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COMMON FIXED POINT THEOREMS FOR MULTIVALUED GENERALIZED (α,β)-NONEXPANSIVE MAPPINGS IN BANACH SPACES

ATIT WIRIYAPONGSANON, WARUNUN INTHAKON,
AND NARAWADEE PHUDOLSITTHIPHAT*

Abstract. The purpose of this paper is to introduce new iteration schemes and establish weak and strong convergence theorems for the common fixed points of multivalued generalized (α,β)-nonexpansive mappings. Furthermore, we provide an illustrative example to demonstrate that our algorithm results in faster convergence compared to several existing iteration processes.

1. Introduction

One of the fastest-growing research areas in nonlinear functional analysis is fixed point theory, which can be applied to find solutions to many problems such as ordinary and partial differential equations, variational inequalities, and nonlinear optimization. For more details, see [4–6, 10, 13, 14, 21, 26, 29, 30], as well as the references therein. As a consequence, several researchers in nonlinear analysis have been actively developing faster and more efficient iterative algorithms. Some well-known iterative schemes were introduced by Mann [15], Ishikawa [11], Agarwal et al. [2], Nakajo and Takahashi [16], Thakur et al. [31, 32], Ritika and Khan [20], and Ullah and Arshad [34]. In 2018, Ullah and Arshad [35] introduced the M-iteration process and provided an example to verify that it converges faster than Picard-S [9] and S iteration [2]. Later, Garodia et al. [8] extended the results of [35] to estimate common fixed points of two mappings, achieving a faster rate of convergence for Suzuki generalized nonexpansive mappings. Nowadays, for supporting good health and well-being, fixed point theory can be applied to image restoration problems and some data classification problems involving noncommunicable diseases (NCDs), see [27, 28, 33] for examples.

In case of multivalued mappings, in 2005, Sastry and Babu [22] studied the convergence of Mann and Ishikawa iterative processes in Hilbert spaces. Later, Panyanak [19] generalized the result of Sastry and Babu to the framework of uniformly convex Banach spaces by assuming the endpoint condition. To avoid this condition, Shehzad and Zegeye [24] defined $P_T(x) = \{y \in T(x) : d(x, Tx) = ||x - y||\}$ for multivalued mapping $T : K \to P(K)$, where $P(K)$ is the class of all nonempty bounded proximinal subsets of K, and presented convergence results of Ishikawa iteration process in a uniformly convex Banach space. Recently, Ullah et al. [36]

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*Corresponding author.

modified the M-iterative process for a multivalued version as follows: let K be a nonempty convex subset of E, $\alpha_n \in (0,1)$.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = l_n, \\ w_n = o_n, \\ m_n = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{1.1}$$

where $u_n \in P_T(x_n),\ o_n \in P_T(m_n)$ and $l_n \in P_T(w_n)$.
This scheme was employed for finding fixed points of multivalued generalized (α,β)-nonexpansive mappings. They also gave an example to guarantee that M-iterative scheme converges faster than the iteration of Noor [17], Picard-Mann hybrid [12], Abbas and Nazir [1] and of Picard-S [9].

Motivated by the research going on in this direction, we instigate a new modified algorithm for approximating common fixed points of two multivalued generalized (α,β)-nonexpansive mappings. Moreover, a numerical example is provided to show that our algorithm leads to a faster convergence comparing to a number of existing iteration processes.

2. Preliminaries

Let T and S be two multivalued mappings defined on a nonempty subset K of a Banach space E. The set of fixed points of T is denoted by

$$F(T) = \{x \in K : x \in Tx\}.$$

A mapping T is said to satisfy the endpoint condition [7, 19] if $Tp = \{p\}$ for any $p \in F(T)$. The set of common fixed points of T and S is denoted by

$$CF(T,S) = \{x \in K : x \in F(T) \text{ and } x \in F(S)\}.$$

We call K a proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$||x-k|| = \inf\{||x-y|| : y \in K\} = d(x,K).$$

It can be seen that set of all proximinal includes weakly compact and convex subsets of a Banach space and closed convex subsets of a reflexive Banach space. Let $CB(K)$ be the class of all nonempty bounded and closed subsets of K. It is not difficult to show that every proximinal subset K of E is closed. Hence, $P(K) \subseteq CB(K)$. The Hausdorff metric on $CB(E)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

for every $A, B \in CB(E)$, where $d(x,B) = \inf_{b\in B} ||x-b||$. A multivalued mapping $T : K \to P(K)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq ||x-y||,$$

for all $x, y \in K$. If $F(T) \neq \emptyset$ and

$$H(Tx,Tp) \leq ||x-p||,$$

for all $x \in K$ and $p \in F(T)$, then T is said to be quasi-nonexpansive.

