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A FAMILY OF SMOOTH NCP FUNCTIONS AND AN INEXACT LEVENBERG-MARQUARDT METHOD FOR NONLINEAR COMPLEMENTARITY PROBLEMS

JINGYONG TANG*, JINCHUAN ZHOU, JAN HAROLD ALCANTARA, AND JEIN-SHAN CHEN

Abstract. In this paper, we introduce a family of new NCP functions which is smooth, coercive and strongly semismooth. Based on new NCP functions, we propose an inexact Levenberg-Marquardt method for solving Nonlinear Complementarity Problem (NCP). Different from existing exact/inexact Levenberg-Marquardt methods, the proposed method adopts a derivative-free line search to ensure its globalization. Moreover, by using the strong semismoothness of new NCP functions, we prove that the proposed method is locally superlinearly/quadratically convergent under a local error bound condition. Some numerical results are reported.

1. Introduction

The Nonlinear Complementarity Problem (NCP) is to find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$x \geq 0,\ y \geq 0,\ y = F(x),\ x^T y = 0, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function and the superscript T denotes the transpose operator. The NCP has received a lot of attention due to its various applications in operations research, economic equilibrium, and engineering design, see [14, 15].

A number of algorithms have been developed for solving the NCP (e.g., [4, 5, 10, 13, 17, 18, 20, 24, 27, 29, 30, 32, 34, 35]). Among them, equation-based methods are most extensively studied and proved to be highly successful. Equation-based methods are usually defined by reformulating the NCP into an equivalent system of equations such as

$$\mathrm{H}(x, y) = 0.$$

In the reformulation of the NCP, a class of functions, called NCP functions, serves an important role. A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an *NCP function* if it satisfies

$$\phi(a, b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0,\ ab = 0. \tag{1.2}$$

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*Corresponding author.

Up to now, a variety of NCP functions have been studied. Among them, the well-known Fischer-Burmeister function is one of the most prominent NCP functions which is defined by

$$\phi^{\rm FB}(a,b) = \sqrt{a^2+b^2} - (a+b),\ \forall (a,b) \in \mathbb{R}^2. \tag{1.3}$$

Notice that $\phi^{\rm FB}$ is not smooth everywhere. With the above characterization of $\phi^{\rm FB}$, the NCP is equivalent to the following nonsmooth equations:

$$\mathrm{H}(x,y) := \begin{pmatrix} F(x)-y \\ \phi^{\rm FB}(x_1,y_1) \\ \vdots \\ \phi^{\rm FB}(x_n,y_n) \end{pmatrix} = 0. \tag{1.4}$$

Some nonsmooth algorithms have been successfully extended to solve (1.4), including nonsmooth inexact Newton method [13], generalized Newton and Gauss-Newton methods [20], nonsmooth Levenberg-Marquardt algorithm [10] and so on.

Based on the Fischer-Burmeister function $\phi^{\rm FB}$ in (1.3), Chen and Pan [5] introduced a family of NCP functions defined by

$$\phi_p^{\rm CP}(a,b) = \|(a,b)\|_p - (a+b),\ \forall (a,b) \in \mathbb{R}^2, \tag{1.5}$$

where $p \in (1,\infty)$ is any fixed real number and $\|(a,b)\|_p$ denotes the p-norm of (a,b), i.e.,

$$\|(a,b)\|_p = \sqrt[p]{|a|^p+|b|^p}.$$

In fact, the function $\phi_p^{\rm CP}$ is obtained by replacing the 2-norm of (a,b) in $\phi^{\rm FB}$ by a more general p-norm. Chen and Pan showed that the function $\phi_p^{\rm CP}$ has several favorable properties analogous to what $\phi^{\rm FB}$ has. Based on $\phi_p^{\rm CP}$, a derivative-free descent algorithm [4, 5] and a regularization semismooth Newton method [6] have been studied for solving the NCP.

In this paper, based on the function $\phi_p^{\rm CP}$ and motivated by the work in [1], we introduce a family of new NCP functions which is defined by

$$\phi_{p,q}(a,b) = \|(a,b)\|_p^q - \mathrm{sgn}(a+b)|a+b|^q,\ \forall (a,b) \in \mathbb{R}^2, \tag{1.6}$$

where $p \in (1,\infty)$ and $q \in [1,\infty)$ are any fixed real numbers and

$$\mathrm{sgn}(t) := \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0. \\ -1 & \text{if } t < 0 \end{cases}$$

Obviously, $\phi_{p,q}$ taking $q = 1$ yields the NCP function $\phi_p^{\rm CP}$. We show that, different from the NCP functions $\phi^{\rm FB}$ and $\phi_p^{\rm CP}$, the function $\phi_{p,q}$ in (1.6) is smooth everywhere for any $q \geq p$. Moreover, we show that $\phi_{p,q}$ is coercive and strongly semismooth under suitable choices of p, q. By using the function $\phi_{p,q}$, we reformulate the NCP as the following smooth equations

$$\mathrm{H}_{p,q}(x,y) := \begin{pmatrix} F(x) - y \\ \phi_{p,q}(x_1, y_1) \\ \vdots \\ \phi_{p,q}(x_n, y_n) \end{pmatrix} = 0 \tag{1.7}$$

and propose an inexact Levenberg-Marquardt method (ILMM) to solve it. The proposed ILMM differentiates itself from the current exact/inexact Levenberg-Marquardt methods by adopting a derivative-free line search to ensure its globalization. Specially, we prove that the iteration sequence generated by the proposed ILMM is bounded when the function F is a uniform P-function, and if the generated iteration sequence has an isolated accumulation point, then the whole sequence converges to this point. Moreover, we prove that the convergence rate of the proposed ILMM is superlinear/quadratic under a local error bound condition which is weaker than the nonsingularity condition. It is worth pointing out that, different from existing exact/inexact Levenberg-Marquardt methods, we obtain the local convergence rate of our ILMM by using the (strong) semismoothness of the function $\mathrm{H}_{p,q}$ in (1.7). We also report some numerical results which indicate that the proposed ILMM is very effective for solving large-scale NCPs.

This paper is organized as follows. In Section 2, we give some properties of the function $\phi_{p,q}$. In Section 3, we propose an inexact Levenberg-Marquardt method to solve the smooth equations (1.7). In Section 4, we analyze the global convergence of the method. In Section 5, we establish the local superlinear/quadratic convergence of the method under a local error bound condition. Numerical results are reported in Section 6. Some conclusions are given in Section 7.

Throughout this paper, $\mathbb{R}^n$ denotes the set of all n dimensional real column vectors. For simplicity, the column vector $(x^T, y^T)^T$ is written as (x, y) where $x, y \in \mathbb{R}^n$. $\|\cdot\|$ denotes the 2-norm. $\mathrm{sgn}(\cdot)$ is the sign function. For any $x^* \in \mathbb{R}^n$, the neighbourhood of x^* is denoted by $N(x^*, \varepsilon) := \{x \in \mathbb{R}^n | \|x - x^*\| \leq \varepsilon\}$ where $\varepsilon > 0$ is a constant. For any differentiable function $f(x) : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes the gradient of f at x, and for any differentiable mapping $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, $\mathrm{J}(x)$ denotes the Jacobian of f at x. For any $\alpha, \beta > 0$, $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) means that $\limsup_{\beta\to 0} \frac{\alpha}{\beta} < \infty$ (respectively, $\limsup_{\beta\to 0} \frac{\alpha}{\beta} = 0$).

2. Properties of the function $\phi_{p,q}(a, b)$

In this section, we show that the function $\phi_{p,q}$ defined by (1.6) is a family of NCP functions and it is smooth, coercive and strongly semismooth. It is worth pointing out that the continuous-type generalization technique adopted in the function $\phi_{p,q}$ was introduced by Alcantara and Chen [1] to generalize the NR function. This generalization subsumes the discrete-type generalization used in [3, 7, 25, 31] where one requires q is a positive odd integer.

