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ON SUPERIOR METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. In this paper, a modification of the super meter is introduced which is called superior meter. The topology induced by the new version of super meter is discussed and it is shown that under suitable condition a neighborhood produced by a superior is an open set. Also, it is investigated, by providing an example, that despite metric spaces, the compactness is not necessarily equivalent to the sequential compactness in superior metric spaces. Of course, it is proved that if a superior meter is upper semicontinuous with respect to the second variable, then the compactness and sequentially compactness are equivalent. Moreover, the relationship between the compactness and totally boundedness and completeness are studied. The Cantor intersection theorem is extended from metric spaces to superior metric spaces. The Banach contraction principle and Caristi fixed-point theorem, in the setting of complete superior metric spaces with mild assumptions, are presented. Finally, some examples support the main results are appeared in the article.

1. Introduction

The concept of metric spaces, which abstract the notion of distance, originated in the early 20th century with the work of Ren 辿 Fr 辿 chet [8] in 1906 and was formalized by Felix Hausdorff [9] in 1914, who coined the term metric space. Fr 辿 chet's work focused on spaces of functions and convergence, while Hausdorff's work provided the formal definition and expanded the scope of metric spaces. The generalization of metric spaces has evolved over time, leading to concepts like generalized b-metric spaces and statistical metric spaces, which relax the traditional requirements of the triangle inequality. The concept of b-metric space or generalizations of it appeared in works of Bourbaki [4], Bakhtin [1], etc. Some examples of b-metric spaces and some fixed and strict fixed point theorems in b-metric spaces can also be found in M. Boriceanu, A. Petruşel and I.A.Rus [3]. In 2015, a new generalization of b-metric spaces, dislocated metric spaces, and modular spaces is introduced by Jleli and Samet [10]. Recently, a new generalization of metric spaces which extends the definition given by Jleli and Samet [10] is introduced by Erdal and Khojastehd [11]. In this paper, first, the concept of supermeter, which is a partial modification of supermeter, is introduced. The concepts of a convergent sequence, Cauchy sequence, open set, and many topological notion in the setting of superior metric spaces are presented. It is proved that the limit point of a sequence in a superior metric space, if exists, is unique which is different from the case where

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we have a super metric space. Despite metric spaces, the diameter of a bounded set is not necessarily equal to the diameter of its closure. In this regard, we have stated a proposition that states that an inequality holds instead of equality. Using this proposition and the other results, we extend Cantor's intersection theorem from metric spaces to superior metric spaces. The equivalence between compactness and sequential compactness is proven by the condition that the superior meter is upper semicontinuous with respect to the second variable. Moreover, the relationship between the compactness and totally boundedness and completeness are studied. The Banach contraction principle and Caristi fixed-point theorem are extended to the (new version) complete superior metric spaces with mild assumptions than the hypotheses given in [11]. Finally, some examples support the main results appeared in the article.

2. Main results

In this section, a different form of super metric space is introduced. Then some well known topological notions and results in theses spaces are examined. Further, some fixed point theorems in the setting of superior metric spaces are given.

The following definition is a generalization of a metric space.

Definition 2.1 ([5,10,12])**.** Let X be a nonempty set. The mapping $D : X \times X \to [0, +\infty)$ is called a quasi meter (b-meter, K-meter), if it satisfies the following conditions

1. $D(x, y) = 0$, if and if $x = y$,
2. $D(x, y) = D(y, x)$, for all $x, y \in X$,
3. there exists $c > 0$ such that, for all $x, y, z \in X$,

$$D(x, y) \leq c(D(x, z) + D(z, y)).$$

Note. It is obvious that if D is a quasi meter, then $c \geq 1$ when X has more one point.

Definition 2.2 ([11])**.** Let X be a nonempty set. The mapping $D : X \times X \to [0, +\infty)$ is said to be super meter,

1. if $D(x, y) = 0$, then $x = y$,
2. $D(x, y) = D(y, x)$, for all $x, y \in X$,
3. there exists $s \geq 1$ such that for all $y \in X$, there exist distinct sequences $(x_n), (y_n) \subset X$ with $D(x_n, y_n) \to 0$, when n tends to infinity, such that

$$\limsup_{n\to\infty} D(y_n, y) \leq s \limsup_{n\to\infty} D(x_n, y).$$

The next definition, known as JS-meter which is a generalization of the definition of a meter and further b-meter.

Definition 2.3 ([10])**.** Let X be a nonempty set. The mapping $D : X \times X \to [0, +\infty)$ is called generalized meter, if it satisfies the following conditions

1. $D(x, y) = 0$, then $x = y$,
2. $D(x, y) = D(y, x)$, for all $x, y \in X$,

3. there exists $c > 0$ such that, if $D(x_n, x) \to 0$, then
$$D(x,y) \leq c \limsup_{n\to\infty} D(x_n, y), \ \forall y \in X.$$

Remark 2.4. It is obvious that condition (iii) of Definition 2.3 is true when the mapping D is lower semicontinuous with respect to the first variable, because if $D(x_n, x) \to 0$ then it follows from the lower semicontinuity of D with respect to the first variable that
$$D(x,y) \leq \liminf_{n\to\infty} D(x_n, y) \leq \limsup_{n\to\infty} D(x_n, y), \ \forall y \in X.$$
This completes the proof of the assertion.

In the following, we are going to introduce a new definition which is slightly different from the definitions of the super meter. Further, it is an extension of the definition of JS-meter.

