = \epsilon = 0$, then a quasi (α, ϵ)-solution will reduce to an exact solution (if exists). Therefore the case of $\alpha \neq 0$ and $\epsilon \neq 0$ is often of interest, for such a reason, we always assume hereafter that $\alpha > 0$ and $\epsilon > 0$.

Now, we are going to explain the geometrical meaning of the mentioned approximate solutions, and give two examples to show their differences.

- *Geometrical meaning of an ϵ-solution.* Let $\epsilon > 0$ be given, by definition of an ϵ-solution to problem (RCP), we have $f(\bar{x}) \leq \inf_{x \in F} f(x) + \epsilon$. Observe that the ϵ-solution set of problem (RCP) is $\{\bar{x} \in F \colon f(\bar{x}) \le \inf_{x\in F} f(x) + \epsilon\}$, i.e., the intersection of $(\inf_{x\in F} f(x) + \epsilon)$-(lower-)level set and the feasible set F.

- *Geometrical meaning of a quasi α-solution.* Let $\alpha > 0$ be given, it follows from the definition that $\bar{x}$ is a quasi α-solution to problem (RCP) if
$$f(\bar{x}) \le f(x) + \alpha\|x - \bar{x}\|, \text{ for all } x \in F.$$
Clearly, a quasi α-solution means $f(x) \ge f(\bar{x}) - \alpha\|x - \bar{x}\| =: g(x)$ for all $x \in F$ by means of their graphs. If in addition f is differentiable over its domain, then $\|\nabla f(\bar{x})\| \le \alpha$ due to [27]. It is also worth mentioning that the notion of a quasi α-solution introduced by Loridan [27] is motivated by the well-known Ekeland's Variational Principle [10].
- *Geometrical meaning of a regular (α, ϵ)-solution.* It is easy to see that the regular (α, ϵ)-solution set of problem (RCP) is the intersection of its ϵ-solution set and quasi α-solution set.
- *Geometrical meaning of a quasi (α, ϵ)-solution.* Let $\alpha > 0$ and $\epsilon > 0$ be given. By definition, it is true that (1.2) can be written as
$$f(\bar{x}) + \alpha\|\bar{x} - \bar{x}\| \le f(x) + \alpha\|x - \bar{x}\| + \epsilon, \quad \text{for all} \quad x \in F;$$
in other words, $\bar{x}$ is an ϵ-solution of function $f(\cdot) + \alpha\|\cdot - \bar{x}\|$ over F. Besides, let $g(x) = f(\bar{x}) - \alpha\|x - \bar{x}\| - \epsilon$, particularly, one has $g(\bar{x}) = f(\bar{x}) - \epsilon$. Thus, a quasi (α, ϵ)-solution says $f(x) \ge g(x)$ for all $x \in F$ in view of their graphs.

Below, we give two simple examples with (detailed) calculations to show the differences of the mentioned approximate solutions. One may see [21] for the calculations of the first three approximate solution sets in Example 1.3, while the calculation of quasi (α, ϵ)-solution set is new as we shall see in the following. Besides, we leave the calculations of Example 1.3 to the reader.

Example 1.3. Consider the following unconstrained convex optimization problem:

$$(\mathrm{P}_1) \qquad \min\ f(x) := x^2 \quad \text{subject to} \quad x \in \mathbb{R}.$$

Observe that $x = 0$ is the (global) optimal solution to problem (P_1). Let $\alpha > 0$ and $\epsilon > 0$ be given, a simple calculation yields that

- $\bar{x}$ is an ϵ-solution to problem (P_1) if $f(\bar{x}) \le f(x) + \epsilon$, for all $x \in \mathbb{R}$, which is equivalent to $f(\bar{x}) \le \inf_{x\in\mathbb{R}} f(x) + \epsilon$, and the ϵ-solution set is $[-\sqrt{\epsilon}, \sqrt{\epsilon}]$;
- $\bar{x}$ is a quasi α-solution to problem (P_1) if $f(\bar{x}) \le f(x) + \alpha\|x - \bar{x}\|$, for all $x \in \mathbb{R}$. Since $f(x) = x^2$ is differentiable over $\mathbb{R}$, it follows from *Geometrical meaning of a quasi α-solution* that $\|\nabla f(\bar{x})\| = \|2\bar{x}\| \le \alpha$, hence the quasi α-solution set is $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$;
- the regular (α, ϵ)-solution set is $[\max\{-\frac{\alpha}{2}, -\sqrt{\epsilon}\}, \min\{\sqrt{\epsilon}, \frac{\alpha}{2}\}]$.

Now, we will calculate the quasi (α, ϵ)-solution set with a detailed calculation. By definition, we have $\bar{x}^2 \le x^2 + \alpha|x - \bar{x}| + \epsilon$, for all $x \in \mathbb{R}$. Let $g(x) = f(\bar{x}) - \alpha|x - \bar{x}| - \epsilon = \bar{x}^2 - \alpha|x - \bar{x}| - \epsilon$, clearly, $g(\bar{x}) = f(\bar{x}) - \epsilon$. See the Fig. 1, since a quasi (α, ϵ)-solution says $f(x) \ge g(x)$ for all $x \in \mathbb{R}$ in the view of their graphs, consider first $x \ge 0$, so the slope of g, i.e., α should be no less than the slope of g_1 at the point a, i.e., $\nabla f(a) = 2a$, as shown in Fig. 1; in other words, $\alpha \ge 2a$. Now, we calculate the point a. Observe that $g_1(x) = 2a(x - a) + a^2$, and point $(\bar{x}, \bar{x}^2 - \epsilon)$ locates in the

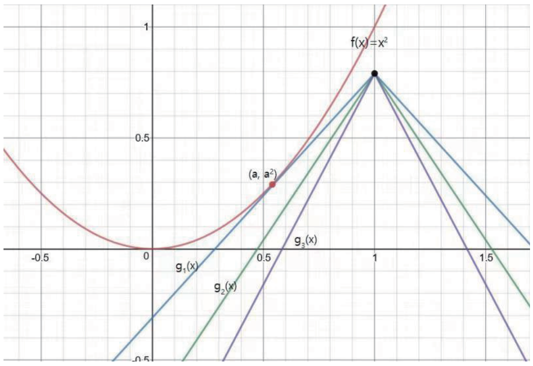

FIGURE 1. A quasi (α, ϵ)-solution $\bar{x}$ means $f(x) \geq g(x)$ for all $x \in \mathbb{R}$ in the view of their graphs.

graph $g_1(x)$, then, we have $a = \bar{x} - \sqrt{\epsilon}$, furthermore $0 \leq \bar{x} \leq \frac{\alpha}{2} + \sqrt{\epsilon}$. A similar argument shows that $-\frac{\alpha}{2} - \sqrt{\epsilon} \leq \bar{x} \leq 0$ if we consider $x \leq 0$.

- Hence, the quasi (α, ϵ)-solution set is $[-\frac{\alpha}{2} - \sqrt{\epsilon}, \frac{\alpha}{2} + \sqrt{\epsilon}]$.