Proposition 2.1. *The function $\phi_{p,q}$ is a family of* NCP *functions, i.e.,*

$$\phi_{p,q}(a, b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0,\ ab = 0. \tag{2.1}$$

Proof. Let

$$f_q(t) := \mathrm{sgn}(t)|t|^q, \ \forall\ q \in [1, +\infty).$$

Then we have

$$\phi_{p,q}(a,b) = f_q(\|(a,b)\|_p) - f_q(a+b). \tag{2.2}$$

Since f_q is a bijective function, it follows that

$$\begin{aligned} \phi_{p,q}(a,b) = 0 &\iff f_q(\|(a,b)\|_p) = f_q(a+b) \\ &\iff \|(a,b)\|_p = a+b \\ &\iff \phi_p^{\mathrm{CP}}(a,b) = 0. \end{aligned}$$

Since ϕ_p^{CP} is a family of NCP functions, we have proved (2.1). $\square$

Proposition 2.2. (i) *If $q \geq p$, then the function $\phi_{p,q}$ is continuously differentiable at any $(a,b) \in \mathbb{R}^2$ whose gradient is given by*

$$\nabla \phi_{p,q}(a,b) = \begin{bmatrix} \nabla_a \phi_{p,q} \\ \nabla_b \phi_{p,q} \end{bmatrix}, \tag{2.3}$$

where

$$\begin{aligned} \nabla_a \phi_{p,q} &= q[\mathrm{sgn}(a)|a|^{p-1}\|(a,b)\|_p^{q-p} - |a+b|^{q-1}], \\ \nabla_b \phi_{p,q} &= q[\mathrm{sgn}(b)|b|^{p-1}\|(a,b)\|_p^{q-p} - |a+b|^{q-1}]. \end{aligned}$$

(ii) *If $p > 2$ and $q \geq 2p$, then $\phi_{p,q}$ is twice continuously differentiable at any $(a,b) \in \mathbb{R}^2$ whose Hessian is given by*

$$\nabla^2 \phi_{p,q}(a,b) = \begin{bmatrix} \nabla^2_{aa}\phi_{p,q} & \nabla^2_{ab}\phi_{p,q} \\ \nabla^2_{ba}\phi_{p,q} & \nabla^2_{bb}\phi_{p,q} \end{bmatrix}, \tag{2.4}$$

where

$$\begin{aligned} \nabla^2_{aa}\phi_{p,q} &= q[(p-1)|a|^{p-2}\|(a,b)\|_p^{q-p} + (q-p)a^{2p-2}\|(a,b)\|_p^{q-2p} \\ &\qquad\qquad\qquad\qquad - (q-1)\mathrm{sgn}(a+b)|a+b|^{q-2}], \\ \nabla^2_{ab}\phi_{p,q} &= \nabla^2_{ba}\phi_{p,q}(a,b) \\ &= q\big[(q-p)\mathrm{sgn}(a)|a|^{p-1}\mathrm{sgn}(b)|b|^{p-1}\|(a,b)\|_p^{q-2p} \\ &\qquad\qquad\qquad\qquad - (q-1)\mathrm{sgn}(a+b)|a+b|^{q-2}\big], \\ \nabla^2_{bb}\phi_{p,q} &= q[(p-1)|b|^{p-2}\|(a,b)\|_p^{q-p} + (q-p)b^{2p-2}\|(a,b)\|_p^{q-2p} \\ &\qquad\qquad\qquad\qquad - (q-1)\mathrm{sgn}(a+b)|a+b|^{q-2}\big]. \end{aligned}$$

Proof. Note that the function $f_q(t) = \mathrm{sgn}(t)|t|^q$ is continuously differentiable when $q > 1$ with $f_q'(t) = q|t|^{q-1}$. Thus, when $q \geq p$, by (2.2) we have for any $(a,b) \neq (0,0)$,

$$\nabla_a \phi_{p,q}(a,b) = q[\mathrm{sgn}(a)|a|^{p-1}\|(a,b)\|_p^{q-p} - |a+b|^{q-1}], \tag{2.5}$$

$$\nabla_b \phi_{p,q}(a,b) = q[\mathrm{sgn}(b)|b|^{p-1}\|(a,b)\|_p^{q-p} - |a+b|^{q-1}]. \tag{2.6}$$

In the following, we show that $\phi_{p,q}$ is differentiable at $(a,b) = (0,0)$ with gradient being zero when $q \geq p$. In fact, since $\phi_{p,q}(a,b) = f_q(\|(a,b)\|_p) - f_q(a+b)$ and f_q is continuously differentiable as $q > 1$, it is sufficient to show that $f_q(\|(a,b)\|_p)$

is differentiable at $(a, b) = (0, 0)$ with gradient being zero. Since $f_q(\|(a, b)\|_p) = \|(a, b)\|_p^q$, we have

$$\lim_{(a,b)\to(0,0)} \frac{f_q(\|(a,b)\|_p)}{\|(a,b)\|_p} = \lim_{(a,b)\to(0,0)} \|(a,b)\|_p^{q-1} = 0, \tag{2.7}$$

where the last step is due to the fact $q > 1$ since $q \geq p$ and $p > 1$. The equation (2.7) means $f_q(\|(a, b)\|_p) = o(\|(a, b)\|_p) = o(\|(a, b)\|_2)$, where the last equation follows from the equivalence of norms in finite dimensional space. This shows that $f_q(\|(a, b)\|_p)$ is differentiable at $(a, b) = (0, 0)$ with gradient being zero and so is $\phi_{p,q}$. Note that $\nabla\phi_{p,q}$ is continuous at $(a, b) = (0, 0)$ according to the formula (2.5) and (2.6). Hence, $\phi_{p,q}$ is continuously differentiable everywhere. Moreover, when $p > 2$ and $q \geq 2p$, by using $(\mathrm{sgn}(t)|t|^\alpha)' = \alpha|t|^{\alpha-1}$ and $(|t|^\alpha)' = \alpha\mathrm{sgn}(t)|t|^{\alpha-1}$ for any $\alpha > 1$, we can prove the result (ii). The proof is completed. $\square$

Chen and Pan proved that the function ϕ_p^{CP} is coercive for any $p > 1$, see [5, Lemma 3.1]. The following proposition shows that this property also holds for the function $\phi_{p,q}$.

Proposition 2.3. *Let $\phi_{p,q}$ be defined by* (1.6). *If $\{(a_k, b_k)\} \subset \mathbb{R}^2$ with $a_k \to -\infty$ or $b_k \to -\infty$, or $a_k \to \infty$ and $b_k \to \infty$, then $|\phi_{p,q}(a_k, b_k)| \to \infty$ as $k \to \infty$.*

Proof. For the sequence $\{(a_k, b_k)\}$, we consider the following three cases.

Case 1. $a_k + b_k > 0$. Let $u_k = \|(a_k, b_k)\|_p$ and $v_k = a_k + b_k$. Then $u_k, v_k > 0$ and

$$\begin{aligned} |\phi_{p,q}(a_k, b_k)| &= |u_k^q - v_k^q| = |u_k - v_k||u_k^{q-1} + u_k^{q-2}v_k + \cdots + u_k v_k^{q-2} + v_k^{q-1}| \\ &\geq |u_k - v_k|u_k^{q-1} = |\phi_p^{\mathrm{CP}}(a_k, b_k)|\|(a_k, b_k)\|_p^{q-1}. \end{aligned}$$

Case 2. $a_k + b_k < 0$. Then

$$|\phi_{p,q}(a_k, b_k)| = |\|(a_k, b_k)\|_p^q + |a_k + b_k|^q| \geq \|(a_k, b_k)\|_p^q.$$

Case 3. $a_k + b_k = 0$. Then $|\phi_{p,q}(a_k, b_k)| = \|(a_k, b_k)\|_p^q$.

Combining these three cases, we obtain that for any $k \geq 0$,

$$|\phi_{p,q}(a_k, b_k)| \geq \min\left\{|\phi_p^{\mathrm{CP}}(a_k, b_k)|, \|(a_k, b_k)\|_p\right\}\|(a_k, b_k)\|_p^{q-1}.$$

Since $\|(a_k, b_k)\|_p \to \infty$ and $|\phi_p^{\mathrm{CP}}(a_k, b_k)| \to \infty$ by [5, Lemma 3.1], the desired result follows. $\square$

As it is well-known, the strongly semismooth property plays a fundamental role in analyzing the local quadratic convergence of smooth/nonsmooth Newton-type methods. Let $G(x) : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function. Then G is differentiable almost everywhere by Rademacher's theorem. Let $D_G \subseteq \mathbb{R}^n$ be the set of points at which G is differentiable. Then the B-subdifferential $\partial_B G(x)$ of G at x is defined by

$$\partial_B G(x) := \left\{V \in \mathbb{R}^{m\times n} | V = \lim_{x^k\to x} G'(x^k),\ \{x^k\} \subseteq D_G\right\}$$

and the Clarke's generalized Jacobian of G at x is $\partial G(x) := \mathrm{conv}(\partial_B G(x))$ [8]. Observe that $\partial G(x) = \{G'(x)\}$ if G is continuously differentiable at x. We say that

G is directionally differentiable at x along the direction d if

$$G'(x; d) := \lim_{t \downarrow 0} \frac{G(x + td) - G(x)}{t}$$

exists where $G'(x; d)$ is called the directional derivative of G at x along the direction d, and G is directionally differentiable at x if G is directionally differentiable at x along any direction $d \neq 0$.

Definition 2.4. *(see [26]) Let $G(x) : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function around $x \in \mathbb{R}^n$. We say that G is semismooth at x if G is directionally differentiable at x and for any $y \to x$ and $V \in \partial G(y)$,*

$$G(y) - G(x) - V(y - x) = o(\|y - x\|).$$

G is further said to be strongly semismooth at x if G is semismooth at x and for any $y \to x$ and $V \in \partial G(y)$,

$$G(y) - G(x) - V(y - x) = O(\|y - x\|^2).$$

G is called (strongly) semismooth on $\mathbb{R}^n$ if it is (strongly) semismooth at each point $x \in \mathbb{R}^n$. Semismooth functions include smooth functions, piecewise smooth functions and convex and concave functions. We also mention that the composition of two (strongly) semismooth functions is a (strongly) semismooth function (see, [14, Proposition 7.4.4]), and a vector-valued function is (strongly) semismooth if and only if its all component functions are (strongly) semismooth (see, [26, Corollary 2.4]).