Definition 2.1 ([37])**.** Consider a multivalued mapping $T : K \to 2^K$. Then, T is called generalized (α,β)-nonexpansive if there exist two positive real constants α,β with $\alpha + \beta < 1$ and for all $x, y \in K$, we have

$$\begin{aligned}&\frac{1}{2}d(x, Tx) \leq ||x - y|| \text{ implies that}\\ &H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + \beta d(x, Tx)\\ &\qquad\qquad + \beta d(y, Ty) + (1 - 2\alpha - 2\beta)||x - y||.\end{aligned}$$

Lemma 2.2 ([37])**.** *Suppose a Banach space E and $\emptyset \neq K \subset E$ and also consider a multivalued mapping $T : K \to CB(K)$. If T is generalized (α,β)-nonexpansive with $F(T) \neq \emptyset$ and satisfies the endpoint condition, then T is quasi-nonexpansive.*

Lemma 2.3 ([37])**.** *Let K be a nonempty subset of a Banach space E and let $T : K \to CB(K)$ be a generalized (α,β)-nonexpansive multivalued mapping. Then,*

$$d(x, Ty) \leq \left(\frac{3 + \alpha + \beta}{1 - \alpha - \beta}\right) d(x, Tx) + ||x - y||, \quad \text{for } x, y \in K.$$

The following results are necessary for proving our main theorems.

Proposition 2.4 ([25])**.** *Let K be a nonempty subset of a metric space E and $T : K \to P(K)$ be a multivalued mapping. Then the following conditions are equivalent:*

(i) *$x \in F(T)$, that is, $x \in Tx$,*
(ii) *$P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$,*
(iii) *$x \in F(P_T)$, that is, $x \in P_T(x)$.*

Further, $F(T) = F(P_T)$.

By Proposition 2.4, it is clear that P_T satisfies the endpoint condition. The following results also play essential roles in this paper.

Lemma 2.5 ([23])**.** *Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in N$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} ||x_n|| \leq r$, $\limsup_{n\to\infty} ||y_n|| \leq r$ and $\lim_{n\to\infty} ||(1 - t_n)x_n + t_n y_n|| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Definition 2.6 ([18])**.** Let E be a Banach space. The space E is said to be endowed with Opial's condition if for any sequence $\{x_n\} \subset E$, with $x_n \rightharpoonup x$, it follows that

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||,$$

where $y \in E$ and $y \neq x$.

Definition 2.7 ([37])**.** A multivalued mapping $T : K \to CB(K)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K that is $x_n \rightharpoonup x$ for some $x \in K$ and $y_n \in T(x_n)$, $n \in \mathbb{N}$, which converges strongly to y then we have $y \in T(x)$.

3. Main results

We begin this section by modifying the iteration process given by (1.1) to approximate common fixed points of two mappings, i.e., let K be a nonempty convex subset of E and $T, S : K \to P(K)$ be two multivalued mappings and $\alpha_n, \beta_n \in [0,1]$. Let $\{x_n\}$ be a sequence defined as:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = l_n, \\ m_n = (1-\alpha_n)x_n + \alpha_n u_n, \\ s_n = (1-\beta_n)m_n + \beta_n v_n, \\ w_n = o_n, \end{cases} \tag{3.1}$$

where $u_n \in P_T(x_n),\ v_n \in P_S(m_n),\ o_n \in P_T(s_n)$ and $l_n \in P_S(w_n)$.

Note that our algorithm can be reduce to M-iteration when $P_T = P_S$ and $\beta_n = 0$, for all $n \in \mathbb{N}$.

Lemma 3.1. *Let K be a nonempty closed convex subset of uniformly convex Banach space E and let $T, S : K \to P(K)$ be two multivalued mappings such that $CF(T,S) \neq \emptyset$. Assume that P_T, P_S are generalized (α,β)-nonexpansive mappings. Suppose that $\{x_n\}$ is a sequence defined by* (3.1) *such that* $0 < a \leq \alpha_n, \beta_n \leq b < 1$. *Then, for $p \in CF(T,S)$,* $\lim_{n\to\infty} ||x_n - p||$ *exists and* $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0 = \lim_{n\to\infty} d(x_n, P_S(x_n))$.