1.3. **Motivations and Contributions.** Very recently, approximate optimality conditions and approximate duality theorems for ϵ-solutions and quasi α-solutions to problem (RCP) were studied by Lee and Lee [22] and Lee and Jiao [21], respectively. In particular, if the feasible set F is given by a linear matrix inequality with data uncertainty (known as robust convex *semidefinite* programming problems), approximate optimality conditions and approximate duality theorems for ϵ-solutions and quasi α-solutions to such a robust model problem were studied due to its special structure by Lee and Lee [23] and Jiao and Lee [18], respectively. Furthermore, if the feasible set F is given by an infinite number of constraints that is a convex system with data uncertainty (known as robust convex *semi-infinite* programming problems), some results for ϵ-solutions of the robust convex semi-infinite programming problems were studied by Lee and Lee [24]. Meanwhile, some recent works on approximate solutions for robust convex optimization problems were also investigated by researchers; see, for example [11, 26, 34]. It is worth mentioning that all the results in the above mentioned papers were obtained by employing the robust version of respective Farkas' lemmas.

On the other hand, in 2008, Beldiman *et al.* [2] introduced the so-called *quasi* (α, ϵ)*-solution* to a standard optimization problem; however, they didn't explore the approximate optimality conditions for such a solution. Since the concept of a quasi (α, ϵ)-solution to problem (RCP) may provide a rather large range for a better starting point when we design algorithms, in contrast to a quasi α-solution; see, for example [9], this motivates us to consider the study on approximate optimality conditions and approximate duality theorems for such a class of approximate solutions to robust convex optimization problems (RCP). Besides, comparing with the works in [18, 21–23], we also examine approximate saddle point theorems. We mainly make contributions to robust convex optimization as follows.

(i) We explore approximate optimality conditions and approximate duality theorems for a quasi (α, ϵ)-solution to problem (RCP) by using the robust version of Farkas' lemma.
(ii) The results of this paper generalize the ones in [21, 22]. Nevertheless, the results on a *quasi* (α, ϵ)*-solution* to problem (RCP) seem new, even though the tools are from [15, 21, 22].
(iii) We propose some results on approximate saddle point theorems as well.

The rest of the paper is organized as follows. Section 2 gives some notations and preliminaries. In Sect. 3, under the fulfilment of the *robust characteristic cone constraint qualification* [15], we examine approximate optimality theorems for a quasi (α, ϵ)-solution to problem (RCP). In Sect. 4, the ϵ-Wolfe type duality is formulated, and approximate duality theorems are established; we also design an example to illustrate the approximate duality results. Finally, we give some results on approximate saddle point theorems in Sect. 5. Conclusions are given in Sect. 6.

2. Preliminaries

In this section, we recall some notations and preliminary results that will be used in the paper; see [4, 7, 31] for more details. We abbreviate $(x_1, x_2, \ldots, x_n)$ by x. The Euclidean space $\mathbb{R}^n$ is equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set A in $\mathbb{R}^n$ is convex whenever $\lambda a_1 + (1-\lambda)a_2 \in A$ for all $\lambda \in [0,1]$, $a_1, a_2 \in A$. Besides, for a given set $A \subset \mathbb{R}^n$, we denote the interior, closure and convex hull generated by A, by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{conv} A$, respectively.

Let f be an extended-real-valued function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. A function f is said to be *proper* if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the *domain* of f by $\operatorname{dom} f$, and $\operatorname{dom} f := \{x \in \mathbb{R}^n \colon f(x) < +\infty\}$. The *epigraph* of f, denoted by $\operatorname{epi} f$, is defined by $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \colon f(x) \leq r\}$. The function f is said to be *convex* if for all $\lambda \in [0,1]$, $f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$ for all $x, y \in \mathbb{R}^n$; equivalently, $\operatorname{epi} f$ is convex. f is said to be *concave* whenever $-f$ is convex.

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function. The (convex) subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) := \{x^* \in \mathbb{R}^n \colon \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}.$$

We set $\partial f(x) = \emptyset$ whenever $x \notin \operatorname{dom} f$. More generally, for any $\epsilon \geq 0$, the ϵ-subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial_\epsilon f(x) := \{x^* \in \mathbb{R}^n \colon \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon,\ \forall y \in \mathbb{R}^n\}.$$

We set $\partial_\epsilon f(x) = \emptyset$ whenever $x \notin \operatorname{dom} f$. We say f is a *lower semicontinuous* function if $\liminf\limits_{y \to x} f(y) \geq f(x)$ for all $x \in \mathbb{R}^n$. As usual, for any proper convex function f on $\mathbb{R}^n$, its *conjugate function* $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) \colon x \in \mathbb{R}^n\}$ for any $x^* \in \mathbb{R}^n$.

Let δ_A be the *indicator function* with respect to a closed convex subset A of $\mathbb{R}^n$, that is, $\delta_A(x) = 0$ if $x \in A$, and $\delta_A(x) = +\infty$ if $x \notin A$.

Definition 2.1. Let $A \subset \mathbb{R}^n$ be a closed convex set and $x \in A$. Then we denote

- $N_A(x)$ to be the *normal cone* to A at $x \in A$, where
$$N_A(x) := \{v \in \mathbb{R}^n : \langle v, y - x\rangle \leq 0,\ \forall y \in A\};$$
- $N_{\epsilon,A}(x)$ to be the ϵ-*normal set* to A at $x \in A$, where
$$N_{\epsilon,A}(x) := \{v \in \mathbb{R}^n : \langle v, y - x\rangle \leq \epsilon,\ \forall y \in A\}.$$

A remarkable result of indicator function δ_A is that its subdifferential coincides with *normal cone*, and its ϵ-subdifferential coincides with ϵ-*normal set.* In summary, see the following proposition.

Proposition 2.2 ([7, Chapter 2])**.** *Let $A \subset \mathbb{R}^n$ be a closed convex set and $x \in A$. Then,*

$$\partial\delta_A(x) = N_A(x),\ \forall x \in A;\ \textit{and}\ \ \partial_\epsilon\delta_A(x) = N_{\epsilon,A}(x),\ \forall x \in A.$$

Now, we recall the following celebrated result due to Jeyakumar (see [13]), which describes the relationship between the epigraph of a conjugate function of a proper lower semicontinuous convex function and its ϵ-subdifferential, and it also plays an important role in deriving the main results in the paper.

Proposition 2.3 ([13])**.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. If $a \in \operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$, then*

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geq 0} \{(v, \langle v, a\rangle + \epsilon - f(a)) : v \in \partial_\epsilon f(a)\}.$$

Next, we recall the epigraphical conditions for the sum rule of the conjugate functions.

Proposition 2.4 ([14])**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. If $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, then*

$$\operatorname{epi}(f + g)^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} g^*).$$

Moreover, if one of the functions f and g is continuous, then

$$\operatorname{epi}(f + g)^* = \operatorname{epi} f^* + \operatorname{epi} g^*.$$

3. Approximate optimality conditions

It is well known that optimality conditions play an important role in both the theory and practice of mathematical optimization. In order to obtain (approximate) optimality conditions, constraint qualifications are indispensable, since a constraint qualification is a condition ensuring that every (approximate) optimal solution to the considered problems satisfies the (approximate) optimality conditions.