Proposition 2.5. *The function $\phi_{p,q}$ defined by* (1.6) *is semismooth on $\mathbb{R}^2$ if $q \geq 1$ and strongly semismooth if $q \geq 2$.*

Proof. Recall that $\phi_{p,q}(a, b) = f_q(\|(a, b)\|_p) - f_q(a + b)$. Since the p-norm function $\|(a, b)\|_p$ is strongly semismooth on $\mathbb{R}^2$ by [14, Proposition 7.4.8], it only needs to show that $f_q(t) = \text{sgn}(t)|t|^q$ is semismooth if $q \geq 1$ and strongly semismooth if $q \geq 2$. This holds automatically because f_q is continuously differentiable as $q \geq 1$ and twice continuously differentiable as $q > 2$ respectively. The remaining case is $q = 2$. In this case, $f_2(t) = \text{sgn}(t)t^2$. It is twice continuously differentiable at $t \neq 0$ and strongly semismooth at $t = 0$ due to

$$\lim_{h \to 0} \frac{|f_2(h) - f_2(0) - f_2'(h)h|}{h^2} = \lim_{h \to 0} \frac{|\text{sgn}(h)h^2 - 2|h|h|}{h^2} = 1.$$

This completes the proof. □

At the end of this section, we illustrate the geometrical interpretation of the function $z = \phi_{p,q}$ with different p and q. From Figures 1–9, we can see that the function $\phi_{p,q}$ with $q = 1$, i.e., $\phi_p^{\text{CP}}(a, b)$, is convex but it is not smooth at $(0, 0)$. While the function $\phi_{p,q}$ with $q \geq p$ is smooth everywhere but it is neither convex nor concave. It is worth pointing out that the convexity of the function $\phi_p^{\text{CP}}(a, b)$ has been proved by Chen and Pan [5, Proposition 3.1 (d)].

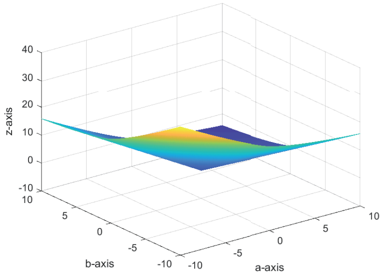

FIGURE 1.

$p = 1.5, q = 1$

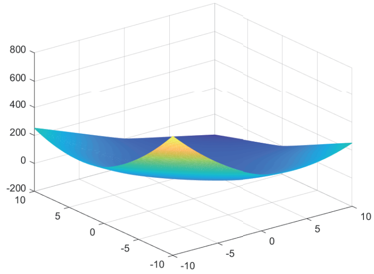

FIGURE 2.

$p = 1.5, q = 2$

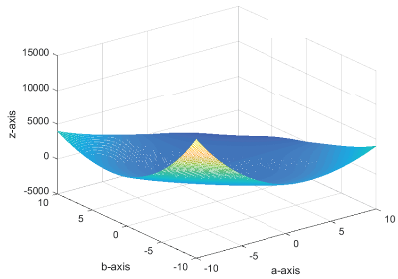

FIGURE 3.

$p = 1.5, q = 3$

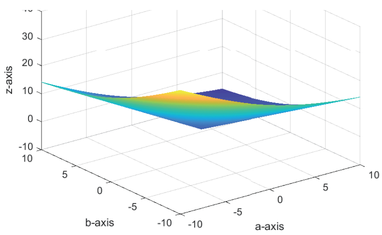

FIGURE 4.

$p = 2, q = 1$

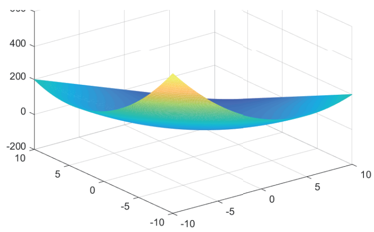

FIGURE 5.

$p = 2, q = 2$

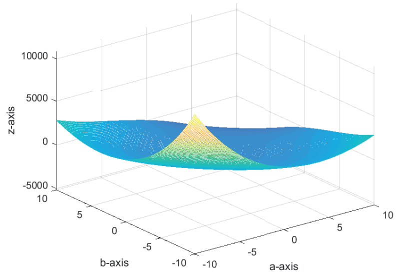

FIGURE 6.
$p = 2, q = 3$

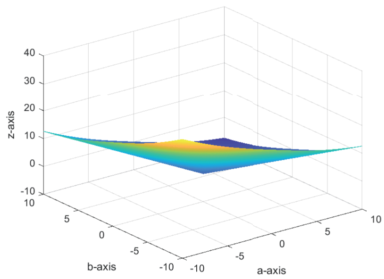

FIGURE 7.
$p = 3, q = 1$

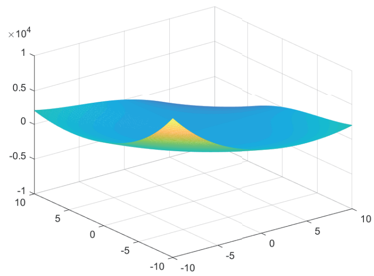

FIGURE 8.
$p = 3, q = 3$

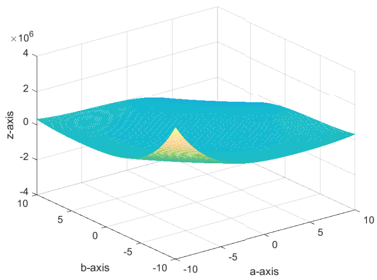

FIGURE 9.
$p = 3, q = 5$

3. An inexact Levenberg-Marquardt method

In the rest of this paper, for simplicity, we denote $z := (x, y)$. In this section, we study an inexact Levenberg-Marquardt method for solving the smooth nonlinear equation (1.7). By Propositions 2.1 and 2.2, we have the following result.

Proposition 3.1. *Let* $\mathrm{H}_{p,q}(z)$ *be given in* (1.7). *Then the following results hold.*

(a) $\mathrm{H}_{p,q}(z) = 0$ *if and only if* $z = (x, y)$ *is the solution of the* NCP.

(b) $\mathrm{H}_{p,q}(z)$ *is continuously differentiable at any* $z \in \mathbb{R}^{2n}$ *with the Jacobian*

$$\mathrm{J}_{p,q}(z) = \begin{bmatrix} F'(x) & -I \\ D_x & D_y \end{bmatrix},$$

where both D_x *and* D_y *are diagonal matrices in* $\mathbb{R}^{n\times n}$ *satisfying*

$$(D_x)_{ii} = q[\mathrm{sgn}(x_i)|x_i|^{p-1}(\|(x_i, y_i)\|_p)^{q-p} - |x_i + y_i|^{q-1}],$$
$$(D_y)_{ii} = q[\mathrm{sgn}(y_i)|y_i|^{p-1}(\|(x_i, y_i)\|_p)^{q-p} - |x_i + y_i|^{q-1}].$$

We define the merit function $\psi_{p,q} : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$\psi_{p,q}(z) := \frac{1}{2}\|\mathrm{H}_{p,q}(z)\|^2. \tag{3.1}$$

Then, $\psi_{p,q}(z)$ is continuously differentiable at any $z \in \mathbb{R}^{2n}$ with

$$\nabla\psi_{p,q}(z) = \mathrm{J}_{p,q}(z)^T\mathrm{H}_{p,q}(z). \tag{3.2}$$

We now describe our method as follows.

Algorithm 3.1 (*An inexact Levenberg-Marquardt method*)

Step 0: Choose $\rho, \tau, \gamma, \delta, \sigma \in (0, 1)$, $\alpha > 0$, $\beta \in (0, 2]$ and $z^0 := (x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\eta, \theta > 0$ be given constants and $\{\eta_k\}$, $\{\theta_k\}$ be given positive sequences satisfying $\sum_{k=0}^{\infty}\eta_k \le \eta < \infty$ and $\theta_k \le \theta$ for all $k \ge 0$. Set $k := 0$.

Step 1: If $\|\nabla\psi_{p,q}(z^k)\| = 0$, then stop.

Step 2: Set

$$\mu_k := \alpha\|\mathrm{H}_{p,q}(z^k)\|^{\beta}. \tag{3.3}$$

Find a search direction $d_k \in \mathbb{R}^{2n}$ which satisfies

$$(\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k) + \mu_k I)d_k = -\nabla\psi_{p,q}(z^k) + r_k, \tag{3.4}$$

where

$$\|r_k\| \le \min\{\rho\|\nabla\psi_{p,q}(z^k)\|, \theta_k\psi_{p,q}(z^k)\}. \tag{3.5}$$

If d_k satisfies

$$\|\mathrm{H}_{p,q}(z^k + d_k)\| \le \tau\|\mathrm{H}_{p,q}(z^k)\|, \tag{3.6}$$

then set $\lambda_k := 1$ and go to Step 4.

Step 3: If the descent condition

$$\nabla\psi_{p,q}(z^k)^T d_k \le -\gamma\|d_k\|^2 \tag{3.7}$$

is not satisfied, then set $d_k := -\nabla\psi_{p,q}(z^k)$. Let l_k be the smallest nonnegative integer l satisfying

$$\|\mathrm{H}_{p,q}(z^k + \delta^l d_k)\| \le (1+\eta_k)\|\mathrm{H}_{p,q}(z^k)\| - \sigma\|\delta^l d_k\|^2. \tag{3.8}$$

Set $\lambda_k := \delta^{l_k}$ and go to Step 4.