Definition 2.5. Let X be a set and $D : X \times X \to \mathbb{R}$ a mapping. We say that D is a superior meter, if the following assertions hold

1. if $D(x, y) = 0$, then $x = y$,
2. $D(x, y) = D(y, x)$, for all $x, y \in X$,
3. there exists $s > 0$ such that, if for distinct sequences $(x_n), (y_n)$ of X, $D(x_n, y_n) \to 0$ then
$$\limsup_{n\to\infty} D(y_n, y) \leq s \limsup_{n\to\infty} D(x_n, y), \ \forall y \in X.$$

The following example shows that Definition 2.5 is different from of the Definition 2.2.

Example 2.6. Let $X = [1, 3]$ and $D(x,y) = \begin{cases} xy & x \neq y \\ 0 & x = y \end{cases}$. It is obvious that for each y in X, there is no $(x_n, y_n) \in X \times X$, such that $D(x_n, y_n) \to 0$. Hence m does not satisfy in Definition 2.2 while it fulfils in the Definition 2.5.

Convention: From now on, we will use condition $D(x, x) = 0$, for all $x \in X$ instead of condition (1) in the definition of a superior meter (that is; Definition 2.5).

Definition 2.7. Let (X, D) be a superior meter. For each $x \in X$ and $r > 0$, the ball centered by x and radius r is denoted by $B(x, r)$ and is defined by
$$B(x, r) = \{y \in X : D(x, y) < r\}.$$
It is obvious by the above convention that $x \in B(x, r)$.

Definition 2.8. A subset A of a superior metric space X is called open, if for all $x \in A$, there exists $r_x > 0$ such that $B(x, r_x) \subset A$, and $A \subset X$ is called closed, if A^c is open.

Remark 2.9. If D is a superior meter and $D(x_n, x) \to 0$, then by taking $y_n = x$, we get
$$D(y_n = x, y) \leq c \limsup_{n\to\infty} D(x_n, y).$$
Consequently, a superior meter is a JS-meter.

In the next result a closed set and the closure of a set of a superior metric space are characterized.

Proposition 2.10. *Let (X, D) be a superior metric space. Then for any nonempty subset $A \subset X$, we have*

(i) *A is closed if and only if for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$.*
(ii) *If we define $\overline{A}$ to be intersection of all closed subsets of X which contains A, then $x \in \overline{A}$ if and only if for any $\varepsilon > 0$, we have $B(x, \varepsilon) \cap A \neq \emptyset$.*

Proof. (i) Assume that A is closed and let $\{x_n\}$ be a sequence in A such that $\lim_{n\to\infty} D(x_n, x) = 0$. We claim that $x \in A$, otherwise $x \notin A$. Since A is closed (that is A^c is open) then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap A = \emptyset$. Since $\{x_n\}$ converges to x, then there exists $N \geq 1$ such that for any $n \geq N$, we have $x_n \in B(x, \varepsilon)$. Hence $x_n \in B(x, \varepsilon) \cap A$, which leads to a contraction. Conversely, assume that for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$. We prove that A is closed. Let $x \notin A$, we need to prove that there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap A = \emptyset$. Assume it is not, i.e. for any $\varepsilon > 0$, we have $B(x, \varepsilon) \cap A \neq \emptyset$, so for any $n \geq 1$, choose $x_n \in B(x, \frac{1}{n}) \cap A \neq \emptyset$. Clearly we have $\{x_n\}$ converges to x. Our assumption on A implies that $x \in A$, which is a contradiction.
(ii) Clearly $\overline{A}$ is the smallest closed subset which contains A. Put $A^* = \{x \in X : \forall \varepsilon > 0, \exists a \in A \textit{ such that } D(x, a) < \varepsilon\}$, by $D(x, x) = 0$, we have $A \subset A^*$. It is obvious that A^* is closed, because if $\{x_n\}$ is a sequence in A^* such that $\{x_n\}$ converges to x, then, since $x_n \in A^*$ for any $n \in \mathbb{N}$ there exists $a_n \in A$ such that $D(x_n, a_n) < \frac{1}{n}$ which implies $D(x_n, a_n) \to 0$. Hence, by the definition of superior meter we get

$$\limsup D(a_n, y) \leq K \limsup D(x_n, y), \ \forall y \in X.$$

Now, if, in the previous inequality, we take $y = x$, then

$$\limsup D(a_n, x) \leq K \limsup D(x_n, x).$$

The right hand of the latest inequality approaches to zero when n tends to infinity which implies $\limsup D(a_n, x) \to 0$. Therefore,

$$0 \leq \liminf D(a_n, x) \leq \limsup D(a_n, x) \to 0.$$

Hence $\lim_{n\to\infty} D(a_n, x) = 0$. This proves one part of (ii). Vice versa, by contradiction assume that $x \notin \overline{A}$. Hence, there exists closed subset B of X such that $A \subset B$ with $x \notin B$. This means $x \in B^c$ and since B^c is open, there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset B^c$. Therefore which $B(x, \varepsilon) \subset A$ which is a contraction. This completes the proof. □

Definition 2.11. If (X, D) is a superior metric space and $\{x_n\}$ is a sequence in X, then $\{x_n\}$ converges to $x \in X$, when $D(x_n, x) \to 0$.

Definition 2.12. Let (X, D) be a superior metric space. The sequence $\{x_n\}$ in X is called a cauchy sequence, if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.