Proof. If $p \in CF(T,S)$, then by Proposition 2.4 and Lemma 2.2, we have

$$\begin{aligned} ||m_n - p|| &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||u_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n H(P_T(x_n), P_T(p)) \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||x_n - p|| \\ &= ||x_n - p||. \end{aligned} \tag{3.2}$$

By the same token, we obtain

$$\begin{aligned} ||s_n - p|| &\leq (1-\beta_n)||m_n - p|| + \beta_n ||v_n - p|| \\ &\leq (1-\beta_n)||m_n - p|| + \beta_n H(P_S(m_n), P_S(p)) \\ &\leq (1-\beta_n)||m_n - p|| + \beta_n ||m_n - p|| \\ &= ||m_n - p||. \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned} ||w_n - p|| &= ||o_n - p|| \\ &\leq H(P_T(s_n), P_T(p)) \\ &\leq ||s_n - p||. \end{aligned} \tag{3.4}$$

By (3.2), (3.3) and (3.4),

$$||x_{n+1} - p|| = ||l_n - p||$$

$$\leq H(P_S(w_n), P_S(p))$$
$$\leq ||w_n - p||$$
$$\leq ||s_n - p|| \tag{3.5}$$
$$\leq ||m_n - p|| \tag{3.6}$$
$$\leq ||x_n - p|| \tag{3.7}$$

By (3.7), we have $\{||x_n - p||\}$ is bounded and nonincreasing, $\lim_{n\to\infty} ||x_n - p||$ exists for $p \in CF(T, S)$. Suppose that

$$\lim_{n\to\infty} ||x_n - p|| = c. \tag{3.8}$$

Then,

$$\limsup_{n\to\infty} ||x_n - p|| \leq c.$$

Since

$$||u_n - p|| \leq H(P_T(x_n), P_T(p)) \leq ||x_n - p||,$$

we obtain

$$\limsup_{n\to\infty} ||u_n - p|| \leq c.$$

By (3.6) and (3.8), we have

$$\begin{aligned} c &= \lim_{n\to\infty} ||x_{n+1} - p|| \\ &\leq \lim_{n\to\infty} ||m_n - p|| \\ &= \lim_{n\to\infty} ||(1 - \alpha_n)(x_n - p) + \alpha_n(u_n - p)|| \\ &\leq c. \end{aligned}$$

From Lemma 2.5, we get

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{3.9}$$

Hence,

$$\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0.$$

By (3.9) and $0 < a \leq \alpha_n \leq b < 1$, we have

$$\begin{aligned} \lim_{n\to\infty} ||x_n - m_n|| &= \lim_{n\to\infty} ||x_n - (1 - \alpha_n)x_n - \alpha_n u_n|| \\ &\leq \lim_{n\to\infty} \alpha_n ||x_n - u_n|| \\ &= 0. \end{aligned} \tag{3.10}$$

From (3.2), we get

$$\limsup_{n\to\infty} ||m_n - p|| \leq \limsup_{n\to\infty} ||x_n - p|| \leq c.$$

Since

$$||v_n - p|| \leq H(P_S(m_n), P_S(p)) \leq ||m_n - p||,$$

we obtain

$$\limsup_{n\to\infty} ||v_n - p|| \leq c.$$

By (3.5), we have

$$\begin{aligned} c &= \lim_{n\to\infty} ||x_{n+1} - p|| \\ &\leq \lim_{n\to\infty} ||s_n - p|| \\ &= \lim_{n\to\infty} ||(1-\beta_n)(m_n - p) + \beta_n(v_n - p)|| \\ &\leq c. \end{aligned}$$

From Lemma 2.5, we get

$$\lim_{n\to\infty} ||m_n - v_n|| = 0.$$

Hence,

$$\lim_{n\to\infty} d(m_n, P_S(m_n)) = 0. \tag{3.11}$$

From (3.10) and (3.11), we can conclude that $\lim_{n\to\infty} d(x_n, P_S(x_n)) = 0$. □

Theorem 3.2. *Let E be a uniformly convex Banach space and K be a nonempty compact and convex subset of E. Let $T, S : K \to P(K)$ be two multivalued mappings such that $CF(T,S) \neq \emptyset$. Assume that P_T, P_S are generalized (α,β)-nonexpansive mappings. Let $\{x_n\}$ be a sequence defined by* (3.1) *such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then, $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Proof. By Lemma 3.1,

$$\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0 \ \text{ and } \ \lim_{n\to\infty} d(x_n, P_S(x_n)) = 0.$$

Since K is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges to $p \in K$. By Lemma 2.3, it follows that

$$d(x_{n_i}, P_T(p)) \leq \left(\frac{3+\alpha_T+\beta_T}{1-\alpha_T-\beta_T}\right) d(x_{n_i}, P_T(x_{n_i})) + ||x_{n_i} - p||,$$

and

$$d(x_{n_i}, P_S(p)) \leq \left(\frac{3+\alpha_S+\beta_S}{1-\alpha_S-\beta_S}\right) d(x_{n_i}, P_S(x_{n_i})) + ||x_{n_i} - p||.$$

Letting $i \to \infty$, we derive $p \in F(P_T) \cap F(P_S)$. By Proposition 2.4, $CF(T,S) = F(P_T) \cap F(P_S)$. Therefore, $\{x_n\}$ converges strongly to $p \in CF(T,S)$. □

Theorem 3.3. *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T, S : K \to P(K)$ be two multivalued mappings such that $CF(T,S) \neq \emptyset$. Assume that P_T, P_S are generalized (α,β)-nonexpansive mappings. Let $\{x_n\}$ be a sequence defined by* (3.1) *such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then, $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if* $\liminf_{n\to\infty} d(x_n, CF(T,S)) = 0$.