Step 4: Set $z^{k+1} := z^k + \lambda_k d_k$. Set $k := k+1$ and go to Step 1.

For Algorithm 3.1, we have the following remarks.

(i) For some k, if $\nabla\psi_{p,q}(z^k) \neq 0$, then by (3.2) we have $\mathrm{H}_{p,q}(z^k) \neq 0$ and hence $\mu_k > 0$. So, the matrix $\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k) + \mu_k I$ is positive definite and the search direction d_k in Step 2 is always obtained. Notice that $d_k \neq 0$. In fact, if $d_k = 0$, then by (3.4) we have $\|r_k - \nabla\psi_{p,q}(z^k)\| = 0$. Since $\|r_k\| \le \rho\|\nabla\psi_{p,q}(z^k)\|$, it follows that $\|\nabla\psi_{p,q}(z^k)\| = \|r_k\| = 0$ which contradicts $\nabla\psi_{p,q}(z^k) \neq 0$.

(ii) The search direction d_k obtained in Step 2 may not be a good descent direction of $\psi_{p,q}$ since it is an approximate solution of Levenberg-Marquardt equation. So, in Step 3, if the descent condition (3.7) is not satisfied, then we reset d_k to be the steepest descent direction of $\psi_{p,q}$. By this way, we can ensure that d_k is always a descent direction of $\psi_{p,q}$.

(iii) To ensure the globalization, existing exact/inexact Levenberg-Marquardt methods (e.g., [2, 9, 11, 12, 19, 21, 23, 29, 33]) usually adopt the Armijo line search rule. While, in Step 3 of Algorithm 3.1, we adopt a derivative-free line search which was introduced by Li and Fukushima [22] to ensure the globalization of Broyden-like method. It is easy to see that the inequality (3.8) holds for all sufficiently large l, because when $l \to \infty$, the left-hand side of (3.8) tends to $\|\mathrm{H}_{p,q}(z^k)\|$ but the right-hand side tends to $(1+\eta_k)\|\mathrm{H}_{p,q}(z^k)\|$. This shows that the line search in Step 3 is well-defined.

4. Global convergence

In the following, we assume that $\nabla\psi_{p,q}(z^k) \neq 0$ for all $k \ge 0$, so that Algorithm 3.1 generates an infinite sequence $\{z^k\}$ with $\|\mathrm{H}_{p,q}(z^k)\| > 0$ for all $k \ge 0$.

Lemma 4.1. *Let $\{z^k\}$ be the iteration sequence generated by Algorithm* 3.1. *Then the sequence $\{\|\mathrm{H}_{p,q}(z^k)\|\}$ is convergent. Moreover, $\|\mathrm{H}_{p,q}(z^k)\| \le e^{\eta}\|\mathrm{H}_{p,q}(z^0)\|$ for all $k \ge 0$.*

Proof. For any $k \ge 0$, if the condition (3.6) holds, then $z^{k+1} = z^k + d_k$ and

$$\|\mathrm{H}_{p,q}(z^{k+1})\| \le \tau\|\mathrm{H}_{p,q}(z^k)\|. \tag{4.1}$$

Otherwise, $z^{k+1} = z^k + \lambda_k d_k$ with λ_k being generated by (3.8) and

$$\|\mathrm{H}_{p,q}(z^{k+1})\| \le (1+\eta_k)\|\mathrm{H}_{p,q}(z^k)\| - \sigma\|\lambda_k d_k\|^2. \tag{4.2}$$

So, we can conclude from (4.1) and (4.2) that for all $k \ge 0$

$$\|\mathrm{H}_{p,q}(z^{k+1})\| \le (1+\eta_k)\|\mathrm{H}_{p,q}(z^k)\|. \tag{4.3}$$

Since $\sum_{k=0}^{\infty}\eta_k \le \eta < \infty$, by (4.3) and [22, Lemma 2.2], we have the first result.

Moreover, due to (4.3), by following the proof of [22, Lemma 2.1], we can obtain the second result. ☐

Theorem 4.2. *Let* $\{z^k = (x^k, y^k)\}$ *be the iteration sequence generated by Algorithm* 3.1*. Then any accumulation point* $z^* := (x^*, y^*)$ *of* $\{z^k\}$ *is a stationary point of the merit function* $\psi_{p,q}(z)$*, i.e.,* $\nabla\psi_{p,q}(z^*) = 0$*. Moreover, if* $\mathrm{J}_{p,q}(z^*)$ *is nonsingular, then* $\mathrm{H}_{p,q}(z^*) = 0$ *and* (x^*, y^*) *is a solution of the* NCP.

Proof. Since z^* is the accumulation point of $\{z^k\}$, there exists a subsequence $\{z^k\}_{k\in K}$ where $K \subset \{0, 1, ...\}$ such that $\lim_{(K\ni)k\to\infty} z^k = z^*$. By continuity, we have

$$\lim_{(K\ni)k\to\infty} \mathrm{H}_{p,q}(z^k) = \mathrm{H}_{p,q}(z^*), \quad \lim_{(K\ni)k\to\infty} \mathrm{J}_{p,q}(z^k) = \mathrm{J}_{p,q}(z^*),$$

$$\lim_{(K\ni)k\to\infty} \psi_{p,q}(z^k) = \psi_{p,q}(z^*), \quad \lim_{(K\ni)k\to\infty} \nabla\psi_{p,q}(z^k) = \nabla\psi_{p,q}(z^*).$$

We assume that $\nabla\psi_{p,q}(z^*) \neq 0$ and will derive a contradiction. Since $\nabla\psi_{p,q}(z^*) \neq 0$, $\|\mathrm{H}_{p,q}(z^*)\| > 0$ which implies $\lim_{(K\ni)k\to\infty} \mu_k = \alpha\|\mathrm{H}_{p,q}(z^*)\|^\beta > 0$. So, there exists a constant ξ such that $\mu_k \geq \xi > 0$ for all $k \in K$. Moreover, $\{\nabla\psi_{p,q}(z^k)\}_{k\in K}$ is bounded since it is convergent. Thus, for any $k \in K$, if d_k is generated in Step 2 by (3.4) with (3.5) and it satisfies the condition (3.7), then

$$\begin{aligned}
\|d_k\| &\leq \|(\mathrm{J}_{p,q}(z^k)^T \mathrm{J}_{p,q}(z^k) + \mu_k I)^{-1}\|(\|\nabla\psi_{p,q}(z^k)\| + \|r_k\|) \\
&\leq \frac{\|\nabla\psi_{p,q}(z^k)\| + \|r_k\|}{\mu_k} \\
&\leq \frac{(1+\rho)\|\nabla\psi_{p,q}(z^k)\|}{\xi} \\
&< \infty.
\end{aligned}$$

Otherwise, $\|d_k\| = \|-\nabla\psi_{p,q}(z^k)\| < \infty$. Hence, the sequence $\{\|d_k\|\}_{k\in K}$ is bounded and it has at least one accumulation point d^*. We may assume that $\lim_{(K_1\ni)k\to\infty} d_k = d^*$ where $K_1 \subset K$ is an infinite subset. Now, we will show $\nabla\psi_{p,q}(z^*)^T d^* = 0$. Since $\{\|\mathrm{H}_{p,q}(z^k)\|\}$ is convergent by Lemma 4.1, we have

$$\lim_{k\to\infty} \|\mathrm{H}_{p,q}(z^k)\| = \|\mathrm{H}_{p,q}(z^*)\| > 0. \tag{4.4}$$

So, if there are infinitely many k for which λ_k is determined by (3.6), then $\|\mathrm{H}_{p,q}(z^{k+1})\| \leq \tau\|\mathrm{H}_{p,q}(z^k)\|$ holds for infinitely many k. This yields $\liminf_{k\to\infty} \|\mathrm{H}_{p,q}(z^k)\| = 0$ which contradicts (4.4). Thus, there exists an index $\bar{k}$ such that λ_k is determined by (3.8) for all $k \geq \bar{k}$. In what follows, we divide the proof into two parts.

Part 1. $\lambda_k \geq c > 0$ for all $k \geq \bar{k}$ and $k \in K_1$ where c is a fixed constant. Then it follows from (3.8) that

$$\sigma c^2\|d_k\|^2 \leq \sigma\|\lambda_k d_k\|^2 \leq (1+\eta_k)\|\mathrm{H}_{p,q}(z^k)\| - \|\mathrm{H}_{p,q}(z^{k+1})\|. \tag{4.5}$$

Since $\lim_{k\to\infty} \eta_k = 0$ and $\lim_{k\to\infty} \|\mathrm{H}_{p,q}(z^k)\| = \|\mathrm{H}_{p,q}(z^*)\|$, by letting $k \to \infty$ with $k \in K_1$ in (4.5), we have $d^* = 0$ and hence $\nabla\psi_{p,q}(z^*)^T d^* = 0$.

