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MULTIPLICITY OF SOLUTIONS FOR DISCRETE POTENTIAL BOUNDARY $p(k)$-LAPLACE KIRCHHOFF TYPE EQUATIONS

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Abstract. In this article, we prove the existence and multiplicity of solutions for discrete $p(k)$-Laplace Kirchhoff type equations with a potential boundary condition. The technical approach for the proof of the existence of solutions is based on variational methods and critical point theory for convex sets.

1. Introduction

In this article, we consider the following nonlinear discrete boundary value problem.

$$\begin{cases} -M(\rho[u])\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) \\ \qquad\qquad +f(k,u(k))=0,\ k\in\mathbb{N}\,[1,T]\,, \\ (a(0,|\triangle u(0)|)\triangle u(0),-a(T,|\triangle u(T)|)\triangle u(T))\in\partial_j\,(u(0),u(T+1))\,, \end{cases} \tag{1.1}$$

where $T\geq 2$ is a positive integer, $\triangle u(k)=u(k+1)-u(k)$ is the forward difference operator, for $a,b\in\mathbb{N}$ with $a<b$, $\mathbb{N}\,[a,b]=\{a,a+1,\ldots,b-1,b\}$, $f:\mathbb{N}\,[1,T]\times\mathbb{R}\to\mathbb{R}$ is Carathéodory and $a(k,\cdot),M:[0,\infty)\to[0,\infty)$ are two continuous functions for all $k\in\mathbb{N}\,[1,T]$, $t\in[0,\infty)$ with the function $t\to M(t)$ nondecreasing, $A_0:\mathbb{N}\,[1,T]\times[0,\infty)\to[0,\infty)$ which satisfies $A_0(k,t)=\int_0^t a(k,\xi)\xi\,d\xi$ and the functional $\rho:\mathbb{R}\to[0,\infty)$ defined by

$$\rho[u]=\sum_{k=1}^{T+1}A_0(k-1,|\triangle u(k-1)|).$$

Moreover, $j:\mathbb{R}\times\mathbb{R}\to(-\infty,\infty)$ is convex, proper (i.e., $D(j):=\{z\in\mathbb{R}\times\mathbb{R}:j(z)<\infty\}\neq\emptyset$), lower semicontinuous (in short, l.s.c.) and ∂_j denotes the subdifferential of j. Recall that for $z\in\mathbb{R}\times\mathbb{R}$, the set $\partial_j(z)$ is defined by

$$\partial_j(z)=\{\zeta\in\mathbb{R}\times\mathbb{R}:j(\sigma)-j(z)\geq\langle\zeta,\sigma-z\rangle,\forall\sigma\in\mathbb{R}\times\mathbb{R}\}, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ stands for the usual inner product in $\mathbb{R}\times\mathbb{R}$.

The nonhomogeneous difference operator

$$\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)), \tag{1.3}$$

which appears on the left-hand side of problem (1.1) is more general than the one that appears in the usual operators with variable exponent. Indeed, if we take

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$a(k,s) = s^{p(k)-2}$, then we obtain the standard discrete $p(\cdot)$-Laplacian operator, that is,

$$\triangle_{p(k-1)}u(k-1) := \triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right).$$

Difference equations represent the discrete counterpart of PDEs and are usually studied with numerical analysis. We refer the reader to [38] for a comprehensive qualitative analysis of nonlinear PDEs with variable exponent using variational and topological methods. The operator (1.3) can be seen as the discrete counterpart of the following anisotropic operator

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,\left|\frac{\partial}{\partial x_i}u\right|\right)\frac{\partial}{\partial x_i}u.$$

The above operator was introduced by I.H. Kim and Y.H. Kim [20] to study a homogeneous Dirichlet boundary problem.

The differential equations with variable exponent have been intensively studied in the last few decades since they can model various phenomena arising from the study of elastic mechanics [48], electrorheological fluids [37,40] and image restoration [13].

The presence of the non-local term $\rho[u]$ is an important feature of this article. Kirchhoff in 1876 (see [21]) suggested a model defined by the following equation.

$$\rho\frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{Ea}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.4}$$

where $\rho > 0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string.

Equation (1.4) takes into account the change of the tension on the string which is caused by the change of its length during the vibration. After that, several physicists also considered such equations for their research in the theory of nonlinear vibrations theoretically or experimentally [9,10,33,35], On the other hand, Kirchhoff's equation received much attention only after Lions in 1978 (see [25]) established an abstract framework for the problem related to the stationary analog of the equation of Kirchhoff type. Some important and interesting results can be found in [2,11]. In recent years, discrete Kirchhoff-type problems have been widely investigated. We refer the readers to papers [12,18,22,34,36,43–45] and references therein. For example, in [45] Yucedag studied the following problem

$$\begin{cases} -M\left(\frac{|\triangle u(k-1)|^{p(k-1)}}{p(k-1)}\right)\triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right) \\ \quad = f(k,u(k)) \quad \text{for } k\in\mathbb{N}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.5}$$

by using the variational approach and the Mountain Pass theorem to obtain the existence of at least one nontrivial solution. In [22], Koné et *al*, proved, by using the minimization method, the existence of a weak solution for the following problem

$$\begin{cases} -M\left(A(k-1,\triangle u(k-1)\right)\triangle(a(k-1,\triangle u(k-1)) \\ \quad = f(k) \quad \text{for } k\in\mathbb{N}[1,T], \\ u(0) = \triangle u(T) = 0. \end{cases} \tag{1.6}$$

Discrete boundary value problems have been intensively studied by many authors in the literature. For papers involving the discrete $p(k)$-Laplacian operator, we refer the readers to the following works [3,5,6,14,15,23,26,30,31]. In the case where $p(k)$ is a constant (also called the discrete p-Laplacian operator), we refer the readers to [1,7,8,17,19,42] and the references therein. The discrete $p(k)$-Laplacian operator has more complicated nonlinearities than the discrete p-Laplacian operator; for example, it is not homogeneous.

In [19], by using critical point theory, the authors considered the existence of ground state periodic solutions of the following difference equations with discrete p-Laplacian.

$$\begin{cases} -\triangle_p x(k-1) = f(k, x(k)) \quad \text{for } k \in \mathbb{N}\,[1,T]\,, \\ x(0) - x(T+1) = 0 = \triangle x(0) - \triangle x(T). \end{cases} \tag{1.7}$$

For discrete problems with $p(k)$-Laplacian operator, in [4] the authors studied the following problem.

$$\begin{cases} -\triangle_{p(k-1)} x(k-1) = f(k, x(k)), \text{ for } k \in \mathbb{N}\,[1,T]\,, \\ \left(h_{p(0)}(\triangle x(0)), -h_{p(T)}(\triangle x(T))\right) \in \partial_j\left(x(0), x(T+1)\right). \end{cases} \tag{1.8}$$

By using the variational approach relying on Szulkin's critical point theory, they proved the existence of ground state as well as mountain pass type solutions for problem (1.8).

For discrete problems with $p(k)$-Laplacian operator, in [24] the authors dealt with the following problem which is a generalization of problem (1.8),

$$\begin{cases} -\triangle\left(a(k-1, \triangle u(k-1))\right) = f(k, u(k)) \quad \text{for } k \in \mathbb{N}\,[1,T]\,, \\ \left(a(0, \triangle u(0)), -a(T, \triangle u(T))\right) \in \partial_j\left(u(0), u(T+1)\right). \end{cases} \tag{1.9}$$

By using the variational technique relying on Szulkin's critical point theory, the authors showed the existence of solutions by ground state and mountain pass methods for the problem (1.9).

In this article, we use the variational methods that rely on Szulkin's critical point theory to investigate the existence of solutions for the problem (1.1) by ground state and mountain pass techniques.

The paper is organized as follows. In Section 2, the variational framework associated with problem (1.1) is established. Some definitions and lemmas essential to prove our main results are also stated. In Section 3, we use the direct variational approach to obtain the existence of at least one nontrivial weak solution for problem (1.1). Finally, in Section 4, we apply the critical point theory to obtain the existence of at least two nontrivial solutions for problem (1.1) with a disturbance term.

2. Preliminaries

In this section, we first establish the variational framework associated with problem (1.1). We connect solutions to (1.1) with critical point theory developed by Szulkin [41].

Throughout this article, we will use the notations.

$$p^+ = \max_{k\in\mathbb{N}[0,T]} p(k),\ \ p^- = \min_{k\in\mathbb{N}[0,T]} p(k),\ \ \overline{p} = \max_{k\in\mathbb{N}[1,T]} p(k),\ \ \underline{p} = \min_{k\in\mathbb{N}[1,T]} p(k).$$

We introduce the space

$$X := \{u : \mathbb{N}\,[0, T+1] \to \mathbb{R}\}$$

equipped with the norm

$$\|u\|_\delta = \left(\sum_{k=1}^{T+1}\left(|\triangle u(k-1)|^{p^-} + \delta |u(k)|^{p^-}\right)\right)^{1/p^-},$$

for some positive constant δ.

On the space X we also consider the norms

$$\|u\| = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-}\right)^{1/p^-},$$

$$\|u\|_{\delta,p^+} = \left(\sum_{k=1}^{T+1}\left(|\triangle u(k-1)|^{p^+} + \delta |u(k)|^{p^+}\right)\right)^{1/p^+}$$

and the Luxemburg norm

$$\|u\|_{\delta,p(\cdot)} = \inf\left\{\lambda > 0 : \sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}\left|\frac{\triangle u(k-1)}{\lambda}\right|^{p(k-1)} + \frac{\delta}{p(k)}\left|\frac{u(k)}{\lambda}\right|^{p(k)}\right) \le 1\right\}$$

for some positive constant δ.