In this section, we establish approximate optimality theorems for a *quasi* (α, ϵ)-*solution* to problem (RCP) under the fulfilment of the closed convex cone constraint qualification [15], that is, the cone

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi}\left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\right)^*$$

is closed and convex. In fact, it follows from [15, Proposition 2.2] that the set $D := \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} (\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*$ is a cone in $\mathbb{R}^{n+1}$, which is called the *robust characteristic cone*. Furthermore, it is also known from [15] that the robust characteristic cone D is convex whenever $g_i(x, \cdot)$ is concave and $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$, is convex; and the robust characteristic cone D is closed whenever the robust Slater condition, i.e., $\{x \in \mathbb{R}^m : g_i(x_0, v_i) < 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} \neq \emptyset$ holds and $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$, is compact.

Definition 3.1. We say a *robust characteristic cone constraint qualification* (RCCCQ, for short) holds if the robust characteristic cone D is closed and convex.

The following lemma was given in [15], which is the robust version of Farkas' lemma for convex functions:

Lemma 3.2 ([15])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$ be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex function. Let $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$ be convex and compact and let $F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\} \neq \emptyset$. Then the following statements are equivalent*:

(i) $\{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} \subseteq \{x \in \mathbb{R}^n : f(x) \geq 0\}$;

(ii) $(0,0) \in \operatorname{epi} f^* + \operatorname{cl} \left(\operatorname{conv} \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^* \right)$.

As mentioned before, the above Lemma 3.2 is known as the robust version of Farkas' lemma for convex functions, and many results are obtained based on it; see, for example, [18, 21–24]. Below, we give a result on quasi (α, ϵ)-solutions to problem (RCP) according to Lemma 3.2.

Theorem 3.3. *Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given, and consider problem* (RCP). *Let $\bar{x} \in F$, where F is defined by* (1.1). *Suppose that the* (RCCCQ) *holds, i.e., the robust characteristic cone D is closed and convex. Then the following statements are equivalent*:

(i) *$\bar{x}$ is a quasi (α, ϵ)-solution to problem* (RCP);

(ii) *there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that for any $x \in \mathbb{R}^n$,*

$$f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) + \alpha \|x - \bar{x}\| + \epsilon \geq f(\bar{x}).$$

Proof. [(i) $\Rightarrow$ (ii)] Let $\bar{x}$ be a quasi (α, ϵ)-solution to problem (RCP), then by definition,

$$f(x) + \alpha \|x - \bar{x}\| + \epsilon \geq f(\bar{x}), \quad \forall x \in F.$$

So $F \subseteq \{x \in \mathbb{R}^n : f(x) + \alpha \|x - \bar{x}\| + \epsilon - f(\bar{x}) \geq 0\}$. Let $\phi(x) = f(x) + \alpha \|x - \bar{x}\| + \epsilon - f(\bar{x})$. It follows from Lemma 3.2 that

$$(0,0) \in \operatorname{epi} \phi^* + \operatorname{cl} \left(\operatorname{conv} \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^* \right). \tag{3.1}$$

The above inclusion (3.1), along with the fulfilment of (RCCCQ), yields that

$$(0,0) \in \operatorname{epi} \phi^* + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^*.$$

So, there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that

$$(0,0) \in \operatorname{epi} \phi^* + \operatorname{epi} \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i) \right)^*. \tag{3.2}$$

Now, by Proposition 2.4,

$$\begin{aligned} \operatorname{epi} \phi^* &= \operatorname{epi} (f(\cdot) + \alpha\| \cdot - \bar{x}\| + \epsilon - f(\bar{x}))^* \\ &= \operatorname{epi} f^* + \operatorname{epi} (\alpha\| \cdot - \bar{x}\| + \epsilon - f(\bar{x}))^*. \end{aligned} \tag{3.3}$$

Since

$$(\alpha\| \cdot - \bar{x}\| + \epsilon - f(\bar{x}))^*(a) = \begin{cases} \alpha\|\bar{x}\| - \epsilon + f(\bar{x}), & \text{if } \|a\| \leq \alpha, \\ +\infty, & \text{else,} \end{cases}$$

which, along with (3.3), yields

$$\operatorname{epi} \phi^* = \operatorname{epi} f^* + \alpha\mathbb{B} \times [\alpha\|\bar{x}\| - \epsilon + f(\bar{x}), +\infty).$$

Hence it follows from (3.2) that,

$$(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) \in \operatorname{epi} f^* + \operatorname{epi} \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i) \right)^* + \alpha\mathbb{B} \times \mathbb{R}_+.$$

Then there exist $u \in \mathbb{R}^n$, $\delta \geq 0$, $w_i \in \mathbb{R}^n$, $\beta_i \geq 0$, $i = 1, \ldots, m$, $b \in \mathbb{B}$ and $r \in \mathbb{R}_+$ such that

$$(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) = (u, f^*(u) + \delta) + \sum_{i=1}^{m} \bar{\lambda}_i (w_i, g_i^*(w_i, \bar{v}_i) + \beta_i) + (\alpha b, r).$$

So, we have

$$\begin{aligned} 0 &= u + \sum_{i=1}^{m} \bar{\lambda}_i w_i + \alpha b, \\ -\alpha\|\bar{x}\| + \epsilon - f(\bar{x}) &= f^*(u) + \delta + \sum_{i=1}^{m} \bar{\lambda}_i ((g_i(\cdot, \bar{v}_i))^*(w_i) + \beta_i) + r. \end{aligned}$$

Hence for any $x \in \mathbb{R}^n$,

$$\begin{aligned} -\left\langle \sum_{i=1}^{m} \bar{\lambda}_i w_i, x \right\rangle - \langle \alpha b, x \rangle - f(x) &= \langle u, x \rangle - f(x) \\ &\leq f^*(u) \\ &= -\alpha\|\bar{x}\| + \epsilon - f(\bar{x}) - \delta \\ &\quad - \sum_{i=1}^{m} \bar{\lambda}_i ((g_i(\cdot, \bar{v}_i))^*(w_i) + \beta_i) - r. \end{aligned}$$

Thus for any $x \in \mathbb{R}^n$,

$$\begin{aligned} f(\bar{x}) &\leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i \langle w_i, x\rangle - \sum_{i=1}^{m} \bar{\lambda}_i (g_i(\cdot, \bar{v}_i))^*(w_i) + \langle \alpha b, x\rangle - \alpha\|\bar{x}\| + \epsilon - \delta \\ &\leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i \big(\langle w_i, x\rangle - (g_i(\cdot, \bar{v}_i))^*(w_i)\big) + \langle \alpha b, x\rangle - \alpha\|\bar{x}\| + \epsilon \\ &\leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) + \alpha\|b\|\|x - \bar{x}\| + \alpha\|b\|\|\bar{x}\| - \alpha\|\bar{x}\| + \epsilon \\ &\leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) + \alpha\|x - \bar{x}\| + \epsilon. \end{aligned}$$

[(i) ⇐ (ii)] Suppose that there exist $\bar{\lambda}_i \geq 0$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that for any $x \in \mathbb{R}^n$, $f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) + \alpha\|x - \bar{x}\| + \epsilon \geq f(\bar{x})$. Then we have for any $x \in F$,

$$f(x) + \alpha\|x - \bar{x}\| + \epsilon \geq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) + \alpha\|x - \bar{x}\| + \epsilon \geq f(\bar{x}).$$

Thus $f(x) + \alpha\|x - \bar{x}\| + \epsilon \geq f(\bar{x})$ for any $x \in F$. Hence $\bar{x}$ is a quasi (α, ϵ)-solution of problem (RCP). □

Now, we are ready to give the approximate optimality condition (both necessary and sufficient) for a quasi (α, ϵ)-solution of problem (RCP) under the fulfilment of (RCCCQ).