Part 2. $\{\lambda_k\}_{k\in K_1}$ has a subsequence converging to zero and we may assume $\lim_{(K_2\ni)k\to\infty}\lambda_k = 0$ where $K_2 \subset K_1$ is an infinite set. From (3.8), we have for all $k \geq \bar{k}$ and $k \in K_2$,

$$\begin{aligned}\|\mathrm{H}_{p,q}(z^k + \delta^{-1}\lambda_k d_k)\| &> (1+\eta_k)\|\mathrm{H}_{p,q}(z^k)\| - \sigma\|\delta^{-1}\lambda_k d_k\|^2 \\ &\geq \|\mathrm{H}_{p,q}(z^k)\| - \sigma\|\delta^{-1}\lambda_k d_k\|^2,\end{aligned}$$

which gives

$$\frac{\|\mathrm{H}_{p,q}(z^k + \delta^{-1}\lambda_k d_k)\| - \|\mathrm{H}_{p,q}(z^k)\|}{\delta^{-1}\lambda_k} > -\sigma\delta^{-1}\lambda_k\|d_k\|^2. \tag{4.6}$$

Multiplying both sides of (4.6) by $\frac{1}{2}[\|\mathrm{H}_{p,q}(z^k+\delta^{-1}\lambda_k d_k)\| + \|\mathrm{H}_{p,q}(z^k)\|]$, we have for all $k \geq \bar{k}$ and $k \in K_2$,

$$\frac{\psi_{p,q}(z^k + \delta^{-1}\lambda_k d_k) - \psi_{p,q}(z^k)}{\delta^{-1}\lambda_k}$$

$$> -\frac{1}{2}\sigma\delta^{-1}\lambda_k\|d_k\|^2[\|\mathrm{H}_{p,q}(z^k + \delta^{-1}\lambda_k d_k)\| + \|\mathrm{H}_{p,q}(z^k)\|]. \tag{4.7}$$

Since $\psi_{p,q}$ is continuously differentiable at z^*, by letting $k \to \infty$ with $k \in K_2$ in (4.7), we have $\nabla\psi_{p,q}(z^*)^T d^* \geq 0$. On the other hand, since d_k is a sufficient descent direction of $\psi_{p,q}$, we have $\nabla\psi_{p,q}(z^*)^T d^* = \lim_{(K_2\ni)k\to\infty}\nabla\psi_{p,q}(z^k)^T d^k \leq 0$. These give $\nabla\psi_{p,q}(z^*)^T d^* = 0$.

Let $\bar{K} := \{k \in K_1 | d_k = -\nabla\psi_{p,q}(z^k)\}$. If $\bar{K}$ is an infinite set, then we have

$$\begin{aligned}\|\nabla\psi_{p,q}(z^*)\|^2 &= \lim_{(\bar{K}\ni)k\to\infty}\|\nabla\psi_{p,q}(z^k)\|^2 \\ &= \lim_{(\bar{K}\ni)k\to\infty} -\nabla\psi_{p,q}(z^k)^T d_k \\ &= -\nabla\psi_{p,q}(z^*)^T d^* \\ &= 0,\end{aligned}$$

which contradicts the assumption $\nabla\psi_{p,q}(z^*) \neq 0$. Otherwise, $\bar{K}$ is a finite set and d_k satisfies (3.7) for all sufficiently large $k \in K_1$. Then, by (3.7) we have

$$\begin{aligned}\gamma\|d^*\|^2 &= \lim_{(K_1\ni)k\to\infty}\gamma\|d_k\|^2 \\ &\leq \lim_{(K_1\ni)k\to\infty} -\nabla\psi_{p,q}(z^k)^T d_k \\ &= -\nabla\psi_{p,q}(z^*)^T d^* \\ &= 0,\end{aligned}$$

which gives $d^* = 0$. By (3.4), we have for all $k \in K_1$,

$$\|r_k - \nabla\psi_{p,q}(z^k)\| \leq \|\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k) + \mu_k I\|\|d_k\|. \tag{4.8}$$

Since

$$\lim_{(K_1\ni)k\to\infty}(\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k)+\mu_k I)=\mathrm{J}_{p,q}(z^*)^T\mathrm{J}_{p,q}(z^*)+\alpha\|\mathrm{H}_{p,q}(z^*)\|^\beta I,$$

by (4.8) and $d^*=0$, we have

$$\lim_{(K_1\ni)k\to\infty}\|r_k-\nabla\psi_{p,q}(z^k)\|=0. \tag{4.9}$$

Since $\|r_k\|\le\rho\|\nabla\psi_{p,q}(z^k)\|$, we can deduce from (4.9) that

$$\|\nabla\psi_{p,q}(z^*)\|=\lim_{(K_1\ni)k\to\infty}\|\nabla\psi_{p,q}(z^k)\|=\lim_{(K_1\ni)k\to\infty}\|r_k\|=0,$$

which also contradicts the assumption $\nabla\psi_{p,q}(z^*)\neq 0$. Therefore, we can conclude that any accumulation point z^* of $\{z^k\}$ satisfies $\nabla\psi_{p,q}(z^*)=0$. The second result follows directly from (3.2). We have completed the proof. □

In Theorem 4.2, we prove that any accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 3.1, if it exists, is a stationary point of the merit function $\psi_{p,q}(z)$. An important question that remains unanswered is whether such an accumulation point exists or not. In the rest of this section, we answer this question by investigating the boundedness of the sequence $\{z^k\}$. For this purpose, we introduce commonly studied uniform P-functions. A function $F:\mathbb{R}^n\to\mathbb{R}^n$ is said to be a uniform P-function on $\mathbb{R}^n$ if there exists a positive scalar $\xi>0$ such that

$$\max_{1\le i\le n}(u_i-v_i)(F_i(u)-F_i(v))\ge\xi\|u-v\|^2,\ \forall\ u,v\in\mathbb{R}^n.$$

A stronger condition than the uniform P-property of F is the so-called strong monotonicity which states that there exists a positive scalar $\xi>0$ such that

$$\langle u-v,F(u)-F(v)\rangle\ge\xi\|u-v\|^2,\ \forall\ u,v\in\mathbb{R}^n.$$

Both uniform P-functions and strong monotone functions are very broad and have very fine properties. By using the coerciveness of the function $\phi_{p,q}$ given in Proposition 2.3, similarly as the proof of [5, Proposition 3.5], we can obtain the following result.

Lemma 4.3. *If F is a uniform P-function, then the level sets*

$$L(C):=\{z=(x,y)\in\mathbb{R}^{2n}|\psi_{p,q}(z)\le C\} \tag{4.10}$$

are bounded for any $C>0$.

Theorem 4.4. *If F is a uniform P-function, then the iteration sequence $\{z^k\}$ generated by Algorithm* 3.1 *is bounded and hence it has at least one accumulation point.*

Proof. By (3.1) and the second result in Lemma 4.1, we have $\psi_{p,q}(z^k)\le e^{2\eta}\psi_{p,q}(z^0)$ for all $k\ge 0$. This implies $\{z^k\}\subset L(e^{2\eta}\psi_{p,q}(z^0))$ where $L(\cdot)$ is the level set defined by (4.10). Then, by Lemma 4.3, we obtain the desired result. □

Theorem 4.5. *Let $\{z^k\}$ be the iteration sequence generated by Algorithm* 3.1*. If $\{z^k\}$ has an isolated accumulation point z^*, then the whole sequence $\{z^k\}$ converges to z^*.*

Proof. Let $\{z^k\}_{k\in K}$ be any subsequence of $\{z^k\}$ such that $\lim_{(K\ni)k\to\infty} z^k = z^*$. By Theorem 4.2, it holds $\lim_{(K\ni)k\to\infty} \nabla\psi_{p,q}(z^k) = \nabla\psi_{p,q}(z^*) = 0$. Since either $\nabla\psi_{p,q}(z^k)^T d_k \le -\gamma\|d_k\|^2$ or $d_k = -\nabla\psi_{p,q}(z^k)$ for any k, we have $\lim_{(K\ni)k\to\infty} \|d_k\| = 0$ which gives $\lim_{(K\ni)k\to\infty} \|z^{k+1} - z^k\| = 0$. Thus, by [14, Proposition 8.3.10], we have the desired result. □

5. Local superlinear/quadratic convergence

In this section, we suppose that the iteration sequence $\{z^k\}$ generated by Algorithm 3.1 converges to z^* and $\mathrm{H}_{p,q}(z^*) = 0$. We prove that $\{z^k\}$ converges to z^* locally superlinearly or quadratically under the following assumption.

Assumption 5.1. There exist constants $\xi > 0$ and $\epsilon > 0$ such that

$$\|\mathrm{H}_{p,q}(z)\| \ge \xi\|z - z^*\|,\ \forall\ z \in N(z^*, \epsilon). \tag{5.1}$$

Assumption 5.1 is a local error bound condition which has been used to analyze the local convergence properties of some smoothing Levenberg-Marquardt methods (e.g., [21,23]). Note that if $\mathrm{J}_{p,q}(z^*)$ is nonsingular, then Assumption 5.1 holds by [31, Lemma 8]. However, the converse is not necessarily true. A simple counterexample is $\mathrm{H}_{p,q}(z) = |z| = 0$ where $z \in \mathbb{R}$. Hence, Assumption 5.1 is weaker than the nonsingularity condition.