Proof. If the sequence $\{x_n\}$ converges to $p \in CF(T,S)$, then $\lim_{n\to\infty} ||x_n - p|| = 0$. Since $0 \leq d(x_n, CF(T,S)) \leq ||x_n - p||$, we have $\liminf_{n\to\infty} d(x_n, CF(T,S)) = 0$. Conversely, assume that $\liminf_{n\to\infty} d(x_n, CF(T,S)) = 0$.
By Lemma 3.1,

$$d(x_{n+1}, CF(T,S)) \leq d(x_n, CF(T,S)).$$

Thus, $\{d(x_n, CF(T,S))\}$ is a nonincreasing sequence which is bounded below implying that $\lim_{n\to\infty} d(x_n, CF(T,S))$ exists. By hypothesis, $\lim_{n\to\infty} d(x_n, CF(T,S)) = 0$.

Next, we want to show $\{x_n\}$ is Cauchy. Let $m, n \in \mathbb{N}$ and $m > n$. From (3.7), it follows that

$$||x_{n+1} - p|| \leq ||x_n - p|| \quad \text{for all} \;\; p \in CF(T,S).$$

Then,

$$||x_m - x_n|| \leq ||x_m - p|| + ||x_n - p|| \leq 2||x_n - p||.$$

Taking infimum all over $p \in CF(T,S)$ on both sides, we obtain

$$||x_m - x_n|| \leq 2\inf\{||x_n - p|| : p \in CF(T,S)\} = 2d(x_n, CF(T,S)).$$

Since $\lim_{n\to\infty} d(x_n, CF(T,S)) = 0$, we can conclude that $\lim_{n\to\infty} ||x_m - x_n|| = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in E. As a result, there exists $z \in E$ such that $\lim_{n\to\infty} ||x_n - z|| = 0$. Next, we will show that $z \in CF(T,S)$. Consider,

$$\begin{aligned} d(z, P_T(z)) &\leq ||z - x_n|| + d(x_n, P_T(x_n)) + H(P_T(x_n), P_T(z)) \\ &\leq ||z - x_n|| + d(x_n, P_T(x_n)) + ||x_n - z||, \end{aligned}$$

and

$$\begin{aligned} d(z, P_S(z)) &\leq ||z - x_n|| + d(x_n, P_S(x_n)) + H(P_S(x_n), P_S(z)) \\ &\leq ||z - x_n|| + d(x_n, P_S(x_n)) + ||x_n - z||. \end{aligned}$$

By Lemma 3.1 and taking $n \to \infty$ on both sides, we infer that

$$d(z, P_T(z)) = 0 \;\; \text{and} \;\; d(z, P_S(z)) = 0.$$

Then,

$$P_T(z) = \{z\} \;\; \text{and} \;\; P_S(z) = \{z\}.$$

By Proposition 2.4, $z \in CF(T,S)$. Therefore, $\{x_n\}$ converges strongly to $z \in CF(T,S)$. □

Definition 3.4. Let $\emptyset \neq K \subseteq E$, where E is a Banach space. Then $T, S : K \to P(K)$ be two multivalued mappings with $CF(T,S) \neq \emptyset$. Then, T and S are said to satisfy condition (I'). If there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$f(d(x, CF(T,S))) \leq d(x, Tx) \;\; \text{or} \;\; f(d(x, CF(T,S))) \leq d(x, Sx),$$

for all $x \in K$.

Theorem 3.5. *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T, S : K \to P(K)$ be two multivalued mappings satisfying condition (I') and $CF(T,S) \neq \emptyset$. Assume that P_T, P_S are generalized (α,β)-nonexpansive mappings. Let $\{x_n\}$ be a sequence defined by* (3.1) *such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then, $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Proof. Let the sequence $\{x_n\}$ be iteratively generated as in (3.1). By Lemma 3.1,

$$\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0 = \lim_{n\to\infty} d(x_n, P_S(x_n)),$$

which imply that

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0 = \lim_{n\to\infty} d(x_n, S(x_n)).$$

Consider,

$$\begin{aligned} ||x_{n+1} - p|| &\leq ||x_n - p|| \\ \inf\{||x_{n+1} - p|| : p \in CF(T,S)\} &\leq \inf\{||x_n - p|| : p \in CF(T,S)\} \\ d(x_{n+1}, CF(T,S)) &\leq d(x_n, CF(T,S)). \end{aligned}$$