On the other hand, the following inequalities hold true.

$$L\|u\|_{\delta,p^+} \le \|u\|_\delta \le 2^{\frac{p^+-p^-}{p^+p^-}} L\|u\|_{\delta,p^+}, \tag{2.1}$$

where $L := (\max\{T+1, \delta\})^{\frac{p^+-p^-}{p^+p^-}}$. Indeed, by using discrete weighted Hölder inequality (see [32]), one obtains

$$\begin{aligned}
&\sum_{k=1}^{T+1} \delta\, |u(k)|^{p^-}\\
&\le \left(\sum_{k=1}^{T+1} \delta \{1\}^{\frac{p^+}{p^+-p^-}}\right)^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T+1} \delta \left(|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}}\\
&\le \delta^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T+1} \delta |u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}.
\end{aligned}$$

Similarly, one has

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \le (T+1)^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+}\right)^{\frac{p^-}{p^+}}.$$

The above inequalities combined with $\frac{p^-}{p^+} \le 1$, imply that

$$\begin{aligned}
\|u\|_\delta^{p^-} &\le (\max\{T+1,\delta\})^{\frac{p^+-p^-}{p^+}} \\
&\times \left(\left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} \right)^{\frac{p^-}{p^+}} + \left(\sum_{k=1}^{T+1} \delta |u(k)|^{p^+} \right)^{\frac{p^-}{p^+}} \right) \\
&\le 2^{1-\frac{p^-}{p^+}} (\max\{T+1,\delta\})^{\frac{p^+-p^-}{p^+}} \\
&\times \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1}^{T+1} \delta |u(k)|^{p^+} \right)^{\frac{p^-}{p^+}} = 2^{\frac{p^+-p^-}{p^+}} L^{p^-} \|u\|_{\delta,p^+}^{p^-}.
\end{aligned}$$

Therefore, $\|u\|_\delta \le 2^{\frac{p^+-p^-}{p^+p^-}} L \|u\|_{\delta,p^+}$.

Moreover, we get from the fact that $\frac{p^+}{p^-} \ge 1$, the following.

$$\begin{aligned}
\|u\|_{\delta,p^+}^{p^+} &\le (\max\{T+1,\delta\})^{\frac{p^--p^+}{p^-}} \\
&\times \left(\left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \right)^{\frac{p^+}{p^-}} + \left(\sum_{k=1}^{T+1} \delta |u(k)|^{p^-} \right)^{\frac{p^+}{p^-}} \right) \\
&\le (\max\{T+1,\delta\})^{\frac{p^--p^+}{p^-}} \\
&\times \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} + \sum_{k=1}^{T+1} \delta |u(k)|^{p^-} \right)^{\frac{p^+}{p^-}} = L^{-p^+} \|u\|_\delta^{p^+}.
\end{aligned}$$

Therefore, $L\|u\|_{\delta,p^+} \le \|u\|_\delta$. Then, relation (2.1) is satisfied.

We also consider another norm in X, that is,

$$\|u\|_\infty := \max_{k \in \mathbb{N}[0,T+1]} |u(k)|, \text{ for all } u \in X.$$

For every $u \in X$, there exists $\tau \in \mathbb{N}[1,T]$ such that

$$|u(\tau)| \le \sum_{k=1}^{T+1} |\triangle u(k-1)| + \sum_{k=1}^{T+1} |u(k)|$$

and by using the discrete Hölder inequality, one has

$$\begin{aligned}
\|u\|_\infty &\le (T+1)^{\frac{p^--1}{p^-}} \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \right)^{\frac{1}{p^-}} \\
&+ (T+1)^{\frac{p^--1}{p^-}} \left(\sum_{k=1}^{T+1} |u(k)|^{p^-} \right)^{\frac{1}{p^-}}
\end{aligned}$$

$$\begin{aligned}
(2.2) \qquad &\leq (T+1)^{\frac{p^- -1}{p^-}} \left(\left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \right)^{\frac{1}{p^-}} + \left(\sum_{k=1}^{T+1} \delta \, |u(k)|^{p^-} \right)^{\frac{1}{p^-}} \right) \\
&\leq 2^{1-\frac{1}{p^-}} (T+1)^{\frac{p^- -1}{p^-}} \left(\sum_{k=1}^{T+1} \left(|\triangle u(k-1)|^{p^-} + \delta |u(k)|^{p^-} \right) \right)^{\frac{1}{p^-}} \\
&= (2T+2)^{\frac{p^- -1}{p^-}} \|u\|_\delta .
\end{aligned}$$

Since X is of finite dimension, therefore there exist constants $0 < L_1 < L_2$ such that

$$L_1 \|u\|_{\delta,p(\cdot)} \leq \|u\|_\delta \leq L_2 \|u\|_{\delta,p(\cdot)}. \tag{2.3}$$

For the variable exponent, we assume that.

$$p(\cdot) : \mathbb{N}\,[0,T] \to (1,\infty)\,. \tag{2.4}$$

We also assume that a and M satisfy the following assumptions.

$(H1)$ There exist $a_1 : \mathbb{N}\,[0,T] \to [0,\infty)$ and a constant $a_2 > 0$ such that

$$|a(k,|\xi|)\xi| \leq a_1(k) + a_2 |\xi|^{p(k)-1},$$

for all $k \in \mathbb{N}\,[0,T]$ and $\xi \in \mathbb{R}$.

$(H2)$ For all $k \in \mathbb{N}\,[0,T]$ and $\xi > 0$,

$$0 \leq a(k,|\xi|)\xi^2 \leq p^+ \int_0^{|\xi|} a(k,s)s\,ds.$$

$(H3)$ There exists a positive constant c such that

$$\min\left\{ a(k,|\xi|),\ |\xi| \frac{\partial a}{\partial \xi}(k,|\xi|) + a(k,|\xi|) \right\} \geq c|\xi|^{p(k)-2},$$

for all $k \in \mathbb{N}\,[0,T]$ and $\xi \in \mathbb{R}$.

$(H4)$ $M : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and there exist two positive constant m_0 and m_1 such that

$$m_0 \leq M(t) \leq m_1, \ \text{ for } \ t > 0.$$

Now, let $\mathcal{A}_{p(\cdot)} : X \to \mathbb{R}$ be given by

$$\mathcal{A}_{p(\cdot)}(u) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \, |\triangle u(k-1)|^{p(k-1)}\,.$$

If $u \in X$, as $p^+ < \infty$, then the following properties hold.

$\|u\|_{\delta,p(\cdot)} > 1$ imply that

$$\|u\|_{\delta,p(\cdot)}^{p^-} \leq \mathcal{A}_{p(\cdot)}(u) + \delta \sum_{k=1}^{T+1} \frac{1}{p(k)} \, |u(k)|^{p(k)} \leq \|u\|_{\delta,p(\cdot)}^{p^+}, \tag{2.5}$$

$\|u\|_{\delta,p(\cdot)} < 1$ imply that

$$\|u\|_{\delta,p(\cdot)}^{p^+} \leq \mathcal{A}_{p(\cdot)}(u) + \delta \sum_{k=1}^{T+1} \frac{1}{p(k)} \, |u(k)|^{p(k)} \leq \|u\|_{\delta,p(\cdot)}^{p^-}. \tag{2.6}$$

Next, we define the functional $A_{p(\cdot)} : X \to \mathbb{R}$ by

$$A_{p(\cdot)}(u) = \widehat{M}(\rho[u]), \quad \text{for all } u \in X, \tag{2.7}$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$.

Using the functional j, we introduce the functional $J : X \to (-\infty, \infty]$ defined by

$$J(u) = j\left(u(0), u(T+1)\right) \quad \text{for all } u \in X. \tag{2.8}$$

Note that, as j is proper, convex and l.s.c, the same properties hold for J.

We denote

$$A = A_{p(\cdot)} + J. \tag{2.9}$$

Let us also define

$$B(u) = \sum_{k=1}^{T} F(k, u(k)) \ \text{ for all } u \in X, \tag{2.10}$$

where $F(k, s) = \int_0^s f(k, t)\, dt$ for every $(k, s) \in \mathbb{N}[1, T] \times \mathbb{R}$.

The energy functional corresponding to problem (1.1) is defined as $\mathcal{E} : X \to (-\infty, \infty]$, with

$$\mathcal{E} = A + B, \tag{2.11}$$

where A is introduced in (2.9) and B defined by (2.10),

We define a critical point of $\mathcal{E}$ as a point $u \in X$ such that

$$\begin{aligned} &M(\rho[u]) \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) \\ &+ \sum_{k=1}^{T} f(k, u(k))v(k) + j'\left((u(0), u(T+1)) ; (v(0), v(T+1))\right) \geq 0, \end{aligned} \tag{2.12}$$

which in turn is a classical solution of problem (1.1) for any $v \in X$. Since we work in a finite-dimensional space, we see that any classical solution of problem (1.1) is, in fact, a strong, i.e. a weak solution.

Let us also define the quotient associated with problem (1.1) as follows.

$$\begin{aligned} \nu_1 \ = \ \inf\Bigg\{ &\left(\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\triangle u(k-1)|^{p(k-1)}\right) \Big/ \left(\sum_{k=1}^{T+1} \frac{1}{p(k)} |u(k)|^{p(k)}\right) : \\ &u \in X \setminus \{0\} \ \text{ and } \ (u(0), u(T+1)) \in D(j)\Bigg\}. \end{aligned} \tag{2.13}$$

Proposition 2.1 ([29])**.** $\nu_1 > 0$.

Next, let $\mathcal{C} : X \to \mathbb{R} \cup \{\infty\}$ be a functional satisfying the following hypothesis.

$(H5)$ $\mathcal{C} = D + E$, where $E : X \to \mathbb{R}$ is a functional of class C^1 on X with $D : X \to \mathbb{R} \cup \{\infty\}$ convex, proper and lower semicontinuous.