Lemma 5.2. *The function $\mathrm{H}_{p,q}$ defined by* (1.7) *is semismooth on $\mathbb{R}^{2n}$ if $q \ge 1$. Moreover, if F' is locally Lipschitzian and $q \ge 2$, then $\mathrm{H}_{p,q}$ is strongly semismooth on $\mathbb{R}^{2n}$.*

Proof. The lemma holds by Proposition 2.5. □

Lemma 5.3. *Suppose that Assumption* 5.1 *holds. Let d_k be the direction generated by* (3.4) *and* (3.5)*. If $q \ge 1$, then for all sufficiently large k,*

$$\|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)d_k\| \le o(\|z^k - z^*\|) + \|\mathrm{J}_{p,q}(z^k)\|\frac{\|r_k\|}{\mu_k}. \tag{5.2}$$

Moreover, if F' is locally Lipschitzian and $q \ge 2$, then for all sufficiently large k,

$$\|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)d_k\| \le O(\|z^k - z^*\|^{1+\frac{\beta}{2}}) + \|\mathrm{J}_{p,q}(z^k)\|\frac{\|r_k\|}{\mu_k}. \tag{5.3}$$

Proof. Let

$$\hat{d}_k := -(\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k) + \mu_k I)^{-1}\nabla\psi_{p,q}(z^k).$$

Then, by (3.4) it holds

$$d_k = \hat{d}_k + (\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k) + \mu_k I)^{-1} r_k. \tag{5.4}$$

For any $k \ge 0$, we now define

$$\varphi_k(d) := \|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)d\|^2 + \mu_k\|d\|^2, \tag{5.5}$$

and consider the following optimization problem:

$$\min_{d\in\mathbb{R}^{2n}} \varphi_k(d). \tag{5.6}$$

It is easy to see that $\hat{d}_k$ is the solution of (5.6) because $\varphi_k(d)$ is convex and $\hat{d}_k$ is a stationary point of $\varphi_k(d)$. Hence, by (5.4) and (5.5), for all sufficiently large k,

$$\begin{aligned}
&\|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)d_k\| \\
&= \|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)\big[\hat{d}_k + (\mathrm{J}_{p,q}(z^k)^T\mathrm{J}_{p,q}(z^k) + \mu_k I)^{-1}r_k\big]\| \\
&\le \|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)\hat{d}_k\| + \|\mathrm{J}_{p,q}(z^k)\|\frac{\|r_k\|}{\mu_k} \\
&\le \sqrt{\varphi_k(\hat{d}_k)} + \|\mathrm{J}_{p,q}(z^k)\|\frac{\|r_k\|}{\mu_k} \\
&\le \sqrt{\varphi_k(z^* - z^k)} + \|\mathrm{J}_{p,q}(z^k)\|\frac{\|r_k\|}{\mu_k} \\
&= \sqrt{\|\mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\|^2 + \mu_k\|z^k - z^*\|^2} \\
&\quad + \|\mathrm{J}_{p,q}(z^k)\|\frac{\|r_k\|}{\mu_k}.
\end{aligned} \tag{5.7}$$

Since $\mathrm{H}_{p,q}$ is semismooth at z^* when $q \ge 1$ and $\mathrm{H}_{p,q}(z^*) = 0$, for all sufficiently large k,

$$\|\mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\| = o(\|z^k - z^*\|). \tag{5.8}$$

Also noticing that $\lim_{k\to\infty}\mu_k = \alpha\|\mathrm{H}_{p,q}(z^*)\|^\beta = 0$, for all sufficiently large k,

$$\mu_k\|z^k - z^*\|^2 = o(\|z^k - z^*\|^2). \tag{5.9}$$

So, by (5.7)–(5.9), we have (5.2). Moreover, if F' is locally Lipschitzian and $q \ge 2$, then $\mathrm{H}_{p,q}$ is strongly semismooth at z^* and so for all sufficiently large k,

$$\|\mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\| = O(\|z^k - z^*\|^2). \tag{5.10}$$

Since $\mathrm{H}_{p,q}$ is semismooth at z^*, $\mathrm{H}_{p,q}$ is locally Lipschitz continuous near z^*. Hence, for all z sufficiently close to z^*,

$$\|\mathrm{H}_{p,q}(z)\| = \|\mathrm{H}_{p,q}(z) - \mathrm{H}_{p,q}(z^*)\| = O(\|z - z^*\|). \tag{5.11}$$

Then, by the definition of μ_k, for all sufficiently large k,

$$\mu_k = \alpha\|\mathrm{H}_{p,q}(z^k)\|^\beta = O(\|z^k - z^*\|^\beta),$$

which gives

$$\mu_k\|z^k - z^*\|^2 = O(\|z^k - z^*\|^{2+\beta}). \tag{5.12}$$

Thus, by (5.7), (5.10) and (5.12), we have (5.3). The proof is completed. □

Lemma 5.4. *Suppose that Assumption* 5.1 *holds and* $q \geq 1$*. Let* d_k *be the direction generated by* (3.4) *and* (3.5)*. If* $\lim_{k\to\infty} \frac{\|r_k\|}{\mu_k} = 0$*, then* $\lim_{k\to\infty} d_k = 0$.

Proof. By (5.4), we have for all $k \geq 0$

$$\begin{aligned}\|d_k\| &= \|\hat{d}_k + (\mathrm{J}_{p,q}(z^k)^T \mathrm{J}_{p,q}(z^k) + \mu_k I)^{-1} r_k\| \\ &\leq \|\hat{d}_k\| + \|(\mathrm{J}_{p,q}(z^k)^T \mathrm{J}_{p,q}(z^k) + \mu_k I)^{-1}\| \|r_k\| \\ &\leq \|\hat{d}_k\| + \frac{\|r_k\|}{\mu_k}.\end{aligned}$$

Thus, we only need to show that $\lim_{k\to\infty} \|\hat{d}_k\| = 0$. In fact, by the definition of μ_k and (5.1), for all sufficiently large k,

$$\mu_k = \alpha \|\mathrm{H}_{p,q}(z^k)\|^\beta \geq \alpha \xi^\beta \|z^k - z^*\|^\beta. \tag{5.13}$$

Since $\hat{d}_k$ is the solution of (5.6), by (5.5), (5.8) and (5.13), for all sufficiently large k,

$$\begin{aligned}\|\hat{d}_k\|^2 &\leq \frac{\varphi_k(\hat{d}_k)}{\mu_k} \\ &\leq \frac{\varphi_k(z^* - z^k)}{\mu_k} \\ &= \frac{\|\mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\|^2}{\mu_k} + \|z^k - z^*\|^2 \\ &\leq \frac{o(\|z^k - z^*\|^2)}{\alpha \xi^\beta \|z^k - z^*\|^\beta} + \|z^k - z^*\|^2.\end{aligned}$$

This gives $\lim_{k\to\infty} \|\hat{d}_k\| = 0$ and completes the proof. □

Theorem 5.5. *Let Assumption* 5.1 *hold and* $q \geq 1$*. Then* $\{z^k\}$ *converges to* z^* *superlinearly if one of the following conditions holds:*

(i) $\beta \in (0, 1)$;
(ii) $\beta = 1$ *and* $\theta_k \to 0$ *as* $k \to \infty$.

Proof. By (3.1), (3.3) and (5.11), for all sufficiently large k,

$$\frac{\psi_{p,q}(z^k)}{\mu_k} = \frac{1}{2\alpha} \|\mathrm{H}_{p,q}(z^k)\|^{2-\beta} = O(\|z^k - z^*\|^{2-\beta}). \tag{5.14}$$

Thus, if $\beta \in (0, 1)$, then by (3.5) and (5.14), for all sufficiently large k,

$$\frac{\|r_k\|}{\mu_k} \leq \frac{\theta_k \psi_{p,q}(z^k)}{\mu_k} \leq \theta O(\|z^k - z^*\|^{2-\beta}) = o(\|z^k - z^*\|).$$

And if $\beta = 1$ and $\theta_k \to 0$ as $k \to \infty$, then by (3.5) and (5.14), for all sufficiently large k,

$$\frac{\|r_k\|}{\mu_k} \leq \frac{\theta_k \psi_{p,q}(z^k)}{\mu_k} = \theta_k O(\|z^k - z^*\|) = o(\|z^k - z^*\|).$$

So, we conclude that if one of the conditions (i) and (ii) holds, then for all sufficiently large k,

$$\frac{\|r_k\|}{\mu_k} = o(\|z^k - z^*\|), \tag{5.15}$$

which together with (5.2) gives

$$\|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)d_k\| = o(\|z^k - z^*\|). \tag{5.16}$$

Moreover, by Lemma 5.4 and (5.15), we have $\lim_{k\to\infty} d_k = 0$ and hence $\lim_{k\to\infty}(z^k + d^k) = z^*$. Since $\mathrm{H}_{p,q}$ is semismooth at x^*, for all sufficiently large k,

$$\|\mathrm{H}_{p,q}(z^k) - \mathrm{H}_{p,q}(z^*) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\| = o(\|z^k - z^*\|), \tag{5.17}$$

$$\|\mathrm{H}_{p,q}(z^k + d^k) - \mathrm{H}_{p,q}(z^*) - \mathrm{J}_{p,q}(z^k + d^k)(z^k + d^k - z^*)\| = o(\|z^k + d^k - z^*\|). \tag{5.18}$$