Theorem 3.4 (Approximate Optimality Condition). *Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given, and consider problem* (RCP). *Let $\bar{x} \in F$, where F is defined by* (1.1). *Suppose that the* (RCCCQ) *holds. Then the following statements are equivalent*:

(i) *$\bar{x}$ is a quasi (α, ϵ)-solution to problem* (RCP);

(ii) $(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) \in \operatorname{epi} f^* + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^* + \alpha\mathbb{B} \times \mathbb{R}_+$;

(iii) *there exist $\bar{\epsilon}_0 \geq 0$, $\bar{\epsilon}_i \geq 0$, $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$, such that*

$$0 \in \partial_{\bar{\epsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\epsilon}_i} (\bar{\lambda}_i g_i(\cdot, \bar{v}_i))(\bar{x}) + \alpha\mathbb{B}, \tag{3.4}$$

$$\sum_{i=0}^{m} \bar{\epsilon}_i - \epsilon \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i). \tag{3.5}$$

Proof. [(i) ⇔ (ii)] is followed from Theorem 3.3.

[(ii) ⇒ (iii)] Let $F := \{x \in \mathbb{R}^n\colon g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \neq \emptyset$. Assume that (ii) holds, i.e.,

$$(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) \in \operatorname{epi} f^* + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^* + \alpha\mathbb{B} \times \mathbb{R}_+.$$

Then, there exist $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$ such that

$$(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) \in \operatorname{epi} f^* + \operatorname{epi} \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i) \right)^* + \alpha\mathbb{B} \times \mathbb{R}_+. \tag{3.6}$$

Since $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$ are continuous functions, by Proposition 2.4, the inclusion (3.6) can be written as

$$(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) \in \operatorname{epi} f^* + \sum_{i=1}^{m} \operatorname{epi} (\bar{\lambda}_i g_i(\cdot, \bar{v}_i))^* + \alpha\mathbb{B} \times \mathbb{R}_+.$$

Furthermore, it follows from Proposition 2.3 that there exist $\epsilon_i \geq 0$, $i = 0, 1, \ldots, m$ such that

$$\begin{aligned}(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})\ ) \in & \bigcup_{\epsilon_0 \geq 0} \{(\xi_0, \langle \xi_0, \bar{x} \rangle + \epsilon_0 - f(\bar{x}))\colon \xi_0 \in \partial_{\epsilon_0} f(\bar{x})\} \\ & + \sum_{i=1}^{m} \bigcup_{\epsilon_i \geq 0} \big\{(\xi_i, \langle \xi_i, \bar{x} \rangle + \epsilon_i - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i))\colon \xi_i \in \partial_{\epsilon_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})\big\} + \alpha\mathbb{B} \times \mathbb{R}_+.\end{aligned}$$

It means that there exist $\bar{v}_i \in \mathcal{V}_i$, $\bar{\lambda}_i \geq 0$, $\bar{\xi}_0 \in \partial_{\bar{\epsilon}_0} f(\bar{x})$, $\bar{\xi}_i \in \partial_{\bar{\epsilon}_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \ldots, m$, $b \in \mathbb{B}$, $r \in \mathbb{R}_+$ and $\bar{\epsilon}_i \geq 0$, $i = 0, 1, \ldots, m$, such that

$$\begin{aligned}(0, -\alpha\|\bar{x}\| + \epsilon - f(\bar{x})) &= (\bar{\xi}_0, \langle \bar{\xi}_0, \bar{x} \rangle + \bar{\epsilon}_0 - f(\bar{x})) \\ &\quad + \sum_{i=1}^{m} (\bar{\xi}_i, \langle \bar{\xi}_i, \bar{x} \rangle + \bar{\epsilon}_i - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)) + (\alpha b, r),\end{aligned}$$

which is equivalent to say that

$$\begin{cases} 0 = \sum_{i=0}^{m} \bar{\xi}_i + \alpha b \\ \sum_{i=0}^{m} \bar{\epsilon}_i - \epsilon \leq \alpha\|\bar{x}\| - \alpha\langle b, \bar{x} \rangle - \epsilon + \sum_{i=0}^{m} \bar{\epsilon}_i + r = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \leq 0. \end{cases}$$

Thus, [(ii) $\Rightarrow$ (iii)] follows.

Now, we claim that [(iii) $\Rightarrow$ (i)], this implication is known as "sufficient condition". Assume that (iii) holds, i.e., there exist $\bar{\epsilon}_0 \geq 0$, $\bar{\epsilon}_i \geq 0$, $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$ such that (3.4) and (3.5) hold. Then, there exist $\bar{\xi}_0 \in \partial_{\bar{\epsilon}_0} f(\bar{x})$, $\bar{\xi}_i \in \partial_{\bar{\epsilon}_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \ldots, m$ and $b \in \mathbb{B}$ such that

$$0 = \bar{\xi}_0 + \sum_{i=1}^{m} \bar{\xi}_i + \alpha b. \tag{3.7}$$

Since $\bar{\xi}_0 \in \partial_{\bar{\epsilon}_0} f(\bar{x})$ and $\bar{\xi}_i \in \partial_{\bar{\epsilon}_i} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \ldots, m$, then

$$f(x) - f(\bar{x}) \geq \langle \bar{\xi}_0, x - \bar{x} \rangle - \bar{\epsilon}_0 \tag{3.8}$$

$$\bar{\lambda}_i g_i(x, \bar{v}_i) - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geq \langle \bar{\xi}_i, x - \bar{x} \rangle - \bar{\epsilon}_i, \quad i = 1, \ldots, m. \tag{3.9}$$

Along with (3.5) and (3.7), summing (3.8) and (3.9) yields that

$$\begin{aligned}f(x) - f(\bar{x}) \geq & - \sum_{i=1}^{m} \left[\bar{\lambda}_i g_i(x, \bar{v}_i) - \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)\right] + \langle \bar{\xi}_0, x - \bar{x} \rangle - \bar{\epsilon}_0 \\ & + \sum_{i=1}^{m} \left(\langle \bar{\xi}_i, x - \bar{x} \rangle - \bar{\epsilon}_i\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=0}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) + \left\langle \sum_{i=0}^{m} \bar{\xi}_i, x - \bar{x} \right\rangle - \sum_{i=0}^{m} \bar{\epsilon}_i \\
&\geq \langle -\alpha b, x - \bar{x} \rangle - \epsilon \\
&\geq -\alpha \|x - \bar{x}\| - \epsilon.
\end{aligned}
$$

Observe that $\bar{x}$ is a quasi (α, ϵ)-solution to problem (RCP), and (i) holds. Thus, the proof is completed. □

Remark 3.5. We mention here that Theorem 3.4 is still true if the robust Slater condition, i.e., $\{x \in \mathbb{R}^m \colon g_i(x_0, v_i) < 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} \neq \emptyset$, holds. Furthermore, Theorem 3.4 also covers [22, Theorem 2.2] and [21, Theorem 2] if $\alpha = 0$ and $\epsilon = 0$, respectively.