Definition 2.2. A point u is said to be a critical point of the functional $\mathcal{C}$ if $u \in X$ such that $D(u) < \infty$ and

$$\left\langle E'(u), v - u \right\rangle + D(v) - D(u) \geq 0, \quad \text{for all } v \in X.$$

Theorem 2.3 ([27]). *If the functional $\mathcal{E} : X \to \mathbb{R}$ is weakly lower semi-continuous and coercive, i.e. $\lim_{\|x\|\to\infty} \mathcal{E}(x) = \infty$, then there exists $x_0 \in X$ such that $\inf_{x\in X} \mathcal{E}(x) = \mathcal{E}(x_0)$. Moreover, if $\mathcal{E}$ has bounded linear Gâteaux derivative on X, then x_0 is also a critical point of $\mathcal{E}$, i.e. $\mathcal{E}'(x_0) = 0$.*

Theorem 2.4 ([28], Theorem 4.10). *Let $\mathcal{E}_{q(\cdot)} \in C^1(X, \mathbb{R})$ and $\mathcal{E}_{q(\cdot)}$ satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighbourhood U of u_0, such that $u_1 \in X \setminus U$,*

$$\max\{\mathcal{E}_{q(\cdot)}(u_0), \mathcal{E}_{q(\cdot)}(u_1)\} < \inf_{u\in\partial U} \mathcal{E}_{q(\cdot)}(u),$$

then there exists a critical point u of $\mathcal{E}_{q(\cdot)}$, i.e. $\mathcal{E}'_{q(\cdot)}(u) = 0$ with

$$\max\{\mathcal{E}_{q(\cdot)}(u_0), \mathcal{E}_{q(\cdot)}(u_1)\} < \mathcal{E}_{q(\cdot)}(u).$$

Theorem 2.5 ([47], Theorem 38). *For the functional $F : M \subseteq X \to (-\infty, \infty)$ with $M \neq \varnothing$, $\min_{u\in M} F(u) = \alpha$ has a solution if the following conditions hold.*

- (i) *X is a real reflexive Banach space,*
- (ii) *M is bounded and weak sequentially closed,*
- (iii) *F is weak sequentially lower semi-continuous on M, i.e., for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, $F(u) \leq \liminf_{n\to\infty} F(u_n)$.*

Proposition 2.6 ([41], Proposition 1.1). *If $\mathcal{C}$ satisfies the condition $(H5)$, then each local minimum of $\mathcal{C}$ is necessarily a critical point of $\mathcal{C}$.*

Lemma 2.7.

- (i) *The functionals $A_{p(\cdot)}$ and B are well-defined on X.*
- (ii) *The functionals $A_{p(\cdot)}$ and B are of class $C^1(X, \mathbb{R})$ and*

$$\left\langle A'_{p(\cdot)}(u), v\right\rangle = M(\rho[u]) \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1), \tag{2.14}$$

$$\left\langle B'(u), v\right\rangle = \sum_{k=1}^{T} f(k, u(k))v(k), \tag{2.15}$$

for all $u, v \in X$.

Proof. (i) Since F is continuous, then

$$|B(u)| = \left|\sum_{k=1}^{T} F(k, u(k))\right| < \infty.$$

By $(H1)$ and $(H4)$, we get

$$\begin{aligned}
\left|A_{p(\cdot)}(u)\right| &= \left|\widehat{M}(\rho[u])\right| \\
&\leq m_1 \left|\int_0^{\rho[u]} d\xi\right| \leq m_1 |\rho[u]| \\
&\leq m_1 \left[a_1^+ \sum_{k=1}^{T+1} |\triangle u(k-1)| + \frac{a_2}{p^-} \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)}\right] < \infty,
\end{aligned}$$

where $a_1^+ = \max_{k\in\mathbb{N}[0,T]} a_1(k)$.
Then, $A_{p(\cdot)}$ and B are well-defined on X.

(ii) Clearly $A_{p(\cdot)}$ and B are in $C^1(X,\mathbb{R})$. In what follows, we prove (2.14) and (2.15).

Choose $u, v \in X$, one has

$$\left\langle A'_{p(\cdot)}(u), v\right\rangle = \lim_{\tau\to 0^+} \frac{A_{p(\cdot)}(u+\tau v) - A_{p(\cdot)}(u)}{\tau}$$

and

$$\left\langle B'(u), v\right\rangle = \lim_{\tau\to 0^+} \frac{B(u+\tau v) - B(u)}{\tau}.$$

Let us denote

$$|\mathrm{g}_\tau| = \left|\frac{\widehat{M}(\rho[u+\tau v]) - \widehat{M}(\rho[u])}{\tau}\right|;$$

we can find $\tau_k \in \mathbb{R}$ with $0 < |\tau_k| < |\tau| < 1$ such that

$$\begin{aligned}
|\mathrm{g}_\tau| &\le \left|M(\rho[u+\tau_k v])\right| \sum_{k=1}^{T+1} |a(k-1, |\triangle u(k-1) + \tau_k \triangle v(k-1)|) \\
&\times(\triangle u(k-1) + \tau_k \triangle v(k-1))\triangle v(k-1)| \\
&\le m_1 \sum_{k=1}^{T+1} \left|a_1(k-1) + a_2|\triangle u(k-1) + \tau_k\triangle v(k-1)|^{p(k-1)-1}\right| |\triangle v(k-1)| \\
&\le m_1 \sum_{k=1}^{T+1} \left|a_1(k-1) + a_2(|\triangle u(k-1)| + |\triangle v(k-1)|)^{p(k-1)-1}\right| |\triangle v(k-1)|.
\end{aligned}$$

By the discrete Hölder inequality (see [16]), on déduit que

$$\begin{aligned}
\sum_{k=1}^{T+1} |\mathrm{g}_\tau| &\le Tm_1 \sum_{k=1}^{T+1} |a_1(k-1)||\triangle v(k-1)| \\
&+a_2 Tm_1 \sum_{k=1}^{T+1} \left|(|\triangle u(k-1)| + |\triangle v(k-1)|)^{p(k-1)-1} |\triangle v(k-1)|\right| \\
&\le C\|a_1\|_{p'(\cdot)} \|\triangle v\|_{p(\cdot)} + a_2 C\|(|\triangle u(k-1)| + |\triangle v(k-1)|)^{p(\cdot)-1}\|_{p'(\cdot)} \\
&\times\|\triangle v\|_{p(\cdot)} < \infty.
\end{aligned}$$

So, it follows that

$$\begin{cases}
\lim_{\tau\to 0^+} \frac{A_{p(\cdot)}(u+\tau v) - A_{p(\cdot)}(u)}{\tau} \\
= \lim_{\tau\to 0^+} \frac{\widehat{M}(\rho[u+\tau v]) - \widehat{M}(\rho[u])}{\tau} \\
= \lim_{\tau_k\to 0^+} M(\rho[u+\tau_k v]) \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1) \\
+\tau_k\triangle v(k-1)|)(\triangle u(k-1) + \tau_k \triangle v(k-1))\triangle v(k-1) \\
= M(\rho[u]) \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1)
\end{cases}$$

and

$$\begin{aligned}\lim_{\tau\to 0^+}\frac{B(u+\tau v)-B(u)}{\tau} &= \lim_{\tau\to 0^+}\sum_{k=1}^{T}\frac{F(k,u(k)+\tau v(k))-F(k,u(k))}{\tau}\\ &= \sum_{k=1}^{T}\lim_{\tau\to 0^+}\frac{F(k,u(k)+\tau v(k))-F(k,u(k))}{\tau}\\ &= \sum_{k=1}^{T} f(k,u(k))v(k).\end{aligned}$$

Thus, we obtain (2.14) and (2.15). □

Now, we recall some auxiliary results to be used throughout the paper.

Lemma 2.8.
(i) *Let $u\in X$ and $\|u\|_\delta>1$. Then,*

$$\sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}|\triangle u(k-1)|^{p(k-1)}+\frac{\delta}{p(k)}|u(k)|^{p(k)}\right)\geq \frac{\|u\|_\delta^{p^-}}{p^+}-\frac{(1+\delta)(T+1)}{p^+}.$$

(ii) *Let $u\in X$ and $\|u\|_\delta<1$. Then,*

$$\sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}|\triangle u(k-1)|^{p(k-1)}+\frac{\delta}{p(k)}|u(k)|^{p(k)}\right)\geq \frac{2^{\frac{p^--p^+}{p^-}}}{p^+L^{p^+}}\|u\|_\delta^{p^+}.$$

Proof. Let $u\in X$ be fixed. By a similar argument as in [15], we define

$$\alpha_k:=\begin{cases}p^+ & \text{if } |\triangle u(k)|\leq 1\\ p^- & \text{if } |\triangle u(k)|>1\end{cases}\qquad\text{and}\quad \beta_k:=\begin{cases}p^+ & \text{if } |u(k)|\leq 1\\ p^- & \text{if } |u(k)|>1,\end{cases}$$

for each $k\in\mathbb{N}[1,T]$.
(i) For $u\in X$ with $\|u\|_\delta>1$, one has

$$\begin{aligned}&\sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}|\triangle u(k-1)|^{p(k-1)}+\frac{\delta}{p(k)}|u(k)|^{p(k)}\right)\\ &\geq \frac{1}{p^+}\left(\sum_{k=1,\alpha_k=p^+}^{T+1}|\triangle u(k-1)|^{p^+}+\sum_{k=1,\alpha_k=p^-}^{T+1}|\triangle u(k-1)|^{p^-}\right)\\ &+\frac{\delta}{p^+}\left(\sum_{k=1,\beta_k=p^+}^{T+1}|u(k)|^{p^+}+\sum_{k=1,\beta_k=p^-}^{T+1}|u(k)|^{p^-}\right)\\ &=\frac{1}{p^+}\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^-}-\sum_{k=1,\alpha_k=p^+}^{T+1}\left(|\triangle u(k-1)|^{p^-}-|\triangle u(k-1)|^{p^+}\right)\right)\end{aligned}$$

$$+\frac{\delta}{p^+}\left(\sum_{k=1}^{T+1}|u(k)|^{p^-}-\sum_{k=1,\beta_k=p^+}^{T+1}\left(|u(k)|^{p^-}-|u(k)|^{p^+}\right)\right)$$
$$\geq\frac{1}{p^+}\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^-}-(T+1)\right)+\frac{\delta}{p^+}\left(\sum_{k=1}^{T+1}|u(k)|^{p^-}-(T+1)\right)$$
$$=\frac{\|u\|_\delta^{p^-}}{p^+}-\frac{(1+\delta)(T+1)}{p^+}.$$