Thus, $\{d(x_n, CF(T,S))\}$ is a nonincreasing sequence which is bounded below, $\lim_{n\to\infty} d(x_n, CF(T,S))$ exists. Since T and S satisfy condition (I'), we obtain that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$f(d(x_n, CF(T,S))) \leq d(x_n, Tx_n) \text{ or } f(d(x_n, CF(T,S))) \leq d(x_n, Sx_n).$$

It follows that

$$0 \leq \lim_{n\to\infty} f(d(x_n, CF(T,S))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

or

$$0 \leq \lim_{n\to\infty} f(d(x_n, CF(T,S))) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Thus,

$$\lim_{n\to\infty} f(d(x_n, CF(T,S))) = 0.$$

Due to the nondecreasing function of f and $f(0) = 0$, we get

$$\lim_{n\to\infty} d(x_n, CF(T,S)) = 0.$$

Therefore, $\{x_n\}$ converges strongly to $p \in CF(T,S)$. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Assume that $T, S : K \to P(K)$ be two multivalued mappings such that $CF(T,S) \neq \emptyset$ and P_T, P_S are generalized (α,β)-nonexpansive mappings. Let $\{x_n\}$ be a sequence defined by* (3.1) *such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Let $I - P_T$ and $I - P_S$ be two demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of T and S.*

Proof. Let $p \in CF(T,S)$. Then, $\lim_{n\to\infty} ||x_n - p||$ exists as proved in Lemma 3.1. Now, we prove that $\{x_n\}$ has a unique weak subsequential limit in $CF(T,S)$. Since E is uniformly convex, it is reflexive. Therefore, there exist subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to some z_1 and z_2 in K, respectively. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$ and $I - P_T$ is demiclosed at zero, then $z_1 \in F(P_T) = F(T)$. Similarly, $z_2 \in F(P_T) = F(T)$. In the same way, we can prove that $z_1 \in F(P_S) = F(S)$ and $z_2 \in F(P_S) = F(S)$. Therefore, $z_1, z_2 \in CF(T,S)$. Next, we prove the uniqueness. To this end, suppose that $z_1 \neq z_2$. Then by Opial's condition, we have

$$\limsup_{n\to\infty} ||x_n - z_1|| = \limsup_{i\to\infty} ||x_{n_i} - z_1||$$

$$\begin{aligned}
&< \limsup_{i\to\infty} ||x_{n_i} - z_2|| \\
&= \limsup_{n\to\infty} ||x_n - z_2|| \\
&= \limsup_{j\to\infty} ||x_{n_j} - z_2|| \\
&< \limsup_{j\to\infty} ||x_{n_j} - z_1|| \\
&= \limsup_{n\to\infty} ||x_n - z_1||,
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ converges weakly to a common fixed point of T and S. □

Now, we employ the following example to confirm that our convergence result is effective.

Example 3.7. Let $E = \mathbb{R}$, $K = [0, \infty)$ and let $T, S : K \to P(K)$ be two multivalued mappings defined by,

$$T(x) = \begin{cases} \{0\}, & \text{if } x \in [0, \frac{1}{2000}], \\ [0, \frac{x}{6}], & \text{if } x \in (\frac{1}{2000}, \infty) - \{\frac{4}{5}\}, \\ [0, \frac{7}{20}], & \text{if } x \in \{\frac{4}{5}\}, \end{cases}$$

and,

$$S(x) = \begin{cases} \{0\}, & \text{if } x \in [0, \frac{1}{1000}], \\ [0, \frac{x}{4}], & \text{if } x \in (\frac{1}{1000}, \infty). \end{cases}$$

Then, P_T and P_S are generalized (α, β)-nonexpansive mappings.

Proof. Let $\alpha = \beta = \frac{1}{4}$ and we have

$$P_T(x) = \begin{cases} \{0\}, & \text{if } x \in [0, \frac{1}{2000}], \\ \{\frac{x}{6}\}, & \text{if } x \in (\frac{1}{2000}, \infty) - \{\frac{4}{5}\}, \\ \{\frac{7}{20}\}, & \text{if } x \in \{\frac{4}{5}\}, \end{cases}$$

and,

$$P_S(x) = \begin{cases} \{0\}, & \text{if } x \in [0, \frac{1}{1000}], \\ \{\frac{x}{4}\}, & \text{if } x \in (\frac{1}{1000}, \infty). \end{cases}$$

Now, we will prove that P_T is a generalized (α, β)-nonexpansive mapping for $\alpha = \beta = \frac{1}{4}$.