4. Approximate duality theorems

In this section, we focus on the study of duality relations between the primal problem (RCP) and its dual model. To this end, we now formulate the ε-Wolfe type dual model problem for the primal problem (RCP).

$$
\text{(WRD)} \qquad \max_{(y,v,\lambda)} \mathcal{L}(y, v, \lambda) \quad \text{subject to} \quad (y, v, \lambda) \in F_D,
$$

where $\mathcal{L}(y, v, \lambda) := f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)$, which is essentially the Lagrangian function, and F_D is the feasible set of (WRD), which is given by

$$
\begin{aligned}
F_D := \Big\{ (y, v, \lambda) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^m_+ \quad : \quad & 0 \in \partial_{\epsilon_0} f(y) + \sum_{i=1}^{m} \partial_{\epsilon_i} (\lambda_i g_i(\cdot, v_i))(y) + \alpha \mathbb{B}, \\
& \lambda_i \geq 0, v_i \in \mathcal{V}_i, \ i = 1, \ldots, m, \alpha \geq 0, \\
& \sum_{i=0}^{m} \epsilon_i \geq \epsilon, \ \epsilon_i \geq 0, \ i = 0, 1, \ldots, m \Big\}.
\end{aligned}
$$

Note that $v := (v_1, \ldots, v_m)$, and $\mathcal{V} := \mathcal{V}_1 \times \cdots \times \mathcal{V}_m$; moreover, $v \in \mathcal{V}$ stands for $(v_1, \ldots, v_m) \in \mathcal{V}_1 \times \cdots \times \mathcal{V}_m$.

Definition 4.1. Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given, then $(\bar{y}, \bar{v}, \bar{\lambda})$ is called a *quasi (α, ϵ)-solution* to (WRD) if for any $(y, v, \lambda) \in F_D$,

$$
f(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geq f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \alpha \|\bar{y} - y\| - \epsilon.
$$

The following theorem shows the approximate weak duality relationship between (RCP) and (WRD).

Theorem 4.2 (Approximate Weak Duality)**.** *Let x and (y, v, λ) be any feasible solutions of* (RCP) *and* (WRD), *respectively. Then,*

$$
f(x) \geq f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \alpha \|x - y\| - \epsilon.
$$

Proof. Let (y, v, λ) be any feasible solution to (WRD). Then, there exist $\alpha \geq 0$, $\epsilon_0 \geq 0$, $\epsilon_i \geq 0$, $\bar{\xi}_0 \in \partial_{\epsilon_0} f(y)$, $\bar{\xi}_i \in \partial \epsilon_i(\lambda_i g_i(\cdot, v_i))(y)$, $i = 1, \ldots, m$ with $\sum_{i=0}^m \epsilon_i \leq \epsilon$, and $b \in \mathbb{B}$ such that

$$\sum_{i=0}^{m} \bar{\xi}_i + \alpha b = 0.$$

Now, along with $\bar{\xi}_0 \in \partial_{\epsilon_0} f(y)$, $\bar{\xi}_i \in \partial \epsilon_i(\lambda_i g_i(\cdot, v_i))(y)$, $i = 1, \ldots, m$, we have

$$\begin{aligned} f(x) - \left(f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \right) &\geq \langle \bar{\xi}_0, x - y \rangle - \epsilon_0 - \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\ &= -\left\langle \sum_{i=1}^{m} \bar{\xi}_i + \alpha b, x - y \right\rangle - \epsilon_0 - \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\ &\geq -\sum_{i=1}^{m} \lambda_i \Big(g_i(x, v_i) - g_i(y, v_i) \Big) - \langle \alpha b, x - y \rangle \\ &\quad - \sum_{i=0}^{m} \epsilon_i - \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\ &\geq -\sum_{i=1}^{m} \lambda_i g_i(x, v_i) - \alpha \|b\| \cdot \|x - y\| - \epsilon \\ &\geq -\alpha \|x - y\| - \epsilon. \end{aligned}$$

Hence, the conclusion follows. □

In what follows, we investigate the approximate strong duality relationship between (RCP) and (WRD).

Theorem 4.3 (Approximate Strong Duality)**.** *Consider problems* (RCP) *and* (WRD)*. Suppose that the* (RCCCQ) *holds. If $\bar{x}$ is a quasi (α, ϵ)-solution to problem* (RCP)*, then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ and $\bar{v} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{v}, \bar{\lambda})$ is a quasi $(\alpha, 2\epsilon)$-solution to problem* (WRD)*.*

Proof. Let $\bar{x} \in F$ be a quasi (α, ϵ)-solution to problem (RCP). It follows from Theorem 3.4 that there exist $\alpha \geq 0$, $\epsilon_0 \geq 0$, $\epsilon_i \geq 0$, $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$, such that (3.4) and (3.5) hold. This implies that $(\bar{x}, \bar{v}, \bar{\lambda})$ is a feasible solution to problem (WRD). According to Theorem 4.2, for any feasible solution (y, v, λ) to problem (WRD),

$$\begin{aligned} \left(f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \right) - \left(f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \right) &\geq -\alpha \|\bar{x} - y\| - \epsilon \\ &\quad + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \\ &\geq -\alpha \|\bar{x} - y\| - \epsilon \\ &\quad + \sum_{i=0}^{m} \epsilon_i - \epsilon \\ &\geq -\alpha \|\bar{x} - y\| - 2\epsilon. \end{aligned}$$

Thus $(\bar{x}, \bar{v}, \bar{\lambda})$ is a quasi $(\alpha, 2\epsilon)$-solution to problem (WRD). □

We close this section by designing an example, which is motivated by [21, Example 2], to illustrate the approximate duality theorems.

Example 4.4. Consider the following convex optimization problem with data uncertainty:

$$(\text{RCP}_1) \qquad \min\ |x_1| + x_2^2 \quad \text{subject to} \quad x_1^2 - 2v_1 x_1 \leq 0,\ v_1 \in [-1, 1].$$

Let $f(x_1, x_2) = |x_1| + x_2^2$ and $g_1((x_1, x_2), v_1) = x_1^2 - 2v_1 x_1$. Then the robust feasible set of (RCP_1) is $F^1 := \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 = 0,\ x_2 \in \mathbb{R}\}$, and the robust quasi (α, ϵ)-solution set of (RCP_1) is $S_F^1 := \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 = 0,\ -\sqrt{\epsilon} - \frac{\alpha}{2} \leq x_2 \leq \sqrt{\epsilon} + \frac{\alpha}{2}\}$. Clearly, the Slater condition fails for (RCP_1). Moreover, it follows from [21, Example 2] that the cone $\bigcup_{\substack{v_1 \in [-1,1] \\ \lambda_1 \geq 0}} \operatorname{epi}(\lambda_1 g_1(\cdot, v_i))^*$ is closed and convex.