(ii) As $|\triangle u(k)|<1$ and $|u(k)|<1$ for each $k\in\mathbb{N}[1,T]$ since $\|u\|_\delta<1$, we deduce that

$$\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\triangle u(k-1)|^{p(k-1)}\geq\frac{1}{p^+}\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}$$

and

$$\sum_{k=1}^{T+1}\frac{\delta}{p(k)}|u(k)|^{p(k)}\geq\frac{\delta}{p^+}\sum_{k=1}^{T+1}|u(k)|^{p^+}.$$

Hence, by the above inequalities and the relation (2.1), we obtain

$$\begin{aligned}\sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}|\triangle u(k-1)|^{p(k-1)}\right.&+\left.\frac{\delta}{p(k)}|u(k)|^{p(k)}\right)\\
&\geq\frac{1}{p^+}\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}+\delta\sum_{k=1}^{T+1}|u(k)|^{p^+}\right)\\
&=\frac{1}{p^+}\|u\|_{\delta,p^+}^{p^+}\\
&\geq\frac{2^{\frac{p^--p^+}{p^-}}}{p^+L^{p^+}}\|u\|_\delta^{p^+}.\end{aligned}$$

□

Proposition 2.9. *If $u\in X$ is a critical point of the functional $\mathcal{E}$ in the sense that*

$$\left\langle B'(u),w-u\right\rangle+A(w)-A(u)\geq 0,\ \textit{for all}\, w\in X, \tag{2.16}$$

then u is a classical solution of problem (1.1).

Proof. Let us fix u, $v\in X$. We consider $w\in X$ defined by $w=u+sv$ for all $s>0$.

Then, dividing the inequality (2.16) by s and passing to the limit as $s\to 0^+$, one has

$$\left\langle B'(u),v\right\rangle+\left\langle A'_{p(\cdot)}(u),v\right\rangle+\left\langle J'(u),v\right\rangle\geq 0,\ \text{ for all } v\in X,$$

where $\langle J'(u),v\rangle=J'(u,v)$ is the directional derivative of the convex function J at u in the direction of v. Thus, from (2.8) and the above inequality, we obtain

$$\left\langle B'(u),v\right\rangle+\left\langle A'_{p(\cdot)}(u),v\right\rangle+j'\left((u(0),u(T+1));(v(0),v(T+1))\right)\geq 0,\ \text{for all } v\in X.$$

Using the relations (2.14) and (2.15), we get

$$M(\rho[u])\sum_{k=1}^{T+1} a(k-1,|\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1)+\sum_{k=1}^{T} f(k,u(k))v(k)$$
$$+j'\left((u(0),u(T+1))\,;(v(0),v(T+1))\right)\geq 0.$$

Then,

$$\sum_{k=1}^{T} f(k,u(k))v(k)-M(\rho[u])\sum_{k=1}^{T}\triangle\left(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)\right)v(k)$$
$$+M(\rho[u])a(T,|\triangle u(T)|)\triangle u(T)v(T+1)-M(\rho[u])a(0,|\triangle u(0)|)\triangle u(0)v(0)$$
$$+j'\left((u(0),u(T+1))\,;(v(0),v(T+1))\right)\geq 0, \tag{2.17}$$

for all $v\in X$. So, it follows that

$$\sum_{k=1}^{T}\left(-M(\rho[u])\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))+f(k,u(k))\right)v(k)=0,$$

for all $v\in X$ with $v(0)=v(T+1)=0$, which imply that

$$-M(\rho[u])\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))+f(k,u(k))=0, \tag{2.18}$$

for all $k\in\mathbb{N}[1,T]$.

It remains to prove that

$$(a(0,|\triangle u(0)|)\triangle u(0),-a(T,|\triangle u(T)|)\triangle u(T))\in\partial_j\left(u(0),u(T+1)\right).$$

Indeed, we multiply the equality (2.18) by $v(k)$. Thus, summing up k from 1 to T and using (2.17), we write

$$j'\left((u(0),u(T+1))\,;(v(0),v(T+1))\right)$$
$$\geq -M(\rho[u])a(T,|\triangle u(T)|)\triangle u(T)v(T+1)$$
$$+M(\rho[u])a(0,|\triangle u(0)|)\triangle u(0)v(0),$$

for all $v\in X$.

Taking $v\in X$ with $v(0)=\breve{u}$ and $v(T+1)=\breve{v}$, where $\breve{u},\breve{v}\in\mathbb{R}$ are arbitrarily chosen, one has

$$\begin{aligned} j'\left((u(0),u(T+1))\,;(\breve{u},\breve{v})\right) \;\geq\; & M(\rho[u])\Big(a(0,|\triangle u(0)|)\triangle u(0)\breve{u}\\ & -\; a(T,|\triangle u(T)|)\triangle u(T)\breve{v}\Big),\end{aligned}$$

for all $\breve{u},\breve{v}\in\mathbb{R}$, which by assumption $(H4)$, we deduce as $M(\cdot)$ is positive that

$$j'\left((u(0),u(T+1))\,;(\breve{u},\breve{v})\right)$$
$$\geq C\left(a(0,|\triangle u(0)|)\triangle u(0)\breve{u}-a(T,|\triangle u(T)|)\triangle u(T)\breve{v}\right)$$
$$\geq a(0,|\triangle u(0)|)\triangle u(0)t\breve{u}-a(T,|\triangle u(T)|)\triangle u(T)t\breve{v}$$
$$\geq a(0,|\triangle u(0)|)\triangle u(0)\hat{u}-a(T,|\triangle u(T)|)\triangle u(T)\hat{v},$$

where

$$C = \begin{cases} m_0 & \text{if } a(0,|\triangle u(0)|)\triangle u(0) - a(T,|\triangle u(T)|)\triangle u(T) \geq 0, \\ m_1 & \text{if } a(0,|\triangle u(0)|)\triangle u(0) - a(T,|\triangle u(T)|)\triangle u(T) \leq 0, \end{cases}$$

$\hat{u} = t\check{u}$ and $\hat{v} = t\check{v}$, where $\hat{u}, \hat{v} \in \mathbb{R}$ are arbitrarily fixed. By Theorem 23.2 in [39], we conclude that

$$(a(0,|\triangle u(0)|)\triangle u(0), -a(T,|\triangle u(T)|)\triangle u(T)) \in \partial_j \left(u(0), u(T+1)\right),$$

which ends the proof. □

Proposition 2.10. *Suppose that the hypothesis* $(H3)$ *holds. Then, the following estimate*

$$\langle a(k,|u|)u - a(k,|v|)v, u-v\rangle \geq \begin{cases} c\left(|u|+|v|\right)^{p(k)-2}|u-v|^2 & \text{if } 1 < p(k) < 2 \\ 4^{2-p^+}c|u-v|^{p(k)} & \text{if } \quad p(k) \geq 2 \end{cases}$$

holds for all $u, v \in \mathbb{R}$ *and* $k \in \mathbb{N}[1,T]$ *such that* $(u,v) \neq (0,0)$.

Proof. Let $u, v \in \mathbb{R}$ such that $(u,v) \neq (0,0)$. Let us define $\varphi(k,u) = a(k,|u|)u$.

By assumption $(H3)$, we obtain, for all $u \in \mathbb{R}\backslash\{0\}$, that

$$\begin{aligned} \frac{\partial\varphi(k,u)}{\partial u} &= |u|\frac{\partial a}{\partial u}(k,|u|) + a(k,|u|) \\ &\geq c|u|^{p(k)-2}. \end{aligned} \tag{2.19}$$

Note that

$$\varphi(k,u) - \varphi(k,v) = \int_0^1 \frac{\partial\varphi(k, v+t(u-v))}{\partial u}(u-v)\, dt. \tag{2.20}$$

Firstly, we assume that $k \in \mathbb{N}[0,T]$ such that $p(k) \geq 2$. By (2.19) and (2.20), we get

$$\begin{aligned} \langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &= \int_0^1 \frac{\partial\varphi}{\partial u}(k, v+t(u-v))(u-v)(u-v)\, dt \\ &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\, dt. \end{aligned}$$

Thus, without loss of generality, we may assume that $|u| \leq |v|$. Then, $|u-v| \leq |u|+|v| \leq 2|v|$. For any $t \in [0,1/4]$, we can write

$$|v+t(u-v)| \geq |v| - \frac{1}{4}|u-v|,$$

which imply as $|u-v| \leq 2|v|$ that

$$|v+t(u-v)| \geq \frac{1}{4}|u-v|.$$

It follows that

$$\begin{aligned} \langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\, dt \\ &\geq 4^{2-p^+}c|u-v|^{p(k)}. \end{aligned}$$

Secondly, we assume that $k \in \mathbb{N}[0,T]$ such that $1 < p(k) < 2$. Using the preceding arguments, we obtain from condition $(H3)$ that, for all $u \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned} \frac{\partial \varphi(k,u)}{\partial u} &= |u|\frac{\partial a}{\partial u}(k,|u|) + a(k,|u|) \\ &\geq c|u|^{p(k)-2}. \end{aligned}$$

Consider $t \in [0,1/4]$, then, the following inequality holds.

$$|tu + (1-t)v| \leq |u| + |v|.$$

Therefore,

$$\begin{aligned} \langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v + t(u-v)|^{p(k)-2}|u-v|^2\,dt \\ &\geq c\left(|u| + |v|\right)^{p(k)-2}|u-v|^2. \end{aligned}$$

Hence, the proof is complete. □

Lemma 2.11. *Assume that $(H1)$, $(H3)$ and $(H4)$ hold. Then, the operator $A'_{p(\cdot)}: X \to X^*$ is strictly monotone on X and verifies the (S_+) condition, i.e., for every sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and $\limsup_{n\to\infty}\langle A'_{p(\cdot)}(u_n) - A'_{p(\cdot)}(u), u_n - u\rangle \leq 0$, one has $u_n \to u$ in X as $n \to \infty$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual X^*.*

Proof. We first show that $A'_{p(\cdot)}$ is a strictly monotone operator.