Remark 2.13. If $\{x_n\}$ is a cauchy sequence, then for each $\varepsilon > 0$, there exist $n_0, m_0 \in \mathbb{N}$, such that $D(x_n, x_m) < \varepsilon$, for all $m \geq m_0$, $n \geq n_0$. This means that

$\sup D(x_n, x_m) < \varepsilon$, for n and m enough large. Hence

$$\lim_{n\to\infty} \sup_{m>n} D(x_n, x_m) = 0,$$

and $\dim(\{x_n\}) = \sup\{x_n : n \in \mathbb{N}\} < \infty$. Therefore,

$$\lim_{n\to\infty} \dim(\{D(x_k, x_{k+1}) : k \geq n\}) = 0.$$

Proposition 2.14. *Let (X, D) be a superior metric space. If $\{x_n\}$ is a cauchy sequence and $G_n = \{x_n, x_{n+1}, \ldots\}$, then $\{G_n\}$ is a decreasing sequence with $\lim_{n\to\infty} diam(G_n) = 0$.*

Proof. It is obvious that $\{G_n\}$ is a decreasing sequence. Also, since $\{x_n\}$ is a cauchy sequence, then, for all $\varepsilon > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $m \geq m_0$ such that $D(x_n, x_m) < \varepsilon$. Hence, for all $n \geq N_0 = \max\{n_0, m_0\}$, we have $D(x_n, x_{n+1}) < \varepsilon$, which implies $diam G_n \leq \varepsilon$, for all $n \geq N_0$. This completes the proof. □

Lemma 2.15. *Suppose that (X, D) is a superior metric space with constant K (which is the smallest K satisfying condition (3) in the definition of a superior meter) and A is a nonempty bounded subset Of X. Then $diam(\overline{A}) \leq K^2 diam(A)$.*

Proof. Let $x, y \in \overline{A}$, then there exits sequences $\{x_n\}, \{y_n\} \subset A$ such that $D(x_n, x) \to 0$ and $D(y_n, y) \to 0$. By definition of superior meter we know that

$$\limsup D(x, z) \leq K \limsup D(x_n, z),\ \forall z \in X.$$

By putting $z = y$ in the last inequality, we get

$$D(x, z) \leq K \limsup D(x_n, y).$$

In a similar way we have

$$\limsup D(y, w) \leq K \limsup D(y_n, w),\ \forall w \in X.$$

Now, condition (3) of Definition 2.5 implies

$$\limsup D(y, x_n) \leq K \limsup D(x_n, y_n) \leq D diam(A),$$

and then $\sup_{x,y\in\overline{A}} m(x, y) \leq K^2 diam(A)$. □

Remark 2.16. If (X, D) is a metric space, then $k = 1$ and by Lemma 2.15 we get $\mathrm{diam}(\overline{A}) = \mathrm{diam}(A)$.

Theorem 2.17. (Cantor's intersection theorem) *A superior metric space (X, D) is complete if and only if every nested sequence of nonempty closed subsets $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ of X with $diam A_n \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} A_n$ contains exactly one point.*

Proof. Let X be a complete metric space. If we choose $x_n \in A_n$, for all $n \in \mathbb{N}$, then it is obvious that $\{x_n\}$ is a sequence in X and $D(x_n, x_m) \leq diam(A_n) \to 0$ as $n \to \infty$. This confirms that $\{x_n\}$ is a cauchy sequence and by the completeness of X the sequence $\{x_n\}$ is convergent to an element of X, say x. Suppose $t \in \mathbb{N}$ is an arbitrary element, then for any $n \geq t$, we have $x_n \in A_t$. Hence, it follows from Proposition 2.10 that $x \in A_t$, since t is arbitrary member of positive integers we deduce that $x \in \bigcap_{n=1}^{\infty} A_n$.

(uniqueness) Let $x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x, y \in A_n$, for all $n \in \mathbb{N}$ and we have $D(x, y) \leq \operatorname{diam}(A_n)$, for all $n \in \mathbb{N}$. But $0 \leq D(x, y) \leq \lim_{n\to\infty} diam(A_n) = 0$. Therefore $D(x, y) = 0$ and this implies $x = y$.

Conversely ($\Leftarrow$) We must prove that (X, D) is complete. For this, take any cauchy sequence $\{x_n\}$ in X. Define $A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. By Proposition 2.14 we get $diam(A_n) \to 0$ as $n \to \infty$. Also $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$. Thus, the family $\{\overline{A_n}\}$ is a nested family of closed sets which by Lemma 2.15 and $\lim_{n\to\infty} diam(A_n) = 0$ we get $\lim_{n\to\infty} diam(\overline{A_n}) = 0$. Consequently by our hypotheses we deduce $\bigcap_{n=1}^{\infty}(\overline{A_n}) = \{x\}$, for some point in X. Then, for all $n \in \mathbb{N}$, it is obvious that $D(x_n, x) \leq diam(\overline{A_n})$ and $diam(\overline{G_n}) \to 0$ as $n \to \infty$ leads to $\{x_n\}$ converges to x. Hence (X, D) is complete and the proof is finished. □

Proposition 2.18. *If (X, D) is a superior metric space and $\{x_n\}$ converges to x, then $\lim_{n\to\infty} D(x_n, x) = 0$ and by the definition of the superior meter, there exists positive number c such that for each $y \in X$,*

$$D(x_n, y) \leq c \limsup_{n\to\infty} D(x, y) = cD(x, y).$$

Now, by taking $y = x_m$, we get $D(x_n, x_m) \leq cD(x, x_m)$, which implies

$$\begin{aligned} 0 &\leq \liminf_{n,m\to\infty} D(x_n, x_m) \leq \limsup_{n,m\to\infty} D(x_n, x_m) \\ &\leq c \limsup_{m\to\infty} D(x, x_m) = c \lim_{m\to\infty} D(x, x_m) = 0. \end{aligned}$$

This means $\lim_{n,m\to\infty} D(x_n, x_m) = 0$. Consequently $\{x_n\}$ is a cauchy sequence.