Below, we formulate the ε-Wolfe type dual model problem for the primal problem (RCP_1).

$$(\text{WRD}_1) \qquad \max_{\left((y_1,y_2),v_1,\lambda_1\right)} \mathcal{L}\big((y_1, y_2), v_1, \lambda_1\big) \quad \text{s.t.} \quad \big((y_1, y_2), v_1, \lambda_1\big) \in F_D^*,$$

where $\mathcal{L}\big((y_1, y_2), v, \lambda\big) = f(y_1, y_2) + \lambda_1 g_1\big((y_1, y_2), v_1\big)$, and F_D^* is the set of all robust feasible solutions to problem (WRD_1), which is given by

$$\begin{aligned} F_D^* := \big\{\big((y_1, y_2), v_1, \lambda_1\big) \in \mathbb{R}^2 \times [-1, 1] \times \mathbb{R}_+ \quad : \quad & 0 \in \partial_{\epsilon_0} f(y_1, y_2) + \\ & \partial_{\epsilon_1}(\lambda_1 g_1(\cdot, v_1))(y_1, y_2) + \alpha\mathbb{B}, \\ & \epsilon_1 \geq 0,\ \epsilon_2 \geq 0,\ \epsilon_1 + \epsilon_2 \leq \epsilon\big\}. \end{aligned}$$

It is worth mentioning that the feasible set F_D^* of the ε-Wolfe type dual model problem (WRD_1) contains the one in [21, Example 2], since the solution type (quasi (α, ϵ)-solutions) studying here is different to the one (quasi α-solutions) in [21, Example 2]. Now, we calculate the set $F_D^* = F_D^a \cup F_D^b \cup F_D^c$, where

$$\begin{aligned} F_D^a &= \{((0, y_2), v_1, \lambda_1) \colon (0, 0) \in \partial_{\epsilon_0} f(0, y_2) + \partial_{\epsilon_1}(\lambda_1 g_1(\cdot, v_1))(0, y_2) + \alpha\mathbb{B}, \\ &\qquad \lambda_1 \geq 0,\ v_1 \in [-1, 1],\ \epsilon_1 \geq 0,\ \epsilon_2 \geq 0,\ \epsilon_1 + \epsilon_2 \leq \epsilon\} \\ &= \Big\{((0, y_2), v_1, \lambda_1) \colon \left|y_2 + \frac{\alpha b_2}{2}\right| \leq \sqrt{\epsilon_0},\ |2v_1\lambda_1 - \alpha b_1| \leq 1 + \sqrt{\lambda_1 \epsilon_1}, \\ &\qquad b_1^2 + b_2^2 \leq 1,\ \lambda_1 \geq 0,\ v_1 \in [-1, 1],\ \epsilon_0 \geq 0,\ \epsilon_1 \geq 0, \epsilon_0 + \epsilon_1 \leq \epsilon\Big\}, \end{aligned}$$

$$\begin{aligned} F_D^b &= \{((y_1, y_2), v_1, \lambda_1) \colon y_1 > 0,\ (0, 0) \in \partial_{\epsilon_0} f(y_1, y_2) + \partial_{\epsilon_1}(\lambda_1 g_1(\cdot, v_1))(y_1, y_2) \\ &\qquad + \alpha\mathbb{B},\ \lambda_1 \geq 0,\ v_1 \in [-1, 1],\ \epsilon_0 \geq 0,\ \epsilon_1 \geq 0, \epsilon_0 + \epsilon_1 \leq \epsilon\} \\ &= \Big\{((y_1, y_2), v_1, \lambda_1) \colon y_1 > 0,\ 1 + 2\lambda_1 y_1 - 2\lambda_1 v_1 - 2\sqrt{\lambda_1 \epsilon_1} + \alpha b_1 \leq 0 \leq 1 \\ &\qquad + 2\lambda_1 y_1 - 2\lambda_1 v_1 + 2\sqrt{\lambda_1 \epsilon_1} + \alpha b_1,\ \left|y_2 + \frac{\alpha b_2}{2}\right| \leq \sqrt{\epsilon_0}, \\ &\qquad b_1^2 + b_2^2 \leq 1,\ \lambda_1 \geq 0,\ v_1 \in [-1, 1],\ \epsilon_0 \geq 0,\ \epsilon_1 \geq 0, \epsilon_0 + \epsilon_1 \leq \epsilon\Big\} \end{aligned}$$

$$
\begin{aligned}
= \ & \Big\{((y_1,y_2),v_1,\lambda_1)\colon y_1>0, \left|y_2+\frac{\alpha b_2}{2}\right| \le \sqrt{\epsilon_0}, \lambda_1 y_1^2 - 2\sqrt{\lambda_1\epsilon_1}y_1 + \alpha b_1 y_1 \\
& \le -\lambda_1 y_1^2 - (1-2\lambda_1 v_1)y_1 \le \lambda_1 y_1^2 + 2\sqrt{\lambda_1\epsilon_1}y_1 + \alpha b_1 y_1, \\
& b_1^2+b_2^2\le 1, \lambda_1\ge 0,\ v_1\in[-1,1],\ \epsilon_0\ge 0,\ \epsilon_1\ge 0, \epsilon_0+\epsilon_1\le\epsilon\Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_D^c = \ & \{((y_1,y_2),v_1,\lambda_1)\colon y_1<0,\ (0,0)\in\partial_{\epsilon_0} f(y_1,y_2)+\partial_{\epsilon_1}(\lambda_1 g_1(\cdot,v_1))(y_1,y_2) \\
& +\alpha\mathbb{B},\ \lambda_1\ge 0,\ \alpha\mathbb{B},\ \lambda_1\ge 0,\ v_1\in[-1,1],\ \epsilon_0\ge 0,\ \epsilon_1\ge 0, \epsilon_0+\epsilon_1\le\epsilon\} \\
= \ & \Big\{((y_1,y_2),v_1,\lambda_1)\colon y_1<0,\ -1+2\lambda_1 y_1 - 2\lambda_1 v_1 - 2\sqrt{\lambda_1\epsilon_1}+\alpha b_1 \le 0 \le \\
& -1+2\lambda_1 y_1 - 2\lambda_1 v_1 + 2\sqrt{\lambda_1\epsilon_1}+\alpha b_1,\ \left|y_2+\frac{\alpha b_2}{2}\right|\le\sqrt{\epsilon_0}, \\
& b_1^2+b_2^2\le 1,\ \lambda_1\ge 0,\ v_1\in[-1,1],\ \epsilon_0\ge 0,\ \epsilon_1\ge 0, \epsilon_0+\epsilon_1\le\epsilon\Big\} \\
= \ & \Big\{((y_1,y_2),v_1,\lambda_1)\colon y_1<0, \left|y_2+\frac{\alpha b_2}{2}\right| \le \sqrt{\epsilon_0}, \lambda_1 y_1^2 - 2\sqrt{\lambda_1\epsilon_1}y_1 + \alpha b_1 y_1 \\
& \le -\lambda_1 y_1^2 + (1+2\lambda_1 v_1)y_1 \le \lambda_1 y_1^2 + 2\sqrt{\lambda_1\epsilon_1}y_1 + \alpha b_1 y_1,\ b_1^2+b_2^2\le 1, \\
& \lambda_1\ge 0,\ v_1\in[-1,1],\ \epsilon_0\ge 0,\ \epsilon_1\ge 0, \epsilon_0+\epsilon_1\le\epsilon\Big\}.
\end{aligned}
$$