Let us introduce the functional $T : X \to \mathbb{R}$ as follows.

$$T(u) = \rho[u] = \sum_{k=1}^{T+1}\int_0^{|\triangle u(k-1)|} a(k-1,\xi)\xi\,d\xi, \text{ for all } u \in X.$$

Thus, $T \in C^1(X,\mathbb{R})$ and its Gâteaux derivative at the point $u \in X$ is

$$\left\langle T'(u), v\right\rangle = \sum_{k=1}^{T+1} a(k-1,|\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1),$$

for all $u, v \in X$.

For all $u, v \in X$ such that $u \neq v$, we infer that

$$\begin{aligned} &\langle T'(u) - T'(v), u-v\rangle \\ &= \sum_{k=1}^{T+1}\left(a(k-1,|\triangle u(k-1)|)\triangle u(k-1) - a(k-1,|\triangle v(k-1)|)\triangle v(k-1)\right) \\ &\times\left(\triangle u(k-1) - \triangle v(k-1)\right). \end{aligned}$$

By Proposition 2.10, one has

$$\begin{aligned} &\langle T'(u) - T'(v), u-v\rangle \\ &\geq \begin{cases} c\sum_{k=1}^{T+1}\overline{u}(k-1)^{p(k-1)-2}|\triangle u(k-1) - \triangle v(k-1)|^2 > 0 & \text{if } 1 < p(k-1) < 2, \\ 4^{2-p^+}c\sum_{k=1}^{T+1}|\triangle u(k-1) - \triangle v(k-1)|^{p(k-1)} > 0 & \text{if } \quad p(k-1) \geq 2, \end{cases} \end{aligned}$$

where $\overline{u}(k-1) = |\triangle u(k-1)| + |\triangle v(k-1)|$. Therefore, T' is strictly monotone. Thus, by Proposition 25.10 in [46], T is strictly convex. Furthermore, as M is nondecreasing, then $\widehat{M}$ is convex in $(0,\infty)$. Thus, for all $u,v\in X$ such that $u\neq v$ and any $s_1,s_2\in(0,1)$ with $s_1+s_2=1$, one has

$$\widehat{M}\left(T(s_1u+s_2v)\right)<\widehat{M}\left(s_1T(u)+s_2T(v)\right)\leq s_1\widehat{M}\left(T(u)\right)+s_2\widehat{M}\left(T(v)\right).$$

Then, it follows that $A_{p(\cdot)}$ is strictly convex and so $A'_{p(\cdot)}$ is strictly monotone in X.

Next, we verify that the operator $A'_{p(\cdot)}$ is of type (S_+). Let $\{u_n\}\subset X$ be a sequence such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } W \text{ as } n\to\infty,\\ \limsup_{n\to\infty}\langle A'_{p(\cdot)}(u_n)-A'_{p(\cdot)}(u),u_n-u\rangle\leq 0.\end{cases}$$

We will prove that $u_n\to u$ in X. From the above inequality and the strict monotony of $A'_{p(\cdot)}$, one has

$$\lim_{n\to\infty}\langle A'_{p(\cdot)}(u_n)-A'_{p(\cdot)}(u),u_n-u\rangle=0.$$

Thus,

$$\lim_{n\to\infty}\langle A'_{p(\cdot)}(u_n),u_n-u\rangle=0,$$

which imply that

$$\lim_{n\to\infty}M(\rho[u_n])\sum_{k=1}^{T+1}a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1)\triangle(u_n-u)(k-1)=0. \tag{2.21}$$

On the other hand, by $(H1)$ and the relation (2.2), one has

$$\begin{aligned}\rho[u_n] &= \sum_{k=1}^{T+1}\int_0^{|\triangle u_n(k-1)|}a(k-1,\xi)\xi\,d\xi\\ &\leq \sum_{k=1}^{T+1}a_1(k-1)|\triangle u_n(k-1)|+\sum_{k=1}^{T+1}\frac{a_2}{p(k-1)}|\triangle u_n(k-1)|^{p(k-1)}\\ &\leq \bar{a_1}(2T+2)^{2-\frac{1}{p^-}}\|u_n\|+\frac{a_2}{p^-}\sum_{k=1}^{T+1}|\triangle u_n(k-1)|^{p(k-1)}\\ &\leq C_1+C_2\sum_{k=1}^{T+1}|\triangle u_n(k-1)|^{p(k-1)},\end{aligned}$$

where C_1 and C_2 are two positive constants.

It is easy to see that

$$\sum_{k=1}^{T+1}|\triangle u_n(k-1)|^{p(k-1)}\leq\|u_n\|^{p^*}=\begin{cases}\|u_n\|^{p^+} & \text{if } \|u_n\|>1,\\ \|u_n\|^{p^-} & \text{if } \|u_n\|<1.\end{cases}$$

So,

$$\rho[u_n]\leq C_1+C_2\|u_n\|^{p^*}\leq C\left(1+\|u_n\|^{p^*}\right),$$

which proves that the sequences $(\rho[u_n])_{n\geq1}$ is bounded. Since M is continuous, up to a subsequence, there is $s_0\geq 0$ such that

$$M\left(\rho[u_n]\right)\to M(t_0)\geq m_0 \text{ as } n\to\infty. \tag{2.22}$$

By (2.21) and (2.22), one has

$$\lim_{n\to\infty}\sum_{k=1}^{T+1} a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1)\triangle(u_n-u)(k-1)=0.$$

Hence,

$$\lim_{n\to\infty}\langle T'(u_n),u_n-u\rangle=0.$$

Thus,

$$\lim_{n\to\infty}\langle T'(u_n)-T'(u),u_n-u\rangle=0. \tag{2.23}$$

Moreover, we get, according to Proposition 2.10,

$$\langle T'(u_n)-T'(u),u_n-u\rangle$$
$$\geq \begin{cases} c\sum_{k=1}^{T+1}\tilde{u}(k-1)^{p(k-1)-2}|\triangle u_n(k-1)-\triangle u(k-1)|^2>0 & \text{if } 1<p(k-1)<2\\ 4^{2-p^+}c\sum_{k=1}^{T+1}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)}>0 & \text{if } \quad p(k-1)\geq 2, \end{cases} \tag{2.24}$$

with $\tilde{u}(k-1)=|\triangle u_n(k-1)|+|\triangle u(k-1)|$.

By the discrete Hölder inequality (see [16]), we know that

$$\begin{aligned} &\sum_{k=1}^{T+1}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)} \\ &=\sum_{k=1}^{T+1}\tilde{u}(k-1)^{\frac{p(k-1)(2-p(k-1))}{2}}\Bigg(\tilde{u}(k-1)^{\frac{p(k-1)(p(k-1)-2)}{2}}|\triangle u_n(k-1) \\ &-\triangle u(k-1)|^{p(k-1)}\Bigg) \\ &\leq K\|\tilde{u}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}}\|\tilde{u}^{\frac{p(\cdot)(p(\cdot)-2)}{2}}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}} \\ &\leq K\|\tilde{u}\|_{p(\cdot)}^{\theta}\left(\sum_{k=1}^{T+1}\tilde{u}(k-1)^{p(k-1)-2}|\triangle u_n(k-1)-\triangle u(k-1)|^2\right)^{\upsilon}, \end{aligned} \tag{2.25}$$

where θ is either $p^-(2-\breve{p})/2$ or $\breve{p}(2-p^-)/2$ and υ is either $p^-/2$ or $\breve{p}/2$ with $\breve{p}=\max_{\{k\in\mathbb{N}[0,T]:1<p(k)<2\}}p(k)$. Then, from (2.23), (2.24) and (2.25), we deduce that

$$\lim_{n\to\infty}\sum_{k=1}^{T+1}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)}=0. \tag{2.26}$$

By combining (2.5) and (2.6) with (2.26), one obtains

$$u_n\to 0 \text{ as } n\to\infty.$$

Therefore, $A'_{p(\cdot)}$ is of type (S_+). The proof of Lemma 2.11 is complete. □

Lemma 2.12. *Assume that $(H1)$, $(H3)$ and $(H4)$ hold. Then, $A_{p(\cdot)}$ is weakly lower semi-continuous, i.e., $u_n\rightharpoonup u$ in X as $n\to\infty$ imply that $A_{p(\cdot)}(u)\leq\liminf_{n\to\infty}A_{p(\cdot)}(u_n)$.*

Proof. Suppose that $u_n \rightharpoonup u$ in X as $n \to \infty$. By lemmas 2.7 and 2.11, $A_{p(\cdot)}$ is convex (see [47, Proposition 42.6]) and so the following inequality holds.

$$A_{p(\cdot)}(u_n) \geq A_{p(\cdot)}(u) + \langle A'_{p(\cdot)}(u), u_n - u \rangle,$$

for any $n \in \mathbb{N}$. Then,

$$\liminf_{n\to\infty} A_{p(\cdot)}(u_n) \geq A_{p(\cdot)}(u) + \liminf_{n\to\infty} \langle A'_{p(\cdot)}(u), u_n - u \rangle,$$

which imply that

$$\liminf_{n\to\infty} A_{p(\cdot)}(u_n) \geq A_{p(\cdot)}(u).$$

Thus, we conclude that $A_{p(\cdot)}$ is weakly lower semi-continuous. The proof is complete. □

3. Existence of a solution

In this section, we study the existence of at least one nontrivial weak solution of problem (1.1),

We first assume the following hypotheses.