Remark 2.19. If (X, d) is a metric space, then it is clear that d is a continuous function on $X \times X$. This means that if $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$ while the following example shows that this fact may fail for a super metric space.

Example 2.20. It is easy to verify that the function $D(x, y) = \begin{cases} xy & x \neq y \\ 0 & x = y \end{cases}$, where $x, y \in X = (0, 2]$, is a super meter, because for each $y \in X$ the sequences $x_n = 1 - e^{-n}$ and $y_n = \frac{1}{2}(1 - e^{-n})$ satisfy in the inequality $D(y_n, y) \leq c \limsup_{n\to\infty} D(x_n, y)$. Also, D is not continuous, because if we take $x_n = 1, y_n = 1 - e^{-n}$ then $D(x_n = 1, 1) = 0$ and $D(y_n, 2) \to 0$. Hence,

$$\lim_{n\to\infty} D(x_n, y_n) = 0 \neq D(1, 2) = 2.$$

We note that in Example 2.20 the sequence $y_n = 1 - e^{-n}$ converges to each element of X. This means the topology induced by a super meter is not necessarily Hausdorff.

The following example shows that the first item of Definition 2.5 is necessary for the continuity of the superior meter m given in Definition 2.5.

Example 2.21. Let $X = \mathbb{Z}$ (the integer numbers) and $m(x, y) = |\max\{x, y\}|$. It is obvious that m is a superior meter. If we take $\{x_n = -n\}_{n\in\mathbb{N}}$ and $\{y_n = -n\}_{n\in\mathbb{N}}$, then $m(x_n, 0) = |\max\{-n, 0\}| = 0 \to 0$ and $m(y_n, 0) = |\max\{-n, 0\}| = 0 \to 0$, while $m(x_n, y_n) = |\max\{-n, -n\}| = n \nrightarrow m(0, 0) = 0$. This shows that m is not sequentially continuous.

Definition 2.22. If (X, D) is a superior metric space and $x \in X$, $r > 0$, then the ball centered at x and radius $r > 0$ is denoted by $B(x, r)$ and it is defined by $B(x, r) = \{y \in X : D(x, y) < r + D(x, x)\}$. We say that $A \subset X$ is open, if for each $x \in A$, there exists $r > 0$ such that $B(x, r) \subset A$.

Remark 2.23. It is obvious that $x \in B(x, r)$. Now if we define the ball centered at x and radius r by $B(x, r) = \{y \in X : D(x, y) < r\}$ then the simple example $D(x, y) = xy$ for $x, y \in X = [1, 2]$ shows that $B(x, r)$ may be empty.

Example 2.24. If $X = [1, 2]$ and $D(x, y) = xy$, for all $x \neq y$ and $D(x, x) = 0$, then

$$\begin{aligned} B(x, r) =& \{y \in X : D(x, y) < r + D(x, x)\} = \{y \in [1, 2] : xy < r + x^2\} \\ =& \left\{y \in [1, 2] : y < \frac{r}{x} + x\right\} = \left[1, \frac{r}{x} + x\right). \end{aligned}$$

It is easy to see that $B(x, r)$ is open, because, if we take $x_0 \in B(x, r) = [1, \frac{r}{x} + x)$, then $B(x_0, r') \subset B(x, r)$, for $0 < r' < x_0(\frac{r}{x} + x - x_0)$.

Example 2.25. If $X = [0, +\infty)$ and

$$D(x, y) = \begin{cases} \frac{x+y}{1+x+y} & x \neq y, x \neq 0, y \neq 0, \\ 0 & x = y, \\ \max\{\frac{x}{2}, \frac{y}{2}\} & \textit{Otherwise}. \end{cases}$$

Therefore

$$\begin{aligned} B(0, r) =& \{y \in [0, +\infty) : D(0, y) < r\} = \left\{y \in [0, +\infty) : \frac{y}{2} < r\right\} \\ =& \{y \in [0, +\infty) : y < 2r\} = [0, 2r). \end{aligned}$$

Now, if $x \neq 0$, then

$$B(x, r) = \{y \in [0, +\infty) : D(x, y) < r\} = \{0, x\} \cup \left(\frac{2r - x}{1 - r}, +\infty\right), \ r > \frac{x}{2}, \ r > 1.$$

Hence $0 \in B(x, r)$, where $r > \frac{x}{2}$, $r > 1$, and $B(0, r') = [0, 2r')$ is not a subset of $B(x, r)$. In this case, $B(x, r)$ is not open.

Proposition 2.26. *If (X, D) is a superior metric space and $D(x_n, x) \to 0$, $D(x_n, x') \to 0$, then $x = x'$.*

Proof. By the definition of superior meter, we conclude $D(x, y) \leq \limsup_{n \to \infty} D(x_n, y)$, for each $y \in X$. Hence $D(x, x') \leq \limsup_{n\to\infty} D(x_n, x') = 0$, which implies $D(x, x') = 0$. This completes the proof. $\square$

Remark 2.27. It follows from Proposition 2.26 that the function D defined in Example 2.20 is not a superior meter while it was a super meter.