Then, for any $(x_1,x_2)\in F^1$ and any $\big((y_1,y_2),v_1,\lambda_1\big)\in F_D^a$,

$$
\begin{aligned}
& f(x_1,x_2) - [f(y_1,y_2)+\lambda_1 g_1(y_1,y_2,v_1) - \alpha\|(x_1,x_2)-(y_1,y_2)\| - \epsilon] \\
& = x_2^2 - y_2^2 + \alpha|x_2-y_2| + \epsilon \\
& \ge \left(x_2^2+\frac{\alpha b_2}{2}\right)^2 - \alpha b_2 x_2 + \alpha b_2 y_2 + \alpha|x_2-y_2| + \epsilon_1 \\
& \ge -\alpha b_2(x_2-y_2) + \alpha|x_2-y_2| + \epsilon_1 \\
& \ge -\alpha|b_2||x_2-y_2| + \alpha|x_2-y_2| + \epsilon_1 \\
& \ge -\alpha|x_2-y_2| + \alpha|x_2-y_2| + \epsilon_1 \ge 0.
\end{aligned}
$$

Moreover, for any $(x_1,x_2)\in F^1$ and any $\big((y_1,y_2),v_1,\lambda_1\big)\in F_D^b$, if $\lambda_1=0$, then

$$
\begin{aligned}
& f(x_1,x_2) - [f(y_1,y_2)+\lambda_1 g_1(y_1,y_2,v_1) - \alpha\|(x_1,x_2)-(y_1,y_2)\| - \epsilon] \\
& = x_2^2 - y_1 - y_2^2 + \alpha\sqrt{y_1^2+(x_2-y_2)^2} + \epsilon \\
& \ge x_2^2 - y_1 + \alpha b_2 y_2 + \frac{\alpha^2 b_2^2}{4} + \alpha\sqrt{y_1^2+(x_2-y_2)^2} + \epsilon_1 \\
& \ge -y_1 - \alpha b_2 x_2 + \alpha b_2 y_2 + \alpha\sqrt{y_1^2+(x_2-y_2)^2} + \epsilon_1 \\
& = \alpha b_1 y_1 - \alpha b_2 x_2 + \alpha b_2 y_2 + \alpha\sqrt{y_1^2+(x_2-y_2)^2} + \epsilon_1 \\
& \ge -\alpha\sqrt{b_1^2+b_2^2}\sqrt{y_1^2+(x_2-y_2)^2} + \alpha\sqrt{y_1^2+(x_2-y_2)^2} + \epsilon_1
\end{aligned}
$$

$$\geq -\alpha\sqrt{y_1^2+(x_2-y_2)^2}+\alpha\sqrt{y_1^2+(x_2-y_2)^2}+\epsilon_1$$
$$= \epsilon_1 \geq 0.$$

Otherwise, if $\lambda > 0$, then

$$\begin{aligned}
&f(x_1,x_2) - [f(y_1,y_2)+\lambda_1 g_1(y_1,y_2,v_1) - \alpha\|(x_1,x_2)-(y_1,y_2)\| - \epsilon]\\
&= x_2^2 - y_1 - y_2^2 - \lambda_1 y_1^2 + 2\lambda_1 v_1 y_1 + \alpha\sqrt{y_1^2+(x_2-y_2)^2}+\epsilon\\
&\geq x_2^2 + \alpha b_2 y_2 + \frac{\alpha^2 b_2^2}{4} - \lambda_1 y_1^2 + (2\lambda_1 v_1 - 1)y_1 + \alpha\sqrt{y_1^2+(x_2-y_2)^2}+\epsilon_1\\
&\geq x_2^2 + \alpha b_2 y_2 + \frac{\alpha^2 b_2^2}{4} + \lambda_1 y_1^2 - 2\sqrt{\lambda_1\epsilon_1}y_1 + \alpha b_1 y_1 + \alpha\sqrt{y_1^2+(x_2-y_2)^2}+\epsilon_1\\
&\geq \alpha(b_1y_1 - b_2x_2 + b_2y_2) + \alpha\sqrt{y_1^2+(x_2-y_2)^2}\\
&\geq -\alpha\sqrt{b_1^2+b_2^2}\sqrt{y_1^2+(x_2-y_2)^2} + \alpha\sqrt{y_1^2+(x_2-y_2)^2}\\
&\geq -\alpha\sqrt{y_1^2+(x_2-y_2)^2} + \alpha\sqrt{y_1^2+(x_2-y_2)^2}\\
&= 0.
\end{aligned}$$

Similarly, we can easily show that for any $(x_1,x_2) \in F^1$ and $\big((y_1,y_2),v_1,\lambda_1\big) \in F_D^c$,

$$f(x_1,x_2) \geq f(y_1,y_2)+\lambda_1 g_1(y_1,y_2,v_1) - \alpha\|(x_1,x_2)-(y_1,y_2)\| - \epsilon.$$

The foregoing calculations imply that for any feasible solution (x_1,x_2) of (RCP_1) and any feasible solution $((y_1,y_2),v_1,\lambda_1)$ of (WRD_1),

$$f(x_1,x_2) \geq f(y_1,y_2)+\lambda_1 g_1((y_1,y_2),v_1) - \alpha\|(x_1,x_2)-(y_1,y_2)\| - \epsilon,$$

that is, Theorem 4.2 (Approximate Weak Duality) holds.

Furthermore, let $(\bar{x}_1,\bar{x}_2) \in S_F^1$ be a quasi (α,ϵ)-solution of (RCP_1). Then $\bar{x}_1 = 0$ and $-\sqrt{\epsilon}-\frac{\alpha}{2} \leq \bar{x}_2 \leq \sqrt{\epsilon}+\frac{\alpha}{2}$. Let $\bar{\lambda}_1 = \frac{\alpha}{2}$, $\bar{v}_1 = b_1$. Then, we can easily check that there exist suitable $\epsilon_0 \geq 0, \epsilon_1 \geq 0$ with $\epsilon_0+\epsilon_1 \leq \epsilon$ and $(b_1,b_2) \in \mathbb{B}$ and $((\bar{x}_1,\bar{x}_2),\bar{v}_1,\bar{\lambda}_1) \in F_D^1$. Moreover, for any $((y_1,y_2),v_1,\lambda_1) \in F_D^1$,

$$\begin{aligned}
[f(\bar{x}_1,\bar{x}_2) \quad &+ \quad \bar{\lambda}_1 g((\bar{x}_1,\bar{x}_2),\bar{v}_1)] - [f(y_1,y_2)+\lambda_1 g((y_1,y_2),v_1)]\\
&\geq \quad -\epsilon - \alpha\|(\bar{x}_1,\bar{x}_2)-(y_1,y_2)\| + \bar{\lambda}_1 g((\bar{x}_1,\bar{x}_2),\bar{v}_1) \qquad (4.1)\\
&\geq \quad -2\epsilon - \alpha\|(\bar{x}_1,\bar{x}_2)-(y_1,y_2)\|, \qquad (4.2)
\end{aligned}$$

where (4.1) is from Theorem 4.2 (Approximate Weak Duality), and (4.2) is based on the condition $-\epsilon \leq \bar{\lambda}_1 g_1((\bar{x}_1,\bar{x}_2),\bar{v}_1)$. This means $((\bar{x}_1,\bar{x}_2),\bar{v}_1,\bar{\lambda}_1)$ is a quasi $(\alpha,2\epsilon)$-solution of (WRD_1), and thus Theorem 4.3 (Approximate Strong Duality) holds.