Moreover, by the continuity of $\mathrm{J}_{p,q}(x)$ we have

$$\lim_{k\to\infty} \|\mathrm{J}_{p,q}(z^k + d^k) - \mathrm{J}_{p,q}(z^k)\| = 0. \tag{5.19}$$

Thus, by (5.17), (5.18) and (5.19), for all sufficiently large k,

$$\begin{aligned}
& \|\mathrm{H}_{p,q}(z^k + d^k) - \mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k)d^k\| \\
\le\ & \|\mathrm{H}_{p,q}(z^k + d^k) - \mathrm{H}_{p,q}(z^*) - \mathrm{J}_{p,q}(z^k + d^k)(z^k + d^k - z^*)\| \\
& + \|(\mathrm{J}_{p,q}(z^k + d^k) - \mathrm{J}_{p,q}(z^k))(z^k + d^k - z^*)\| \\
& + \|\mathrm{H}_{p,q}(z^k) - \mathrm{H}_{p,q}(z^*) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\| \\
=\ & o(\|z^k + d^k - z^*\|) + o(\|z^k - z^*\|).
\end{aligned} \tag{5.20}$$

On the other hand, by (5.1), for all sufficiently large k,

$$\begin{aligned}
\xi\|z^k + d^k - z^*\| &\le \|\mathrm{H}_{p,q}(z^k + d^k)\| \\
&\le \|\mathrm{H}_{p,q}(z^k + d^k) - \mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k)d^k\| \\
&\quad + \|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k)d^k\|,
\end{aligned} \tag{5.21}$$

which together with (5.16) and (5.20) gives

$$\|z^k + d^k - z^*\| = o(\|z^k - z^*\|). \tag{5.22}$$

Hence, by (5.1), (5.11), and (5.22), for all sufficiently large k,

$$\|\mathrm{H}_{p,q}(z^k + d^k)\| = O(\|z^k + d^k - z^*\|) = o(\|z^k - z^*\|) = o(\|\mathrm{H}_{p,q}(z^k)\|).$$

This indicates that for all sufficiently large k, the direction d_k generated by (3.4) and (3.5) always satisfies the condition (3.6) and hence $\lambda_k = 1$. Consequently, for all sufficiently large k, $z^{k+1} = z^k + d^k$, which together with (5.22) proves the theorem. □

Theorem 5.6. *Let Assumption* 5.1 *hold. Suppose that F' is locally Lipschitzian near z^* and $q \geq 2$. If we choose $\beta = 2$ and $\theta_k = O(\|\mathrm{H}_{p,q}(z^k)\|^2)$, then $\{z^k\}$ converges to z^* quadratically.*

Proof. By (3.1), (3.3), (3.5) and (5.11), for all sufficiently large k,

$$\frac{\|r_k\|}{\mu_k} \leq \frac{\theta_k \psi_{p,q}(z^k)}{\mu_k} = \frac{\theta_k}{2\alpha} = O(\|z^k - z^*\|^2), \tag{5.23}$$

which together with (5.3) gives

$$\|\mathrm{H}_{p,q}(z^k) + \mathrm{J}_{p,q}(z^k) d_k\| = O(\|z^k - z^*\|^2). \tag{5.24}$$

By Lemma 5.2, $\mathrm{H}_{p,q}$ is strongly semismooth at z^*. Thus, for all sufficiently large k,

$$\|\mathrm{H}_{p,q}(z^k) - \mathrm{H}_{p,q}(z^*) - \mathrm{J}_{p,q}(z^k)(z^k - z^*)\| = O(\|z^k - z^*\|^2).$$

Moreover, by Lemma 5.4 and (5.23), we have $\lim_{k\to\infty} d_k = 0$ and so $\lim_{k\to\infty}(z^k + d^k) = z^*$. Therefore, the inequality (5.20) becomes

$$\|\mathrm{H}_{p,q}(z^k + d^k) - \mathrm{H}_{p,q}(z^k) - \mathrm{J}_{p,q}(z^k) d^k\| = o(\|z^k + d^k - z^*\|) + O(\|z^k - z^*\|^2). \tag{5.25}$$

By (5.21), (5.24) and (5.25), for all sufficiently large k,

$$\xi\|z^k + d^k - z^*\| \leq o(\|z^k + d^k - z^*\|) + O(\|z^k - z^*\|^2),$$

which gives

$$\|z^k + d^k - z^*\| = O(\|z^k - z^*\|^2).$$

Then, by following the proof of Theorem 5.5, we can obtain the quadratic convergence. □

6. Numerical results

In this section, we report some numerical results of Algorithm 3.1. All experiments were carried out on a PC with CPU of Inter(R) Core(TM)i7-7700 CPU @ 3.60 GHz and RAM of 8.00GB. The codes are written in MATLAB and run in MATLAB R2018a environment. The parameters used in Algorithm 3.1 are chosen as $\rho = 0.01,\ \tau = 0.5, \gamma = 0.8,\ \delta = 0.8,\ \sigma = 0.2,\ \alpha = 10^{-3},\ \beta = 1,\ \eta_k = 0.85^k, \theta_k = \frac{1}{2^{k+1}}$. In Step 2, GMRES is used as the linear solver to find the inexact direction d_k. Moreover, we use $\|\mathrm{H}_{\tau,q}(z^k)\| \leq 10^{-5}$ as the stopping criterion.

6.1. Algorithm 3.1 for solving LCP.

In this subsection, we apply Algorithm 3.1 to solve the linear complementarity problem (LCP):

$$\text{(LCP)}\ x \geq 0,\ y \geq 0,\ y = Mx + a,\ x^T y = 0, \tag{6.1}$$

in which $M \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}^n$ are given matrix and vector. In our experiments, we investigate the following two LCPs:

(i) Let M be the block diagonal matrix with $\frac{N_1^T N_1}{\|N_1^T N_1\|}, ..., \frac{N_4^T N_4}{\|N_4^T N_4\|}$ as block diagonals, i.e., $M = \mathbf{diag}(\frac{N_i^T N_i}{\|N_i^T N_i\|})$ in which $N_i = \mathbf{rand}(\frac{n}{4}, \frac{n}{4})$ for $i = 1, ..., 4$. Take $a = \mathbf{rand}(n, 1)$. In this case, the function $F(x) = Mx + a$ has the Cartesian P_0-property.

Table 1 Numerical results of Algorithm 3.1 for LCP (i)

p	q	n	mIT	aIT	aCPU	aHK
1.5	2.0	1000	3	3.8	1.58	1.1256e-06
		2000	3	3.8	8.37	1.5427e-06
		3000	3	4.2	26.16	3.3168e-06
	2.5	1000	3	4.2	1.70	4.1813e-06
		2000	3	4.4	9.80	2.0970e-06
		3000	4	4.9	30.78	3.0403e-06
	3.0	1000	4	5.0	1.98	1.4613e-06
		2000	5	5.4	12.03	4.5947e-07
		3000	5	5.7	37.48	2.2154e-06
	4.0	1000	5	6.3	2.63	2.3774e-06
		2000	5	6.4	14.32	2.2187e-06
		3000	5	7.1	44.80	2.8037e-06
2.0	2.0	1000	3	3.0	1.22	1.0820e-06
		2000	3	3.0	6.67	1.5592e-06
		3000	3	3.9	24.74	3.3452e-06
	2.5	1000	4	4.1	1.73	1.5391e-06
		2000	4	4.3	9.73	1.0381e-06
		3000	4	4.7	30.29	8.9666e-07
	3.0	1000	4	5.0	2.11	1.7055e-06
		2000	4	5.2	12.32	1.6543e-06
		3000	4	5.8	37.41	1.1828e-06
	4.0	1000	5	6.3	2.63	2.0269e-06
		2000	5	6.7	14.9	1.9541e-06
		3000	5	7.1	45.18	3.0181e-06
3.0	3.0	1000	5	5.3	2.16	1.4450e-06
		2000	5	5.5	12.49	8.0427e-07
		3000	5	5.9	38.01	1.0286e-06
	3.5	1000	5	5.9	2.52	1.7069e-06
		2000	6	6.2	14.23	7.9032e-07
		3000	6	6.6	42.34	2.2772e-06
	4.0	1000	6	6.5	2.61	1.5799e-06
		2000	6	6.9	15.51	8.0136e-07
		3000	6	7.3	46.58	2.3059e-06
	5.0	1000	6	7.6	3.08	2.6571e-06
		2000	7	8.0	17.69	1.8766e-06
		3000	7	8.2	52.73	3.9742e-06