Proposition 2.28. *Let (X, D) be a superior metric space (or super metric space) and τ is the topology induced by D (Definition 2.22). If D is upper semicontinuous with respect to the second variable, then $B(x, r)$ is open (That is, $B(x, r) \in \tau$) for each $x \in X$ and $r > 0$ and X is C_1 (countable base in each point of X).*

Proof. It follows by the definition of upper semicontinuity of D with respect to the second variable the level set $B(x,r) = \{y \in X : D(x,y) < r\}$ is open. Hence, $B(x,r)$ is an element of τ and by the definition of an open set via a superior meter (super meter) we deduce that the family $\{B(x,r) : x \in X, r > 0\}$ is a base of τ and moreover, for each $x \in X$, the set $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local base at x, consequently, X is C_1 (first countable). This completes the proof. $\square$

Remark 2.29. If (X,D) is a superior metric space (or super metric space) and A is a subset of X then by using Proposition 2.28 we can define a limit point (accumulation point) for a subset of X. We say that x in X is a limit point of the subset A of X when for each positive number r, the intersection $A \cap B(x,r) \setminus \{x\}$ is nonempty. The set of all limit points of A is denoted by $A^{'}$ and it is easy to verify that if $x \in A^{'}$ then $A \cap B(x,r) \setminus \{x\}$ is an infinite set which shows that a finite set does not have a limit point. Moreover, $A^{'}$ is closed because if $x \in (A^{'})^c$ then there exists $r > 0$ such that $A \cap B(x,r) \setminus \{x\} = \emptyset$. Hence for each $z \in B(x,r) \setminus \{x\}$ we have $z \notin A$ and there exists $r_z > 0$ with $B(z, r_z) \subset B(x,r)$ which implies $A \cap B(z, r_z) \setminus \{z\} = \emptyset$. Therefore, $A^{'} \cap B(x,r) = \emptyset$. This means $B(x,r) \subset (A^{'})^c$ and the proof is finished.

Definition 2.30. Let (X,D) be a superior metric space and A a nonempty subset of X. The diameter of A is denoted by $\dim(A)$ and is defined by $\dim(A) = \sup\{D(x,y) : x, y \in A\}$. The set X is called totally bounded if for each $\epsilon > 0$ there exist subsets $A_1, ..., A_n$ of X with $\dim(A_k) < \epsilon$, for all $k \in \{1, 2, ..., n\}$.

Lemma 2.31. *Let $\{x_n\}$ be a cauchy sequence in superior metric space X and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging to $x \in X$. Then $\{x_n\}$ converges to x.*

Proof. The third property of the definition of superior implies

$$\liminf_{k\to\infty} m(x_k, x) \le \limsup_{k\to\infty} m(x_k, x) \le s \limsup_{k\to\infty} m(x_{n_k}, x),\ \forall x \in X,$$

and by letting $k \to \infty$ we conclude $\lim_{n\to\infty} m(x_n, x) = 0$. $\square$

Proposition 2.32. *Let (X,D) be a superior (super) metric space. If D is upper semicontinuous with respect to the second variable and X is compact then it is sequentially compact.*

Proof. Suppose $\{x_n\}$ be a sequence in X. If the set $A = \{x_1, x_2, \ldots\}$ (the range of $\{x_n\}$) is finite, then one of the points, say x_{i_0}, satisfies $x_{i_0} = x_j$ for infinitely many $j \in \mathbb{N}$. Hence $\{x_{i_0}\}$ is a subsequence of $\{x_n\}$ which converges to the point a_{i_0} in X and in this case the proof is completed. Now if the range of $\{x_n\}$ is infinite then we claim that the set $A = \{x_1, x_2, \ldots\}$ has an accumulation, otherwise, by Remark 2.29, for each $x \in X$ there exists $r_x > 0$ such that $A \cap B(x, r_x)$ is a finite set. It is obvious $X \subset \cup_{x\in X} B(x, r_x)$ and since X is compact there exist $x_1, ..., x_n$ of X such that $X \cup_{i\in\{1,2,...,n\}} B(x_i, r_{x_i})$ which is a contradicted by the infiniteness of the sequence $\{x_n\}$. Conversely, let X is sequentially compact. We are going to prove X is compact. On the contrary, if X is not compact, then we claim that for each $\varepsilon > 0$, there exists $x_1, \ldots, x_n \in X$ such that $X \subseteq B(x_1, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon)$, because, Otherwise, there exists $\varepsilon > 0$ such that for each $x_1, \ldots, x_n \in X$, we have $X \nsubseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$. Choose $x_1 \in X$, then $X \nsubseteq B(x_1, \varepsilon)$, and there exists $x_2 \in$

$X \backslash B(x_1, \varepsilon)$. By induction we can construct $\{x_n\}$ such that $x_{n+1} \in X \backslash \bigcup_{k=1}^{n} B(x_k, \varepsilon)$ for all $n \geq 1$. Thus, we have $D(x_n, x_m) \geq \varepsilon$, for all $n, m \geq 1$, $n \neq m$. This implies there is no convergent subsequence of $\{x_n\}$, because if $D(x_{n_k}, x) \to 0$, then by the definition of the superior meter, we have

$$\varepsilon \leq \limsup D(x_{n_k}, x_{n_{k'}} = y) \leq \limsup D(x_{n'_k}, x),$$

which is a contraction.