$(H6)$ $\liminf_{|s|\to\infty} \frac{p(k)\int_0^s f(k,t)\,dt}{|s|^{p(k)}} > \nu_1$, for any $k \in \mathbb{N}[1,T]$.

$(H7)$ There exist two constants $\theta \in (1,\infty)$ and $M > 0$ such that

$$F(k,s) \geq \theta f(k,s)s, \quad \text{for all } k \in \mathbb{N}[1,T] \text{ with } |s| \geq M.$$

$(H8)$ $j(0,0) = 0$.

The main result of this section is the following.

Theorem 3.1. *Assume that* $(H1)$-$(H4)$ *and* $(H6)$-$(H8)$ *hold true. Then, problem* (1.1) *has at least one nontrivial weak solution.*

Proof. By Lemma 2.7, $A_{p(\cdot)}$ and B are of class $C^1(X,\mathbb{R})$; therefore, $\mathcal{E}$ is of class $C^1(X,\mathbb{R})$. From Lemma 2.12, $A_{p(\cdot)}$ is weakly lower semicontinuous. Furthermore, since

$$J(u) \leq \lim_{y\to u} \inf J(y),$$

then

$$\begin{aligned} A_{p(\cdot)}(u) + J(u) &\leq \lim_{y\to u} \inf J(y) + A_{p(\cdot)}(u) \\ &\leq \lim_{y\to u} \inf J(y) + \lim_{y\to u} \inf A_{p(\cdot)}(y) \leq \lim_{y\to u} \inf A(y). \end{aligned}$$

Consequently, $A(u) \leq \lim_{y\to u} \inf A(y)$. This implies that A is weakly lower semi-continuous. Hence, by the continuity of B, we obtain that $\mathcal{E}$ is weakly lower semi-continuous. Thus, it suffices to prove that $\mathcal{E}$ is coercive on X.

Using $(H6)$, we can find $\epsilon, \rho > 0$ such that

$$\int_0^s f(k,t)\,dt \geq \frac{1}{p(k)}(\nu_1 - \epsilon)|s|^{p(k)} \text{ for all } k \in \mathbb{N}[1,T] \text{ and all } s \in \mathbb{R} \text{ with } |t| > \rho.$$

Since $\nu_1 > 0$, we can assume that $\epsilon < \nu_1$. Since $s \to \int_0^s f(k,t)\,dt - \frac{1}{p(k)}(\nu_1 - \epsilon)|s|^{p(k)}$ is continuous on $[-\rho,\rho]$, there exists $C_\rho > 0$ such that

$$\int_0^s f(k,t)\,dt - \frac{1}{p(k)}(\nu_1 - \epsilon)|s|^{p(k)} \quad \geq \quad -C_\rho \text{ for all } k \in \mathbb{N}[1,T] \text{ and all } s \in [-\rho,\rho].$$

Hence,

$$\int_0^s f(k,t)\,dt \geq \frac{1}{p(k)}(\nu_1-\epsilon)\,|s|^{p(k)} - C_\rho \ \text{ for any } \ (k,s)\in \mathbb{N}[1,T]\times\mathbb{R}.$$

Using the above estimate and assumption $(H4)$, one has

$$\begin{aligned}\mathcal{E}(u) &= \widehat{M}(\rho[u]) + \sum_{k=1}^{T}\int_0^{u(k)} f(k,s)\,ds + J(u)\\ &\geq m_0\int_0^{\rho[u]} d\xi + (\nu_1-\epsilon)\sum_{k=1}^{T}\frac{1}{p(k)}|u(k)|^{p(k)} - C_\rho T + J(u).\end{aligned}$$

For $u\in X$ such that $\|u\|_\delta > 1$, the above inequality and relations (2.3) and (2.5) imply that

$$\begin{aligned}\mathcal{E}(u) &\geq m_0\rho[u] + (\nu_1-\epsilon)\sum_{k=1}^{T}\frac{1}{p(k)}|u(k)|^{p(k)} - C_\rho T + J(u)\\ &\geq m_0 c\mathcal{A}_{p(\cdot)}(u) + (\nu_1-\epsilon)\sum_{k=1}^{T}\frac{1}{p(k)}|u(k)|^{p(k)} - C_\rho T + J(u)\\ &\geq \min\left\{m_0 c, \frac{\nu_1-\epsilon}{\delta}\right\}\frac{\|u\|_\delta^{p^-}}{L_2^{p^-}} - C_\rho T + J(u).\end{aligned}$$

On the other hand, relation (2.8) imply the existence of three constants $l_1, l_2, l_3 \geq 0$ such that

$$\begin{aligned}\mathcal{E}(u) &\geq \min\left\{m_0 c, \frac{\nu_1-\epsilon}{\delta}\right\}\frac{\|u\|_\delta^{p^-}}{L_2^{p^-}} - C_\rho T - l_1|u(0)| - l_2|u(T+1)| - l_3\\ &\geq \min\left\{m_0 c, \frac{\nu_1-\epsilon}{\delta}\right\}\frac{\|u\|_\delta^{p^-}}{L_2^{p^-}} - K_1\|u\|_\infty - K_2,\end{aligned}$$

where $K_1 = l_1 + l_2$ and $K_2 = C_\rho T + l_3$.

Since on X, all norms are equivalent, there exists $K_3 > 0$ such that

$$\mathcal{E}(u) \geq \min\left\{m_0 c, \frac{\nu_1-\epsilon}{\delta}\right\}\frac{\|u\|_\delta^{p^-}}{L_2^{p^-}} - K_3\|u\|_\delta - K_2.$$

As $p^- > 1$, then $\mathcal{E}(u)\to\infty$ as $\|u\|_\delta\to\infty$. Thus, $\mathcal{E}$ is coercive on X.

Now, let $u^*\in X$ a global minimum of $\mathcal{E}$ which is a weak solution of problem (1.1).

We show that u^* is not trivial. Letting $k_0\in\mathbb{N}[1,T]$ and $t\in(0,1)$ be a fixed real such that $t < \left(\frac{\gamma}{m_1\left(a_1^+ + \frac{a_2}{p^-}\right)}\right)^{\frac{1}{1-\frac{1}{\theta}}}$. We define a function $v\in X$ by

$$v(k) := \begin{cases} t & \text{if } k = k_0,\\ 0 & \text{if } k\in\mathbb{N}[1,T]-\{k_0\}.\end{cases}$$

By assumption $(H7)$, one has

$$\theta f(k,s)s - F(k,s) \leq 0,$$

for all $(k,s) \in \mathbb{N}[1,T] \times \mathbb{R}$ with $|s| \geq M$. From this, for $|s| \geq M$ and $k \in \mathbb{N}[1,T]$, we get

$$\frac{\partial}{\partial s}\left(\frac{F(k,s)}{|s|^{1/\theta}}\right) = \frac{f(k,s)|s|^{1/\theta} - 1/\theta|s|^{1/\theta-2}sF(k,s)}{|s|^{2/\theta}} = \frac{\theta f(k,s)s - F(k,s)}{\theta|s|^{1/\theta}s} \leq 0.$$

Therefore, there exist $\gamma, R > 0$ such that

$$F(k,s) \leq -\gamma|s|^{1/\theta} \text{ for all } (k,s) \in \mathbb{N}[1,T] \times \mathbb{R} \text{ with } |s| \geq R.$$

Then, from (2.10), one has

$$B(v) = F(k_0,t) \leq -\gamma t^{1/\theta}.$$

Moreover, it follows from $(H1)$, $(H4)$ and (2.7) that

$$A_{p(\cdot)}(v) = \widehat{M}(\rho[v]) \leq m_1\left(a_1^+|t| + \frac{a_2}{p(k_0-1)}|t|^{p(k_0-1)}\right) \leq m_1\left(a_1^+ t + \frac{a_2}{p^-}t^{p^-}\right).$$

Thus, from $(H8)$ and (2.11), one deduces that

$$\begin{aligned}\mathcal{E}(v) &\leq m_1\left(a_1^+ t + \frac{a_2}{p^-}t^{p^-}\right) + J(v) - \gamma t^{1/\theta}\\ &\leq m_1\left(a_1^+ + \frac{a_2}{p^-}\right)t - \gamma t^{1/\theta}.\end{aligned}$$

Thus, $\mathcal{E}(v) < 0$, for any $t < \left(\frac{\gamma}{m_1\left(a_1^+ + \frac{a_2}{p^-}\right)}\right)^{\frac{1}{1-\frac{1}{\theta}}}$. Therefore, $\mathcal{E}(u^*) < 0$ and so u^* is a nontrivial weak solution of problem (1.1). The conditions of Theorem 2.3 are satisfied and then, the existence of at least one nontrivial weak solution to problem (1.1) is established. The proof of Theorem 3.1 is complete. □

4. Multiple solutions

In this section, we focus on the existence of at least two nontrivial solutions of the problem

$$\begin{cases} -M(\rho[u])\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) + q(k)|u(k)|^{p(k)-2}u(k) \\ +f(k,u(k)) = 0, \quad k \in \mathbb{N}[1,T], \\ (a(0,|\triangle u(0)|)\triangle u(0), -a(T,|\triangle u(T)|)\triangle u(T)) \in \partial_j(u(0),u(T+1)), \end{cases} \tag{4.1}$$

where $T \geq 2$ is a positive integer and $q : \mathbb{N}[1,T] \to (0,\infty)$ is a given function. Problem (4.1) follows from Problem (1.1) by disturbance through the introduction of the term $q(k)|u(k)|^{p(k)-2}u(k)$.

Define the functional $A_{q(\cdot),p(\cdot)} : X \to \mathbb{R}$ by

$$A_{q(\cdot),p(\cdot)}(u) := \widehat{M}(\rho[u]) + \sum_{k=1}^{T}\frac{q(k)}{p(k)}|u(k)|^{p(k)}. \tag{4.2}$$

Then, it is easy to see that $A_{q(\cdot),p(\cdot)} \in C^1(X,\mathbb{R})$ with

$$\begin{aligned} \left\langle A'_{q(\cdot),p(\cdot)}(u), v\right\rangle &= M(\rho[u]) \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) \\ &+ \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k), \end{aligned} \tag{4.3}$$

for all $u, v \in X$.