Case(1) : If $x, y \in [0, \frac{1}{2000}]$, then

$$\begin{aligned}
H(P_T(x), P_T(y)) = 0 \le &\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&+ \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) \\
&+ \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y||.
\end{aligned}$$

Case(2) : If $x, y \in (\frac{1}{2000}, \infty) - \{\frac{4}{5}\}$, then

$$\begin{aligned}
&\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&\qquad + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x-y|| \\
&= \frac{1}{4}\left|x - \frac{y}{6}\right| + \frac{1}{4}\left|y - \frac{x}{6}\right| + \frac{1}{4}\left|x - \frac{x}{6}\right| + \frac{1}{4}\left|y - \frac{y}{6}\right| \\
&\geq \frac{1}{4}\left|\Big(x - \frac{y}{6}\Big) - \Big(y - \frac{x}{6}\Big)\right| + \frac{1}{4}\left|\Big(x - \frac{x}{6}\Big) - \Big(y - \frac{y}{6}\Big)\right| \\
&= \frac{1}{4}\left|\frac{7x}{6} - \frac{7y}{6}\right| + \frac{1}{4}\left|\frac{5x}{6} - \frac{5y}{6}\right| \\
&= \frac{6}{2}\left|\frac{x}{6} - \frac{y}{6}\right| \\
&\geq \left|\frac{x}{6} - \frac{y}{6}\right| \\
&= H(P_T(x), P_T(y)).
\end{aligned}$$

Case(3) : If $x \in [0, \frac{1}{2000}]$ and $y \in (\frac{1}{2000}, \infty) - \{\frac{4}{5}\}$, then

$$\begin{aligned}
&\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&\qquad + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x-y|| \\
&= \frac{1}{4}\left|x - \frac{y}{6}\right| + \frac{1}{4}|y - 0| + \frac{1}{4}|x - 0| + \frac{1}{4}\left|y - \frac{y}{6}\right| \\
&= \frac{1}{4}\left|x - \frac{y}{6}\right| + \frac{11y}{24} + \frac{x}{4}. \qquad (3.12)
\end{aligned}$$

Consider two cases of $|x - \frac{y}{6}|$,

$$|x - \frac{y}{6}| = \begin{cases} x - \frac{y}{6}, & \text{if } x \geq \frac{y}{6}, \\ \frac{y}{6} - x, & \text{if } x < \frac{y}{6}. \end{cases}$$

For the first case (i.e., $x \geq \frac{y}{6}$), from (3.12) implies that:

$$\begin{aligned}
&\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&\qquad + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x-y|| \\
&= \frac{x}{4} - \frac{y}{24} + \frac{11y}{24} + \frac{x}{4} \\
&= \frac{x}{2} + \frac{5y}{12} \\
&> \frac{2y}{12} \\
&= H(P_T(x), P_T(y)).
\end{aligned}$$

For the second case (i.e., $x < \frac{y}{6}$), from (3.12) implies that:

$$\begin{aligned}
&\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&\qquad + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y|| \\
&= \frac{y}{24} - \frac{x}{4} + \frac{11y}{24} + \frac{x}{4} \\
&= \frac{y}{2} \\
&> \frac{y}{6} \\
&= H(P_T(x), P_T(y)).
\end{aligned}$$

Case(4) : If $x \in [0, \frac{1}{2000}]$ and $y \in \{\frac{4}{5}\}$, then

$$\begin{aligned}
&\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&\qquad + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y|| \\
&= \frac{1}{4}\left|x - \frac{7}{20}\right| + \frac{1}{4}\left|\frac{4}{5} - 0\right| + \frac{1}{4}|x - 0| + \frac{1}{4}\left|\frac{4}{5} - \frac{7}{20}\right| \\
&= \frac{1}{4}\left|x - \frac{7}{20}\right| + \frac{1}{5} + \frac{x}{4} + \frac{9}{80} \\
&= \frac{7}{80} - \frac{x}{4} + \frac{x}{4} + \frac{25}{80} \\
&= \frac{8}{20} \\
&\geq \frac{7}{20} \\
&= H(P_T(x), P_T(y)).
\end{aligned}$$

Case(5) : If $x \in (\frac{1}{2000}, \infty) - \{\frac{4}{5}\}$ and $y \in \{\frac{4}{5}\}$, then

$$\begin{aligned}
&\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) \\
&\qquad + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y|| \\
&= \frac{1}{4}\left|x - \frac{7}{20}\right| + \frac{1}{4}\left|\frac{4}{5} - \frac{x}{6}\right| + \frac{1}{4}\left|x - \frac{x}{6}\right| + \frac{1}{4}\left|\frac{4}{5} - \frac{7}{20}\right| \\
&= \frac{1}{4}\left|x - \frac{7}{20}\right| + \frac{1}{4}\left|\frac{4}{5} - \frac{x}{6}\right| + \frac{5x}{24} + \frac{9}{80} \\
&\geq \frac{1}{4}\left|\Big(x - \frac{7}{20}\Big) + \Big(\frac{4}{5} - \frac{x}{6}\Big)\right| + \frac{5x}{24} + \frac{9}{80} \\
&= \frac{1}{4}\left|\frac{x}{6} + \frac{9}{20}\right| + \frac{5x}{24} + \frac{9}{80}
\end{aligned}$$

$$= \frac{x}{24} + \frac{9}{80} + \frac{5x}{24} + \frac{9}{80}$$
$$= \frac{x}{4} + \frac{9}{40}$$
$$> \left|\frac{x}{6} - \frac{7}{20}\right|$$
$$= H(P_T(x), P_T(y)).$$