Let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of X. We are going to prove that only finitely many O_α cover X. First, we assert that there exists $\varepsilon_0 > 0$ such that for each $x \in X$ there exists $\alpha \in I$ with $B(x, \varepsilon_0) \subset O_\alpha$. If the assertion is false, then for any $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that for any $i \in I$, $B(x_\varepsilon, \varepsilon) \nsubseteq O_i$. Hence, in particular for $\varepsilon = \frac{1}{n}$ ($n \in \mathbb{N}$) there exists $x_n \in X$ such that $B(x_n, \frac{1}{n}) \nsubseteq O_\alpha$, for all $\alpha \in I$. Since X is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}$ which converges to some point $x \in X$. On the other hand $\{O_i\}_{i \in I}$ covers X, then there exists $i_0 \in I$ such that $x \in O_{i_0}$. It follows from the openness of O_{i_0} that there exists $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subset O_{i_0}$ and $D(x_{n_k}, x) \to 0$. By the definition of superior meter and $D(x_{n_k}, x) \to 0$, we get

$$D(y, x) \leq c \limsup D(x_{n_k}, y) \leq \frac{c}{n_k}, \ \forall y \in B(x_{n_k}, \frac{1}{k}).$$

By taking k large enough, we get $\frac{c}{n_k} < \varepsilon_0$, which conculdes $m(y, x) < \varepsilon_0$, for all $y \in B(x_{n_k}, \frac{1}{n_k})$. This means $B(x_{n_k}, \frac{1}{n_k}) \subset B(x, \varepsilon_0) \subset O_\alpha$, which is a contradiction by the construction $\{x_n\}$. Consequently, there exists ε_0 such that for each $x \in X$, there exists $\alpha_n \in I$ such that $B(x, \varepsilon_0) \subset O_{\alpha_n}$. By ε_0 there exists $(x_1, \ldots, x_n) \in X$ such that $A \subseteq B(x_1, \varepsilon_0) \cup \ldots \cup B(x_n, \varepsilon_0)$. But for any $i = 1, \ldots, n$, there exists $\alpha_i \in I$ such that $B(x_i, \varepsilon_0) \subset \alpha_i$ which implies $A \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$. This completes the proof.

□

Remark 2.33. If (X, d) is a metric space then d is upper semicontinuous with respect to the second variable (even continuous). Hence, by Proposition 2.32 the compactness of X implies the sequentially compact. If we take $X = [1, 2]$ and define $D(x, y) = xy$ for all $x, y \in X$ then (X, D) is a superior metric space which is not a metric space and D is upper semicontinuous with respect to the second variable. Moreover, if $x \in X$ and $r > 0$ then

$$B(x, r) = \{y \in X : D(x, y) < r\} = \{y \in X : xy < r\} = \left\{y \in X : y < \frac{r}{x}\right\},$$

which equals to the empty set when $r < x$. Therefore, by the Definition 2.8, each subset of X is open. This means the topology induced by the superior meter D is discrete topology and in this topology just finite sets are compact and a sequence is convergent if and if after an index it becomes a constant sequence.

Theorem 2.34. *Let (X, D) be a compact superior metric space and D an upper semicontinuous with respect to the second variable. Then X is complete.*

Proof. Let $\{x_n\}$ be a cauchy sequence in X. It follows from the compactness of X and Proposition 2.32 that there exists a subsequence $\{x_{n_k}\}$ which converges to

$x \in X$. Hence, by Lemma 2.31 the sequence $\{x_n\}$ converges to x. This completes the proof. □

Theorem 2.35. *Let E be a totally bounded subset of a superior metric space (X, D). Then every sequence $\{a_n\}$ in E contains a cauchy subsequence.*

Proof. Since E is totally bounded, we can decompose E into a finite number of subsets of diameter less than $\varepsilon_1 = 1$. One of these sets call it A_1, must contain an infinite number of the terms of the sequence. Hence there exists $i_1 \in \mathbb{N}$ such that $a_{i_1} \in A_1$. Now, A_1 is totally bounded and can be decomposed into a finite number subsets of diameter less than $\varepsilon_2 = \frac{1}{2}$. Similarly, one of these sets, call it A_2, must contain an infinite number of the terms of the sequence, thus there exists $i_2 \in \mathbb{N}$ such that $i_2 > i_1$ and $a_{i_2} \in A_2$. Furthermore $A_2 \subset A_1$. We continue in this manner and obtain a nested sequence of sets $E \supset A_1 \supset A_2 \supset \dots$ with $diam(A_n) < \frac{1}{n}$ and a subsequence $\{a_{i_1}, a_{i_2}, \dots\}$ of $\{a_n\}$ with $\{a_{i_n}\}$. We claim that $\{a_{i_n}\}$ is cauchy sequence. Given $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such hat $\frac{1}{n_0} < \varepsilon$ and

$$diam(A_{n_0}\} < \frac{1}{n_0} < \varepsilon.$$

Therefore, for all $i_n, i_m > i_{n_0}$ we have $a_{i_n}, a_{i_m} \in A_{n_0}$ and

$$D(a_{i_n}, a_{i_m}) < \varepsilon,$$

which implies the sequence $\{a_{i_n}\}$ is Cauchy. □

Theorem 2.36. *Let (X, D) be a superior (super) metric space (X, D). If X is totally bounded and complete, then it is sequentially compact.*