Let us put $A_{q(\cdot)} = A_{q(\cdot),p(\cdot)} + J$ and define functional $\mathcal{E}_{q(\cdot)}$ on X by

$$\mathcal{E}_{q(\cdot)}(u) = A_{q(\cdot)}(u) + B(u), \text{ for all } u \in X, \tag{4.4}$$

where J is defined by (2.8) and B introduced in (2.10).

Defining the following constants

$$\overline{\delta} := \nu_1 + \overline{q} \quad \text{and} \quad \underline{\delta} := \nu_1 + \underline{q}. \tag{4.5}$$

We suppose the following additional hypotheses.

($H9$) There exist $s_1, r, \mu, m_0, m_1 \in (0,\infty)$, with $m_0 \le m_1$, $1/\mu > \frac{m_1}{m_0}p^+$, such that

$$j'(z,z) \le \frac{1}{\mu} j(z) + r, \text{ for all } z \in D(j)$$

and

$$F(k,s) \ge \mu f(k,s)s, \text{ for all } k \in \mathbb{N}[1,T] \text{ with } |s| > s_1.$$

($H10$) $\liminf_{|s|\to 0} \frac{f(k,s)}{|s|^{p^+-1}} > \overline{\delta}$, for any $k \in \mathbb{N}[1,T]$.

One has the following result.

Theorem 4.1. *Assume that* ($H1$)-($H4$) *and* ($H8$)-($H10$) *hold. Then, the problem* (4.1) *has at least two nontrivial solutions.*

For the proof of Theorem 4.1, one needs the following lemma.

Lemma 4.2. *Assume that the hypotheses of Theorem 4.1 are satisfied. Then, the functional* $\mathcal{E}_{q(\cdot)}$ *given by* (4.4) *satisfies the Palais-Smale condition in the sense of Szulkin in* $\left(X, \|\cdot\|_{\overline{\delta},p(\cdot)}\right)$*, i.e., every sequence* $\{u_n\} \subset X$ *such that* $\mathcal{E}_{q(\cdot)}(u_n) \to c \in \mathbb{R}$ *as* $n \to \infty$ *and*

$$\left\langle B'(u_n), v-u_n\right\rangle + A_{q(\cdot)}(v) - A_{q(\cdot)}(u_n) \ge -\varsigma_n \|v-u_n\|_{\overline{\delta},p(\cdot)}, \text{ for all } v \in X, \tag{4.6}$$

where $\varsigma_n \to 0$*, possesses a convergent subsequence.*

Proof. Let $\{u_n\} \subset X$ be a (PS)-sequence, i.e., $\mathcal{E}_{q(\cdot)}(u_n) \to c \in \mathbb{R}$ as $n \to \infty$ and the relation (4.6) is fulfilled, with $\varsigma_n \to 0$. Since X is a finite-dimensional space, it is enough to show that $\{u_n\}$ is bounded. Indeed, assuming that $\{u_n\} \subset D(\mathcal{E}_{q(\cdot)}) = D(J)$ and that $\|u_n\|_{\overline{\delta},p(\cdot)} > 1$ for each $n \in \mathbb{N}$, then, by ($H9$) and the relation (2.8), one has

$$J(v) - \mu J'(v,v) \ge -r_1, \text{ for all } v \in D(J), \tag{4.7}$$

with $r_1 = \mu r$. Moreover, by assumption $(H9)$ again, we deduce that for all $n \in \mathbb{N}$,

$$\sum_{k=1,|u_n(k)|>s_1}^{T} [\mu f(k,u_n(k))u_n(k) - F(k,u_n(k))] \leq 0.$$

Hence, we can write

$$\begin{aligned} -B(u_n) + \mu \left\langle B'(u_n), u_n \right\rangle &= \sum_{k=1}^{T} [\mu f(k,u_n(k))u_n(k) - F(k,u_n(k))] \\ &\leq \sum_{k=1,|u_n(k)|\leq s_1}^{T} [\mu f(k,u_n(k))u_n(k) - F(k,u_n(k))] \\ &\leq \sum_{k=1}^{T} \max_{|s|\leq s_1} |\mu f(k,s)s - F(k,s)| := MT, \end{aligned} \tag{4.8}$$

where $M = \max\{|\mu f(k,s)s - F(k,s)| : k \in \mathbb{N}[1,T], |s| \leq s_1\}$. Since $\mathcal{E}_{q(\cdot)}(u_n) \to c \in \mathbb{R}$ as $n \to \infty$, there is a constant $N > 0$ such that

$$|\mathcal{E}_{q(\cdot)}(u_n)| \leq N, \text{ for all } n \in \mathbb{N}. \tag{4.9}$$

Furthermore, setting $v = u_n + tu_n$ in (4.6), dividing by $t > 0$ and letting $t \to 0^+$, we obtain

$$\left\langle B'(u_n), u_n \right\rangle + \left\langle A'_{q(\cdot),p(\cdot)}(u_n), u_n \right\rangle + J'(u_n,u_n) \geq -\varsigma_n \|u_n\|_{\overline{\delta},p(\cdot)}, \text{ for all } n \in \mathbb{N}. \tag{4.10}$$

Now, using (4.9) and (4.10), we deduce the following.

$$\begin{aligned} N + \mu\varsigma_n \|u_n\|_{\overline{\delta},p(\cdot)} &\geq A_{q(\cdot),p(\cdot)}(u_n) + J(u_n) + B(u_n) + \mu\varsigma_n \|u_n\|_{\overline{\delta},p(\cdot)} \\ &\geq A_{q(\cdot),p(\cdot)}(u_n) - \mu \left\langle A'_{q(\cdot),p(\cdot)}(u_n), u_n \right\rangle + B(u_n) \\ &- \mu \left\langle B'(u_n), u_n \right\rangle + J(u_n) - \mu J'(u_n,u_n) \end{aligned}$$

and by (4.7) and (4.8), one has

$$r_1 + MT + N + \mu\varsigma_n \|u_n\|_{\overline{\delta},p(\cdot)} \geq A_{q(\cdot),p(\cdot)}(u_n) - \mu \left\langle A'_{q(\cdot),p(\cdot)}(u_n).u_n \right\rangle,$$

which implies according to $(H3)$, $(H4)$, (4.2) and (4.3) that

$$\begin{aligned} &r_1 + MT + N + \mu\varsigma_n \|u_n\|_{\overline{\delta},p(\cdot)} \\ &\geq m_0 \int_0^{\rho[u_n]} d\xi - \mu m_1 p^+ \rho[u_n] + \sum_{k=1}^{T+1} \left(\frac{1}{p(k)} - \mu \right) q(k)|u_n(k)|^{p(k)} \\ &\geq \left(m_0 - \mu m_1 p^+\right) \rho[u_n] + \sum_{k=1}^{T+1} (1 - \mu p(k)) \frac{q(k)}{p(k)} |u_n(k)|^{p(k)} \\ &\geq \min\left\{c\left(m_0 - \mu m_1 p^+\right), \left(1 - \mu p^+\right)\right\} \left(\mathcal{A}_{p(\cdot)}(u) + \sum_{k=1}^{T+1} \frac{q(k)}{p(k)} |u_n(k)|^{p(k)} \right), \end{aligned}$$

for n large enough. Then, we get by (2.5), (2.13) and (4.5) that

$$\begin{aligned}\|u_n\|_{\overline{\delta},p(\cdot)}^{p^-} &\leq \mathcal{A}_{p(\cdot)}(u_n) + (\nu_1 + \overline{q}) \sum_{k=1}^{T+1} \frac{1}{p(k)} |u_n(k)|^{p(k)} \\ &\leq 2\left(\mathcal{A}_{p(\cdot)}(u_n) + \sum_{k=1}^{T+1} \frac{q(k)}{p(k)} |u_n(k)|^{p(k)}\right),\end{aligned}$$

which implies that

$$r_1 + MT + N + \mu\varsigma_n \|u_n\|_{\overline{\delta},p(\cdot)} \geq \frac{1}{2} \min\left\{c\left(m_0 - \mu m_1 p^+\right), \left(1 - \mu p^+\right)\right\} \|u_n\|_{\overline{\delta},p(\cdot)}^{p^-}.$$

Since $1/\mu > \frac{m_1}{m_0} p^+$, this is a contradiction. It follows that $\{u_n\}$ is bounded in X. Hence, $\mathcal{E}_{q(\cdot)}$ satisfies the Palais-Smale condition. □

The proof of Theorem 4.1

Proof. Note that $\mathcal{E}_{q(\cdot)}(0) = 0$. Moreover, from Lemma 4.2, $\mathcal{E}_{q(\cdot)}$ satisfies the Palais-Smale condition. The rest of the proof is divided into two steps.