Case(6) : If $x, y \in \{\frac{4}{5}\}$, then

$$\frac{1}{4}d(x, P_T(y)) + \frac{1}{4}d(y, P_T(x)) + \frac{1}{4}d(x, P_T(x)) + \frac{1}{4}d(y, P_T(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y||$$
$$= \frac{1}{4}\left|\frac{4}{5} - \frac{7}{20}\right| + \frac{1}{4}\left|\frac{4}{5} - \frac{7}{20}\right| + \frac{1}{4}\left|\frac{4}{5} - \frac{7}{20}\right| + \frac{1}{4}\left|\frac{4}{5} - \frac{7}{20}\right|$$
$$= \left|\frac{4}{5} - \frac{7}{20}\right|$$
$$\geq 0$$
$$= H(P_T(x), P_T(y)).$$

Therefore, P_T is generalized (α, β)-nonexpansive. Moreover, P_S is generalized (α, β)-nonexpansive for $\alpha = \beta = \frac{1}{4}$. Indeed,

Case(1) : If $x, y \in [0, \frac{1}{1000}]$, then

$$H(P_S(x), P_S(y)) = 0 \leq \frac{1}{4}d(x, P_S(y)) + \frac{1}{4}d(y, P_S(x)) + \frac{1}{4}d(x, P_S(x)) + \frac{1}{4}d(y, P_S(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y||.$$

Case(2) : If $x, y \in (\frac{1}{1000}, \infty)$, then

$$\frac{1}{4}d(x, P_S(y)) + \frac{1}{4}d(y, P_S(x)) + \frac{1}{4}d(x, P_S(x)) + \frac{1}{4}d(y, P_S(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y||$$
$$= \frac{1}{4}\left|x - \frac{y}{4}\right| + \frac{1}{4}\left|y - \frac{x}{4}\right| + \frac{1}{4}\left|x - \frac{x}{4}\right| + \frac{1}{4}\left|y - \frac{y}{4}\right|$$
$$\geq \frac{1}{4}\left|\Big(x - \frac{y}{4}\Big) - \Big(y - \frac{x}{4}\Big)\right| + \frac{1}{4}\left|\Big(x - \frac{x}{4}\Big) - \Big(y - \frac{y}{4}\Big)\right|$$
$$= \frac{1}{4}\left|\frac{5x}{4} - \frac{5y}{4}\right| + \frac{1}{4}\left|\frac{3x}{4} - \frac{3y}{4}\right|$$
$$= 2\left|\frac{x}{4} - \frac{y}{4}\right|$$
$$\geq \left|\frac{x}{4} - \frac{y}{4}\right|$$
$$= H(P_S(x), P_S(y)).$$

Case(3) : If $x \in [0, \frac{1}{1000}]$ and $y \in (\frac{1}{1000}, \infty)$, then

$$\begin{aligned}
&\frac{1}{4}d(x, P_S(y)) + \frac{1}{4}d(y, P_S(x)) \\
&\qquad + \frac{1}{4}d(x, P_S(x)) + \frac{1}{4}d(y, P_S(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y|| \\
&= \frac{1}{4}\left|x - \frac{y}{4}\right| + \frac{1}{4}|y - 0| + \frac{1}{4}|x - 0| + \frac{1}{4}\left|y - \frac{y}{4}\right| \\
&= \frac{1}{4}\left|x - \frac{y}{4}\right| + \frac{7y}{16} + \frac{x}{4}. \qquad (3.13)
\end{aligned}$$

Consider two cases of $|x - \frac{y}{4}|$,

$$|x - \frac{y}{4}| = \begin{cases} x - \frac{y}{4}, & \text{if } x \geq \frac{y}{4}, \\ \frac{y}{4} - x, & \text{if } x < \frac{y}{4}. \end{cases}$$

For the first case (i.e., $x \geq \frac{y}{4}$), from (3.13) implies that:

$$\begin{aligned}
&\frac{1}{4}d(x, P_S(y)) + \frac{1}{4}d(y, P_S(x)) + \frac{1}{4}d(x, P_S(x)) \\
&\qquad + \frac{1}{4}d(y, P_S(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y|| \\
&= \frac{x}{4} - \frac{y}{16} + \frac{7y}{16} + \frac{x}{4}x \\
&= \frac{x}{2} + \frac{3y}{8} \\
&> \frac{2y}{8} \\
&= H(P_S(x), P_S(y)).
\end{aligned}$$