Proof. If $\{x_n\}$ is an arbitrary sequence in X then by Theorem 2.35 there exists a Cauchy subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which, by the completeness of X, converges to an element of X. This completes the proof. □

Theorem 2.37. *Let (X, D) be a superior (super) metric space (X, D). If X is totally bounded and complete, then it is countable compact.*

Proof. If we suppose the result is false, then X has an open cover $\mathbf{C} = \{G_n\}_{n \in \mathbb{N}}$ where it does not have a finite subcover. Hence, each finite subset $\{G_k\}_{k=1,\dots,m}$ of $\mathbf{C}$ can not cover X. This means, for each positive integer m there exists $x_m \in X \setminus \cup \{G_k\}_{k=1,\dots,m}$. Thus, we obtain a sequence $\{x_m\}$ with $x_m \in X \setminus \cup \{G_k\}_{k=1,\dots,m}$ which, by Theorem 2.35, contains a Cauchy subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Now by the completeness of X, the sequence $\{x_{n_k}\}$ converges to an element of X which is a contraction and this completes the proof. □

Remark 2.38. If we take $X = (0, 2]$ and $D(x, y) = xy$ for all $x, y \in X$ then (X, D) is a super metric space which is not totally bounded because $x_n = 2 - \frac{1}{n}$ does not have a cauchy sequence. Moreover, X is not sequentially compact because the sequence $x_n = 2 - \frac{1}{n}$ does not have a convergence subsequence. Also if we replace $X = (0, 2]$ by $X = [0, 2]$, then each sequence $\{x_n\}$ of $X = [0, 2]$ converges to 0 which implies it is sequentially compact. Thus, the sequentially compactness cannot imply the totally boundedness in general. Further, for $x \in (0, 1]$ we have

$$B(x, r) = \{y \in X : D(x, y) < r\} = \{y \in [0, 2] : D(x, y) < r\} = \left[0, \frac{r}{x}\right) \cup \{x\}$$

and $B(0,r) = X$. Hence, the family $\{B(2, 4 - \frac{1}{n}) : n \in \mathbb{N}\}$ is an open cover of X which can not be covered by a finite subcover of it. This means X is not compact. This example also denotes that the converse of Theorem 2.32 is not true.

Note. Remark 2.38 shows that it is not possible to establish a characterization of the compactness based on the completeness and totally boundedness in the setting of super (superior) metric spaces while we have the following characterization of a compact set in the setting of metric spaces:

$$\text{compact} \Longleftrightarrow \text{sequentially compact} \Longleftrightarrow \text{countably compact}.$$

The next result extends the Banach contraction principle [2] from complete metric spaces to the complete superior metric spaces.

Theorem 2.39. *Let (X, D) be a complete superior metric space and $T : X \to X$ a mapping with the following property*

$$D(Tx, Ty) \leq kD(x, y), \tag{2.1}$$

where $k \in [0, 1)$. Then T has a unique fixed point and each Picard iteration ($T^n(x_0)$, $x_0 \in X$) converges to the fixed point.

Proof. **Uniqueness.** If x_1, x_2 are two distinct fixed points of T, then $Tx_1 = x_1$, $Tx_2 = x_2$. By (2.1), we get $D(Tx_1, Tx_2) = D(x_1, x_2) \leq kD(x_1, x_2)$, which is contradicted by $k < 1$ and $x_1 \neq x_2$. This completes the proof of the uniqueness.

Existence of fixed point. Choose $x_0 \in X$ and define the sequence $x_{n+1} = Tx_n$, for $n \geq 0$. If for some $n \geq 0$, $x_{n+1} = x_n$, then x_n is a fixed of T and the proof is completed. Thus, we suppose that $x_{n+1} \neq x_n$, for each $n \geq 0$. By (2.1), we have $D(x_{n+1}, x_n) \leq k^n D(x_1, x_0)$, for each $n \geq 1$, and by tending n to infinity, $D(x_{n+1}, x_n) \to 0$. Thus

$$\limsup_{n\to\infty} D(x_n, x_{n+2}) \leq c \limsup_{n\to\infty} D(x_{n+1}, x_{n+1}) = 0,$$

$\limsup_{n\to\infty} D(x_n, x_{n+2}) = 0$, which the properties of D imply that

$$\limsup_{n\to\infty} D(x_n, x_{n+3}) \leq c \limsup_{n\to\infty} D(x_{n+2}, x_{n+2}) = 0.$$

Hence,

$$\limsup_{n\to\infty} D(x_n, x_{n+3}) = 0 = \limsup_{n\to\infty} D(x_n, x_{n+3}).$$

By continuing this process, we get

$$\limsup_{n\to\infty} D(x_n, x_{n+m}) = 0, \forall m \in \mathbb{N}.$$

This means $\{x_n\}$ is a cauchy sequence and by the completeness of X, there exists $x \in X$ such that $D(x_n, x) \to 0$ which the inequality given by 2.1 implies $D(x_{n+1} = Tx_n, Tx) \to 0$. Consequently, the sequence $\{x_n\}$ converges to x and $T(x)$ which by Proposition 2.26 (the uniqueness of limit) we deduce that $Tx = x$, and the proof is completed. $\square$

The following examples improves the Example 2.7 in [11] for supporting Theorem 2.39 in the setting of super metric spaces and further it shows that Theorem 2.39 is a real improvement of Banach contraction principle [2].