Table 2 Numerical results of Algorithm 3.1 for LCP (ii)

p	q	n	mIT	aIT	aCPU	aHK
1.5	2.0	1000	3	4.9	2.01	9.6543e-07
		2000	3	4.9	10.99	2.9507e-06
		3000	3	6.1	31.95	2.2055e-06
	2.5	1000	3	5.1	2.08	1.6982e-06
		2000	3	5.7	12.80	7.8356e-07
		3000	3	7.0	44.83	1.7204e-06
	3.0	1000	3	5.4	2.17	3.2059e-06
		2000	3	6.2	12.80	1.1019e-06
		3000	3	7.5	48.29	2.7454e-06
	4.0	1000	3	6.4	2.83	6.4125e-07
		2000	3	6.9	16.33	3.3298e-06
		3000	3	8.5	54.06	3.0430e-06
2.0	2.0	1000	2	4.1	1.70	1.3339e-06
		2000	3	4.7	10.79	8.5056e-07
		3000	3	5.7	37.92	1.1304e-06
	2.5	1000	2	4.5	2.01	1.9688e-06
		2000	3	5.1	11.89	1.5761e-06
		3000	3	6.4	41.98	2.0109e-06
	3.0	1000	2	5.0	2.04	1.0729e-06
		2000	3	5.4	12.20	1.6718e-06
		3000	3	7.1	45.86	2.3872e-06
	4.0	1000	2	5.6	2.66	2.0955e-06
		2000	3	6.5	14.55	9.9662e-07
		3000	3	8.1	52.34	2.5604e-06
3.0	3.0	1000	2	4.6	1.87	2.2813e-06
		2000	3	5.3	11.90	8.2407e-07
		3000	3	6.8	43.80	1.9457e-06
	3.5	1000	2	5.0	2.05	1.7447e-06
		2000	3	5.6	12.73	1.5802e-06
		3000	3	7.4	47.25	1.6054e-06
	4.0	1000	2	5.5	2.42	4.9046e-07
		2000	3	6.0	13.41	2.4359e-06
		3000	3	7.6	48.91	3.3615e-06
	5.0	1000	2	6.0	2.43	1.9316e-06
		2000	3	6.9	15.69	1.6752e-06
		3000	3	8.5	54.22	2.1936e-06

Table 3 Numerical results of Algorithm 3.1 ($p = 2$, $q = 2$) for LCP

LCP	n	aIT	aCPU	aHK
(i)	5000	3.0	75.52	1.4415e-06
	6000	3.0	129.35	8.0437e-07
	7000	3.0	222.79	2.9783e-07
	8000	3.1	446.83	9.7375e-07
(ii)	5000	4.3	106.82	2.6809e-06
	6000	4.4	190.79	9.7585e-07
	7000	5.2	364.81	2.2891e-06
	8000	5.2	739.36	5.5599e-07

(ii) Let $M = \mathbf{diag}\big(\frac{N_i}{\|N_i\|} - \mathbf{eye}(n/4)\big)$ in which $N_i = \mathbf{rand}(\frac{n}{4}, \frac{n}{4})$ for $i = 1, ..., 4$. Take $a = \mathbf{rand}(n, 1)$. In this case, the function $F(x) = Mx + a$ has no Cartesian P_0-property.

In the experiments, we generate 10 problem instances for each size n. We use $x^0 = (1, 0, ..., 0)^T$ and $y^0 = Mx^0 + a$ as the starting point. Numerical results are listed in Tables 1 and 2 in which p and q are parameters used in the NCP function $\phi_{p,q}$, **mIT** denotes the smallest value of the iteration numbers, **aIT** denotes the average value of the iteration numbers, **aCPU** denotes the average value of the CPU time in seconds and **aHK** denotes the average value of $\|\mathrm{H}_{p,q}(z^k)\|$ when the algorithm terminates among the 10 testing.

From Tables 1 and 2, we can see that Algorithm 3.1 is very effective for solving LCPs even though these problems have no Cartesian P/P_0-property. Moreover, from our numerical implementations, we find that the performance of our algorithm becomes worse when p and q increase. This indicates that Algorithm 3.1 may cause numerical difficulty and ill-conditional effects for large values of p and q. A reasonable interpretation for this is that the values of elements in $\nabla\phi_{p,q}(a, b)$ may be very large/small when p and q are larger. This may make Jacobian matrix $\mathrm{J}_{p,q}(z)$ to be close to singular or badly scaled so that the performance of Algorithm 3.1 is worse. Table 3 gives some numerical results of Algorithm 3.1 with $p = 2$ and $q = 2$ from which we can see that Algorithm 3.1 is fairly capable for solving large-scale LCPs with a limited amount of work.

6.2. **Algorithm 3.1 for solving NCP.** In this subsection, we apply Algorithm 3.1 to solve the NCP defined by (1.1). The test problem is generated in the way proposed by Gomes-Ruggiero et al [16]. Let $f(x) = (f_1(x), ..., f_n(x))$ be a differentiable nonlinear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $x^* = (0, 1, 0, 1, ...)^T$. For $i = 1, ..., n$, set

$$F_i(x) = \begin{cases} f_i(x) - f(x_i^*) + 1, & \text{if } i \text{ is odd}, \\ f_i(x) - f(x_i^*), & \text{otherwise}. \end{cases}$$

Obviously, x^* is a nondegenerate solution of this NCP. In this example, for the function f, we consider the following four cases:

(a) $f_i(x) = e^{x_i} - 1,\ i = 1, ..., n;$

(b) $f_i(x) = \sin(x_i) - x_i^2,\ i = 1, ..., n;$

(c) $f_i(x) = x_i^2 - i,\ i = 1, ..., n;$

(d) $f_i(x) = \begin{cases} x_i^2 + x_{i+1}, & i = 1, ..., n-1, \\ x_i^2, & i = n. \end{cases}$

In the experiments, we choose with $p = 2$ and $q = 2$ and use $x_0 = y_0 = (1, ..., 1)^T$ as the starting point. First, to observe local convergence behavior of our algorithm, we apply Algorithm 3.1 to solve these NCPs with the size $n = 1000$. Table 4 gives the value of $\|\mathrm{H}_{p,q}(z^k)\|$ at the k-th iteration from which we can clearly see the local fast convergence rate of our algorithm.

Table 4 The value of $\|\mathrm{H}_{p,q}(z^k)\|$ at the k-th iteration

	NCP(a)	NCP (b)	NCP (c)	NCP (d)
$k = 1$	15.8076	28.3128	14.2327	14.8954
$k = 2$	2.5974	5.5398	4.9297	7.1101
$k = 3$	0.3398	0.3643	0.7105	1.0330
$k = 4$	0.0019	0.0042	0.0083	0.0298
$k = 5$	3.2660e-08	7.3018e-07	1.5768e-06	9.9516e-06
$k = 6$	5.0250e-15	2.1499e-14	6.1935e-14	1.5354e-12

Next, we apply Algorithm 3.1 to solve these NCPs with different sizes. Numerical results are listed in Table 5 in which **IT** denotes the iteration numbers, **CPU** denotes the CPU time in seconds and **HK** denotes the value of $\|\mathrm{H}_{p,q}(z^k)\|$ when the algorithm terminates. From Table 5, we may see that Algorithm 3.1 is very effective for solving large-scale NCPs.

Table 5 Numerical results of Algorithm 3.1 for NCP

NCP	n	IT	CPU	HK
(a)	5000	5	235.80	5.6812e-08
	6000	5	373.69	5.8987e-08
	7000	5	572.07	6.0467e-08
	8000	5	989.94	6.1401e-08
(b)	5000	4	179.65	1.2951e-10
	6000	4	309.81	1.6573e-10
	7000	4	458.72	2.0575e-10
	8000	4	616.21	8.2120e-14
(c)	5000	5	218.55	3.1020e-06
	6000	5	329.52	3.3193e-06
	7000	5	545.96	3.5074e-06
	8000	5	817.39	3.6727e-06
(d)	5000	5	257.61	3.8963e-12
	6000	6	419.20	4.1944e-12
	7000	6	671.42	4.4223e-12
	8000	6	929.52	4.5971e-12

7. Conclusions

We have introduced a new NCP function $\phi_{p,q}(a, b)$ defined by (1.6) and showed that it is smooth, coercive and strongly semismooth. By the equivalent reformulation, we have proposed an inexact Levenberg-Marquardt method (ILMM) to solve the NCP which is designed based on a derivative-free line search technique. We have proved that the proposed ILMM has global convergence without any additional condition. Moreover, under a local error bound condition which is weaker than the nonsingularity condition, we have proved that the proposed ILMM has local superlinear/quadratic convergence rate. We have also reported some numerical results which indicate that the proposed ILMM is very effective for solving large-scale LCPs and NCPs. In the terms of Jordan algebra, Ma et al. [25] proposed a generalized Fischer-Burmeister $\phi^q_{\mathrm{D-FB}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which is defined by

$$\phi^q_{\mathrm{D-FB}}(x, y) := (\sqrt{x^2 + y^2})^q - (x + y)^q,$$

where $q > 1$ is a positive odd integer. Ma et al. [25] showed that $\phi^q_{\mathrm{D-FB}}$ is a complementarity function associated with the second-order cone and it is continuously differentiable everywhere. Based on results established in [25], the ILMM studied in this paper may be extended to solve general large-scale second-order cone complementarity problems which is an interesting issue deserves further research.

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Jingyong Tang
College of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China
E-mail address: tangjy@xynu.edu.cn

Jinchuan Zhou
College of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China
E-mail address: jinchuanzhou@163.com

Jan Harold Alcantara
Center for Advanced Intelligence Project, RIKEN, Tokyo 103-0027, Japan
E-mail address: janharold.alcantara@riken.jp

Jein-Shan Chen
Department of Mathematics, National Taiwan Normal University, Taipei 116059, Taiwan
E-mail address: jschen@math.ntnu.edu.tw