5. Approximate saddle point results

Let $\alpha \geq 0$ be given. The α-Lagrangian function (with respective to a reference point $\bar{y} \in \mathbb{R}^n$) associated to problem (RCP), denoted by $\mathcal{L}_\alpha\colon \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}_+^m \to \mathbb{R}$ is

$$\mathcal{L}_\alpha(y,v,\lambda) := f(y) + \sum_{i=1}^m \lambda_i g_i(y,v_i) + \alpha\|y-\bar{y}\|.$$

In what follows, we give the definition of the so-called quasi (α, ϵ)-saddle point to the α-Lagrangian function $\mathcal{L}_\alpha$.

Definition 5.1. Let $\epsilon \geq 0$ be given. A point $(\bar{y}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^m_+$ is said to be a *quasi* (α, ϵ)*-saddle point* to the α-Lagrangian function $\mathcal{L}_\alpha$ if for any $(y, v, \lambda) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^m_+$,

$$\mathcal{L}_\alpha(\bar{y}, v, \lambda) - \epsilon \leq \mathcal{L}_\alpha(\bar{y}, \bar{v}, \bar{\lambda}) \leq \mathcal{L}_\alpha(y, \bar{v}, \bar{\lambda}) + \epsilon.$$

The following results are approximate saddle point theorems, which tell us the necessary and sufficient relationships between a quasi (α, ϵ)-solution to problem (RCP) and a quasi (α, ϵ)-saddle point to the α-Lagrangian function $\mathcal{L}_\alpha$.

Theorem 5.2 (Necessity)**.** *Suppose that the* (RCCCQ) *holds. If* $\bar{y} \in F$ *is a quasi* (α, ϵ)*-solution to problem* (RCP)*, then there exist* $\bar{v} \in \mathcal{V}$ *and* $\bar{\lambda} \in \mathbb{R}^m_+$ *such that* $(\bar{y}, \bar{v}, \bar{\lambda})$ *is a quasi* (α, ϵ)*-saddle point to the* α*-Lagrangian function* $\mathcal{L}_\alpha$*.*

Proof. Let $\bar{y} \in F$ be a quasi (α, ϵ)-solution to problem (RCP), then for all $y \in F$,

$$f(y) + \alpha\|y - \bar{y}\| + \epsilon \geq f(\bar{y}).$$

Since the (RCCCQ) holds, by Theorem 3.3, there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that

$$f(y) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(y, \bar{v}_i) + \alpha\|y - \bar{y}\| + \epsilon \geq f(\bar{y}), \quad \forall y \in \mathbb{R}^n. \tag{5.1}$$

Note that we always have $\bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \leq 0$ since $\bar{y} \in F$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$.

It follows from (5.1) that

$$\begin{aligned} \mathcal{L}_\alpha(y, \bar{v}, \bar{\lambda}) + \epsilon &\geq f(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) + \alpha\|\bar{y} - \bar{y}\| \\ &= \mathcal{L}_\alpha(\bar{y}, \bar{v}, \bar{\lambda}). \end{aligned} \tag{5.2}$$

Now, let $y = \bar{y}$ in (5.1), we have $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) + \epsilon \geq 0$, then also

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geq -\epsilon + \sum_{i=1}^{m} \lambda_i g_i(\bar{y}, v_i), \tag{5.3}$$

for any $\lambda_i \geq 0$, $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$. Furthermore, along with (5.3), we have

$$\begin{aligned} \mathcal{L}_\alpha(\bar{y}, v, \lambda) - \epsilon &= f(\bar{y}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{y}, v_i) + \alpha\|\bar{y} - \bar{y}\| - \epsilon \\ &\leq f(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) + \alpha\|\bar{y} - \bar{y}\| \\ &= \mathcal{L}_\alpha(\bar{y}, \bar{v}, \bar{\lambda}). \end{aligned}$$

This, together with (5.2), yields the desired result. □

Theorem 5.3 (Sufficiency)**.** *If* $(\bar{y}, \bar{v}, \bar{\lambda})$ *is a quasi* (α, ϵ)*-saddle point to the* α*-Lagrangian function* $\mathcal{L}_\alpha$*, then* $\bar{y}$ *is a quasi* $(\alpha, 2\epsilon)$*-solution to problem* (RCP)*.*

Proof. let $(\bar{y}, \bar{v}, \bar{\lambda})$ be a quasi (α, ϵ)-saddle point to the α-Lagrangian function $\mathcal{L}_\alpha$. Then it follows from the first inequality in Definition 5.1 that

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{y}, v_i) - \epsilon \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i).$$

Particularly, if $\bar{y} \in F$, then the above inequality yields

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geq \sum_{i=1}^{m} \lambda_i g_i(\bar{y}, v_i) - \epsilon \geq -\epsilon. \tag{5.4}$$

On the other hand, from the second inequality in Definition 5.1, we have

$$f(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \leq f(y) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(y, \bar{v}_i) + \alpha\|\bar{y} - \bar{y}\| + \epsilon. \tag{5.5}$$

Then, along with (5.4) and (5.5), for any $y \in F$,

$$\begin{aligned} f(y) + \alpha\|y - \bar{y}\| + \epsilon &\geq f(y) + \alpha\|y - \bar{y}\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(y, \bar{v}_i) + \epsilon \\ &\geq f(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \\ &\geq f(\bar{y}) - \epsilon. \end{aligned}$$

Thus, $f(\bar{y}) \leq f(y) + \alpha\|y - \bar{y}\| + 2\epsilon$ for any $y \in F$. Consequently, $\bar{y}$ is a quasi $(\alpha, 2\epsilon)$-solution to problem (RCP). □

6. Conclusions

In this paper, we studied approximate optimality theorems (both necessary and sufficient) and approximate duality results for a quasi (α, ϵ)-solution to problem (RCP). The results generalized the ones in [21, 22]. Besides, we explored some approximate saddle point results to the α-Lagrangian function $\mathcal{L}_\alpha$ associated to problem (RCP). It will be also meaningful to examine some characterizations of a quasi (α, ϵ)-solution in some other mathematical settings.

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Manuscript received January 13, 2023
revised December 20, 2023

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