Step 1. We prove that there exists $M > 0$ such that the functional $\mathcal{E}_{q(\cdot)}$ has a local minimum $u_0 \in B_M$, where $B_M = \left\{u \in X \text{ such that } \|u\|_{\overline{\delta},p(\cdot)} < M\right\}$. For that, by Theorem 2.5, we obtain that $u_0 \in \overline{B_M}$. Moreover, we suppose that $\mathcal{E}_{q(\cdot)}(u_0) = \min_{u\in\overline{B_M}} \mathcal{E}_{q(\cdot)}(u)$ and we claim that $\mathcal{E}_{q(\cdot)}(u_0) < \inf_{u\in\partial B_M} \mathcal{E}_{q(\cdot)}(u)$. Since f satisfies $(H10)$, then for any given $0 < \varepsilon - \overline{\delta} < \frac{\min\{1, m_0 c\} p^+}{2(T+1)K_4^{p^+}}$, there exist $\sigma > 0$ such that

$$f(k,s) \geq \left(\overline{\delta} - \varepsilon\right) |s|^{p^+-1} \text{ for all } k \in \mathbb{N}[1,T] \text{ and } s \in \mathbb{R} \text{ with } |s| \leq \sigma.$$

For $0 < s \leq \sigma$, we get that

$$\begin{aligned}F(k,s) &\geq \left(\overline{\delta} - \varepsilon\right) \int_0^s |t|^{p^+-1}\, dt \\ &= \left(\overline{\delta} - \varepsilon\right) \int_0^s t^{p^+-1}\, dt = \frac{1}{p^+}\left(\overline{\delta} - \varepsilon\right) |s|^{p^+}\end{aligned}$$

and for $-\sigma \leq s < 0$, we still have

$$\begin{aligned}F(k,s) &\geq \left(\overline{\delta} - \varepsilon\right) \int_s^0 |t|^{p^+-1}\, dt \\ &= \left(\overline{\delta} - \varepsilon\right) \int_s^0 (-t)^{p^+-1}\, dt = \frac{1}{p^+}\left(\overline{\delta} - \varepsilon\right) |s|^{p^+}.\end{aligned}$$

So, it follows that for any given $0 < \varepsilon - \overline{\delta} < \frac{\min\{1, m_0 c\} p^+}{2(T+1)K_4^{p^+}}$, there exist $\sigma > 0$ with

$$F(k,t) \geq \frac{1}{p^+}\left(\overline{\delta} - \varepsilon\right) |s|^{p^+} \text{ for all } k \in \mathbb{N}[1,T] \text{ and } s \in \mathbb{R}, \text{ with } |s| \leq \sigma. \tag{4.11}$$

Using $(H8)$, one has that $\mathcal{E}_{q(\cdot)}(0) = A_{q(\cdot),p(\cdot)}(0) + B(0) + J(0) = 0$.

From (2.8), one has

$$J(u) = j\left(u(0), u(T+1)\right) = j(0,0) = 0, \text{ for all } u \in D(J). \tag{4.12}$$

Let $\zeta = M^{p^+}\left(\frac{\min\{1,m_0c\}}{2} - (\varepsilon-\overline{\delta})\frac{1}{p^+}(T+1)K_4^{p^+}\right)$.
Then, $\|u\|_{\overline{\delta},p(\cdot)} = M$ for all $u \in \partial B_M$. For $u \in X$ with $\|u\|_{\overline{\delta},p(\cdot)} < 1$, using $(H3)$, $(H4)$, (4.5), (4.11) and (4.12), one deduces that

$$\begin{aligned}\mathcal{E}_{q(\cdot)}(u) &= \widehat{M}(\rho[u]) + \sum_{k=1}^{T+1}\frac{q(k)}{p(k)}|u(k)|^{p(k)} + \sum_{k=1}^{T+1}\int_0^{u(k)} f(k,s)\,ds \\ &\geq m_0c\mathcal{A}_{p(\cdot)}(u) + \sum_{k=1}^{T+1}\frac{q(k)}{p(k)}|u(k)|^{p(k)} - (\varepsilon-\overline{\delta})\frac{1}{p^+}\sum_{k=1}^{T+1}|u(k)|^{p^+} \\ &\geq \min\{1,m_0c\}\left(\mathcal{A}_{p(\cdot)}(u) + \sum_{k=1}^{T+1}\frac{q(k)}{p(k)}|u(k)|^{p(k)}\right) - (\varepsilon-\overline{\delta})\frac{1}{p^+}(T+1)\|u\|_\infty^{p^+}.\end{aligned}$$

Next, from (2.6), (2.13) and (4.5), it follows that

$$\begin{aligned}\|u\|_{\overline{\delta},p(\cdot)}^{p^+} &\leq \mathcal{A}_{p(\cdot)}(u) + (\nu_1+\overline{q})\sum_{k=1}^{T+1}\frac{1}{p(k)}|u(k)|^{p(k)} \\ &\leq 2\left(\mathcal{A}_{p(\cdot)}(u) + \sum_{k=1}^{T+1}\frac{q(k)}{p(k)}|u(k)|^{p(k)}\right).\end{aligned} \tag{4.13}$$

By the equivalence of the norms on X, there exists $K_4 > 0$ such that

$$\|u\|_\infty \leq K_4\|u\|_{\overline{\delta},p(\cdot)}, \text{ for all } u \in X. \tag{4.14}$$

Then, from (4.13) and (4.14), we deduce that

$$\begin{aligned}\mathcal{E}_{q(\cdot)}(u) &\geq \frac{\min\{1,m_0c\}}{2}\|u\|_{\overline{\delta},p(\cdot)}^{p^+} - (\varepsilon-\overline{\delta})\frac{1}{p^+}(T+1)K_4^{p^+}\|u\|_{\overline{\delta},p(\cdot)}^{p^+} \\ &= \|u\|_{\overline{\delta},p(\cdot)}^{p^+}\left(\frac{\min\{1,m_0c\}}{2} - (\varepsilon-\overline{\delta})\frac{1}{p^+}(T+1)K_4^{p^+}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\inf_{u\in\partial B_M}\mathcal{E}_{q(\cdot)}(u) &\geq M^{p^+}\left(\frac{\min\{1,m_0c\}}{2} - (\varepsilon-\overline{\delta})\frac{1}{p^+}(T+1)K_4^{p^+}\right) \\ &= \zeta > 0 = \mathcal{E}_{q(\cdot)}(0) \geq \mathcal{E}_{q(\cdot)}(u_0).\end{aligned}$$

Therefore, $u_0 \in B_M$ and so $\mathcal{E}'_{q(\cdot)}(u_0) = 0$.

Step 2. We show that there exists $u_1 \in X$ with $\|u_1\|_{\overline{\delta},p(\cdot)} > M$ such that $\mathcal{E}_{q(\cdot)}(u_1) < \inf_{u\in\partial B_M}\mathcal{E}_{q(\cdot)}(u)$, where $B_M = \Big\{u \in X \text{ such that } \|u\|_{\overline{\delta},p(\cdot)} < M\Big\}$.

By hypothesis $(H9)$, one has

$$\begin{aligned}\left(\frac{\int_0^s f(k,t)\,dt}{|s|^{1/\mu}}\right)'_s &= \frac{f(k,s)|s|^{1/\mu} - 1/\mu|s|^{1/\mu-2}s\int_0^s f(k,t)\,dt}{|s|^{2/\mu}} \\ &= \frac{\mu f(k,s)s - \int_0^s f(k,t)\,dt}{\mu|s|^{1/\mu}s} \leq 0.\end{aligned}$$

This leads to the existence of some constants $\vartheta, \eta > 0$ such that

$$\int_0^s f(k,t)\,dt \leq -\vartheta |s|^{1/\mu} \text{ for all } k \in \mathbb{N}[1,T] \text{ and all } s \in \mathbb{R} \text{ with } |t| > \eta.$$

Since $s \to \int_0^s f(k,t)dt + \vartheta |s|^{1/\mu}$ is continuous on $[-\eta, \eta]$, there is a constant $C_\eta > 0$ such that

$$\int_0^s f(k,t)\,dt + \vartheta |s|^{1/\mu} \quad \leq \quad C_\eta \text{ for all } k \in \mathbb{N}[1,T] \text{ and all } s \in [-\eta, \eta].$$

Consequently,

$$\int_0^s f(k,t)\,dt \quad \leq \quad -\vartheta |s|^{1/\mu} + C_\eta \text{ for any } (k,s) \in \mathbb{N}[1,T] \times \mathbb{R}. \tag{4.15}$$

Let $w \in X \setminus \{0\}$ be such that $w(0) = w(T+1) = 0$. Moreover, since $m_1 \geq m_0$, we deduce by condition $(H4)$ that

$$\widehat{M}(s) \leq m_1 s \leq m_1 s^{\frac{m_1}{m_0}}. \tag{4.16}$$

From $(H8)$, one has

$$J(sw) = 0, \text{ for all } s \in \mathbb{R}. \tag{4.17}$$

Then, if follows from $(H4)$ and (4.15)-(4.17) that

$$\begin{aligned}
\mathcal{E}_{q(\cdot)}(sw) &= A_{q(\cdot),p(\cdot)}(sw) + B(sw) + J(sw) \\
&= \widehat{M}(\rho[sw]) + \sum_{k=1}^{T+1} \frac{q(k)}{p(k)} |sw(k)|^{p(k)} + \sum_{k=1}^{T+1} \int_0^{sw(k)} f(k,s)\,ds \\
&\leq m_1 s^{\frac{m_1}{m_0} p^+} \rho[w] + s^{p^+} \sum_{k=1}^{T+1} \frac{q(k)}{p(k)} |w(k)|^{p(k)} \\
&\quad -\vartheta s^{1/\mu} \sum_{k=1}^{T+1} |u(k)|^{1/\mu} + C_\eta (T+1),
\end{aligned}$$

where $s \geq 1$.

Since $1/\mu > \frac{m_1}{m_0} p^+$, then, $\mathcal{E}_{q(\cdot)}(sw) \to -\infty$ as $s \to \infty$. Thus, there exists a contant $s_0 > M$ sufficiently large such that $u_1 = s_0 w \in X$, $u_1 \notin \overline{B_M}$ and $\mathcal{E}_{q(\cdot)}(u_1) < 0$. Therefore,

$$\max\{\mathcal{E}_{q(\cdot)}(u_0), \mathcal{E}_{q(\cdot)}(u_1)\} < \inf_{u \in \partial B_M} \mathcal{E}_{q(\cdot)}(u).$$

From Theorem 2.4, there exists a critical point u_* of $\mathcal{E}_{q(\cdot)}$, i.e. $\mathcal{E}'_{q(\cdot)}(u_*) = 0$. Thus, our problem (4.1) has at least two nontrivial solutions. □

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