For the second case (i.e., $x < \frac{y}{4}$), from (3.13) implies that:

$$\begin{aligned}
&\frac{1}{4}d(x, P_S(y)) + \frac{1}{4}d(y, P_S(x)) + \frac{1}{4}d(x, P_S(x)) \\
&\qquad + \frac{1}{4}d(y, P_S(y)) + \Big(1 - \frac{2}{4} - \frac{2}{4}\Big)||x - y|| \\
&= \frac{y}{16} - \frac{x}{4} + \frac{7y}{16} + \frac{x}{4} \\
&= \frac{y}{2} \\
&> \frac{y}{4} \\
&= H(P_S(x), P_S(y)).
\end{aligned}$$

Therefore, P_S is a generalized (α, β)-nonexpansive mapping. Moreover, we can see that $0 \in CF(T, S)$. Next, we give three different initial values $x_1 = 0.8, x_1 = 1.0$ and $x_1 = 10.0$. Table 1 and Figure 1 illustrate the convergence behavior of $\{x_n\}$. We can see that the sequence generated by our algorithm converges to 0. □

TABLE 1

Step	when $x_1 = 0.8$	when $x_1 = 1.0$	when $x_1 = 10.0$
1	0.8000000000000000	1.0000000000000000	10.00000000000000
2	0.0232747395833333	0.0268012152777777	0.268012152777777
3	0.0006237913061071	0.0007183051403658	0.007183051403658
4	0.0000000000000000	0.0000000000000000	0.000000000000000
5	0.0000000000000000	0.0000000000000000	0.000000000000000
6	0.0000000000000000	0.0000000000000000	0.000000000000000
7	0.0000000000000000	0.0000000000000000	0.000000000000000
8	0.0000000000000000	0.0000000000000000	0.000000000000000
9	0.0000000000000000	0.0000000000000000	0.000000000000000
10	0.0000000000000000	0.0000000000000000	0.000000000000000

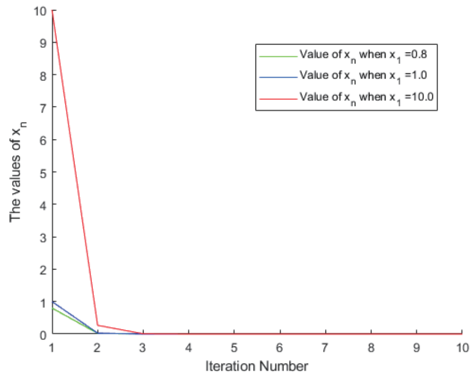

FIGURE 1

Finally, Table 2 and Figure 2 show that the sequence generated by our algorithm converges the fastest as compared to other algorithms by setting $S = T$ with the initial value 10.

TABLE 2

Step	New Iteration	M-Iteration	Abbas Iteration	Picard-Mann
1	10	10	10	10
2	1.7409×10^{-1}	2.1991×10^{-1}	5.2734×10^{-1}	1.3194
3	3.0308×10^{-3}	4.8359×10^{-3}	2.7809×10^{-2}	1.7409×10^{-1}
4	0	1.0635×10^{-4}	1.4665×10^{-3}	2.2971×10^{-2}
5	0	0	4.8374×10^{-5}	3.0308×10^{-3}
6	0	0	0	3.9990×10^{-4}
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
10	0	0	0	0

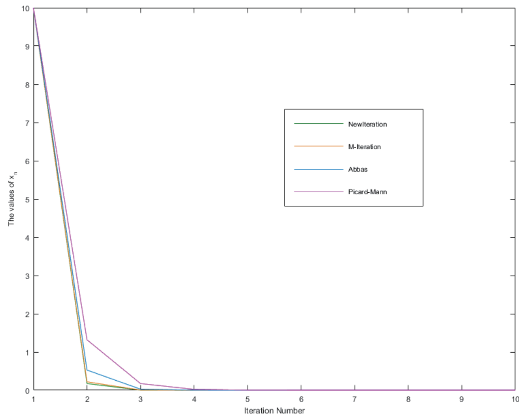

FIGURE 2

4. Conclusions

We have provided weak and strong convergence theorems of common fixed point for two multivalued generalized (α, β)-nonexpansive mappings in uniformly convex Banach spaces. We also presented a numerical example to support our main result.

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A. Wiriyapongsanon
Faculty of Sport and Health Sciences, Thailand National Sports University Lampang Campus, Lampang 52100, Thailand
E-mail address: rtitkp99@gmail.com

W. Inthakon
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
E-mail address: warunun.i@cmu.ac.th

N. Phudolsitthiphat, corresponding author
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
E-mail address: narawadee.nanan@cmu.ac.th