Example 2.40. Define $T : X = [1,3] \to X$ by $T(x) = \begin{cases} 2 & x \neq 3 \\ 1 & x = 3 \end{cases}$. Then, it is easy to check that m is a super meter, because there are no sequences $\{x_n\}$ and $\{y_n\}$ of X such that $m(x_n, y_n) \to 0$. Moreover, $m(Tx, Ty) = m(2, 1) = 2 \leq \alpha \cdot 3 \cdot x$, which is true when $\frac{2}{3} \leq \alpha$. Hence T satisfies all the conditions of the Theorem 2.39, hence T has a unique fixed point which equals to $x = 2$, while if we take $m(x, y) = |x-y|$ (the Euclidian meter on X), then m is not a Banach contraction (that is the inequality given by the relation 2.1) because it is enough we take $x = 3 - \frac{1}{n}$, $y = 3$, then

$$|Tx - Ty| = |2 - 1| \leq \alpha \cdot \left|3 + \frac{1}{n} - 3\right| = \frac{\alpha}{n},$$

which is impossible for $\alpha \in [0, 1)$ and large enough n.

The following theorem establishes an existence result of a fixed point for the self mapping T defined on a super metric space. Moreover, it improves Theorem 2.8 of [11] by relaxing the continuity of T. Further, it can be viewed as a generalization of the corresponding result given in [12] from K-metric spaces to superior metric spaces.

Theorem 2.41. *Let (X, D) be a complete superior metric space and $T : X \to X$ a closed mapping (its range is closed). If there exists function $\varphi : X \to [0, +\infty)$ such that*

$$D(x, Tx) \leq \varphi(x) - \varphi(Tx), \ \forall x \in X, \tag{2.2}$$

and φ is lower semicontinuous, then the fixed points set of T is nonempty.

Proof. Choose $x_0 \in X$ and define $x_{n+1} = Tx_n$, for all $n \geq 0$. If, for some $n \geq 0$, $x_{n+1} = x_n$, then x_n is a fixed point of T and the proof is finished. Hence, we suppose $x_{n+1} \neq x_n$, for each $n \geq 0$. It follows from (2.2) that $D(x_n, x_{n+1} = Tx_n) \leq \varphi(x_n) - \varphi(x_{n+1} = Tx_n)$ and so

$$\sum_{n=0}^{m} D(x_n, x_{n+1}) \leq \sum_{n=0}^{m} (\varphi(x_n) - \varphi(x_{n+1})) = \varphi(x_1) - \varphi(x_{m+1}) \leq \varphi(x_1).$$

This means the sequence of partial sums of the series $\sum_{n=0}^{\infty} D(x_n, x_{n+1})$ is bounded. Hence, $\sum_{n=0}^{\infty} D(x_n, x_{n+1})$ is convergent (note $D(x_n, x_{n+1}) > 0$, for each $n \geq 0$). Thus $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$. It is easy to verify that $\{x_n\}$ is a cauchy sequence (the proof is similar to the proof in Theorem 2.39). Therefore, since X is complete, there exist $x \in X$ such that $\lim_{n\to\infty} D(x_n, x) = 0$. Now, since $(x_n, x_{n+1} = Tx_n) \in G_r(T)$, $(x_n, x_{n+1}) \to (x, x)$ and $G_r(T)$ is closed, we get $(x, x) \in G_r(T)$, which implies $x = Tx$ and the proof is completed. □

The following example shows that the hypotheses of Theorem 2.41 improve the corresponding results given in complete metric spaces, in other words if in Theorem 2.41 we replace the complete superior metric space by a complete metric space then the hypotheses of Theorem 2.41 are mild than the hypotheses given them (for instance, see [5–7, 12]).

Example 2.42. Let $X = [0, 1]$ and $D : X \times X \to \mathbb{R}$ defined by $D(x, y) = |x - y|$, for all $x, y \in X$. Define $T : X \to X$, by $T(x) = \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$, and $\varphi : X \to [0, +\infty)$, by $\varphi(x) = T(x)$, for all $x \in [0, 1]$. It is abvious $D(x, Tx) = |x - \frac{1}{x}| = \frac{1}{x} - x \leq \varphi(x) - \varphi(Tx) = \frac{1}{x} - x$, for all $0 < x \leq 1$, and $D(0, T0) = 0 \leq \varphi(0) - \varphi(T0) = 0$. Moreover, X is a complete super metric space, and φ is lower semicontinuous. T is not continuous on X while T is closed. Furthere, the fixed point set of T equals to $\{0, 1\}$. Finally, T is not a Banach contraction and for $n \geq 2$, $T^n(x) = x$, for each $x \in [0, 1]$ which shows that T is not pointwise contraction in the sense of [6].

3. Conclusions

The definition of superior meter is introduced. The topology induced by a superior meter is discussed and it is shown that under suitable condition the neighborhood produced by a superior is an open set. Also, it is investigated, by providing examples, that despite metric spaces, compactness is not necessarily equivalent to sequential compactness in a superior metric spaces. Of course, it is proved that if a superior meter is upper semicontinuous with respect to the second variable, then the compactness and sequentially compactness are equivalent. Moreover, the relationship between the compactness and totally boundedness and completeness are studied. The Cantor′s intersection theorem is extended from complete metric spaces to complete superior metric spaces. The Banach′ contraction principle and Caristi′ fixed-point theorem, in the setting of complete superior(super) metric spaces, with mild assumptions are presented.

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