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EXISTENCE OF RENORMALIZED SOLUTIONS TO THE DIRICHLET PROBLEM FOR THE ELLIPTIC EQUATION WITH NONSTANDARD GROWTH AND MEASURE DATA

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Abstract. In this paper, we establish the existence of a renormalized solution to the Dirichlet problem for the elliptic equation involving variable exponent and measure μ which does not charge the sets of null variable $p(\cdot)$-capacity. Also, we prove the conditional result: assuming there exists a weak solution $u \in W_{1,\,0}^{p(\cdot)}(\Omega)$ to the Dirichlet problem for an elliptic equation with variable exponent growth and measure data then this weak solution belongs to $L^{\infty}(\Omega)$.

1. Introduction

We consider the elliptic boundary problem with measure data

$$-div\left(a\left(x,\ \nabla u\right)\right)+d\left(x\right)\left|u\right|^{p(x)-2}u=\mu, \tag{1.1}$$

$$u|_{\partial\Omega}=0, \tag{1.2}$$

where $\Omega \subseteq R^n$, $n \geq 3$ be a smooth bounded domain with its $\partial\Omega$ Lipschitz boundary; $p \in P^{\log}(\Omega)$ is a log-Holder continuous function such that $1 < p_m = \inf\{p(x), x \in \Omega\}$ and $p_S = \sup\{p(x), x \in \Omega\} < \infty$. We assume that $d : \Omega \to R$ is a measurable bounded function such that $d(x) \geq 0,\ x \in \Omega$, and μ is a Radon measure with bounded total variation on the domain Ω, and we assume that the given measure μ does not charge the sets of zero variable exponent elliptic capacity. Then, the measure μ can be presented in the form of the following decomposition:

$$\mu = F - div(\Theta)$$

where $F \in L^1(\Omega)$ and $\Theta_1 \in \left(L^{q(\cdot)}(\Omega)\right)^n$. We rewrite the equation (1.1) in the expanded form

$$-div\left(a\left(x,\ \nabla u\right)\right)+d\left(x\right)\left|u\right|^{p(x)-2}u=F-div\left(\Theta_1\right).$$

In a recent paper [23], Ying, Fengping, and Shulin established sufficient conditions under which there is the equivalence of entropy solutions and renormalized solutions to the general nonlinear elliptic equations in Musielak-Orlicz spaces, also the authors proved their existence. In the case of constant exponent $p(x) = p \in (1,\ n],\quad x \in \Omega$, the bounded problem

$$-div\left(a\left(x,\ \nabla u\right)\right)=\mu,$$
$$u|_{\partial\Omega}=0$$

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was thoroughly investigated by G. D. Maso, F. Murat, L. Orsina, A. Prignet [1], the authors described crucial decomposition results for measures μ, introduced the notion of renormalized solution to the case of general measures with bounded total variation for such elliptic problem, and showed the renormalized solvability by an approximation procedure employing strong convergence in $\in W^{p}_{1,\,0}(\Omega)$ of truncated approximations. The concept of renormalized solutions was introduced by R.J. Perna and P.L. Lions [16], where the large-data Cauchy problem for Boltzmann equations was studied, the authors obtained the stability of the sequences of solutions and proved the global existence of at least one solution to the Cauchy problem applying the compactness method for velocity averages, and the analysis of subsolutions and supersolutions. In [18–20, 26, 27], the properties of fractional differential operators in the Musielak-Orlicz-Sobolev space; in [18] Q. Xiong, Z. Zhang studied the solvability of the elliptic differential double obstacle problems with nonstandard growth and involving measure data. In [9], the asymptotic behavior of the renormalized solution to the Dirichlet problem for p-Laplacian type equations was studied. The equations involving variable exponent were studied recently in the works of M. Ding, C. Zhang, S. Zhou [2], D. Liu, B. Wang, and P. Zhao [13], D. Liu and P. Zhao [14], Li, G. Motreanu, D. Wu, H. Zhang [9], F. Yao [21], J.X. Yin, J.K. Li, Y.Y. Ke [22], and Q. H. Zhang [25].

The Dirichlet boundary problem for nonlocal Laplacian equation

$$-div\left(\frac{\nabla u}{|\nabla u|}\right) = F$$

with nonnegative L^1-data F was investigated by D. Li, and C. Zhang [9], where the asymptotic behavior of renormalized solutions to the p-Laplacian was studied [9], also see the literature therein.

In section 2, we collect preliminary definitions and assumptions under which the Dirichlet problem (1.1)-(1.2) is studied. In Section 3, using the Stampacchia results of [2], we show that a weak solution u to the Dirichlet problem (1.1)-(1.2) belongs to $L^{\infty}(\Omega)$. In Section 4, the main result of the paper, which establishes the existence of a renormalized solution to (1.1)-(1.2) is proven.

2. Preliminary information and hypotheses

A continuous function $p\ :\ \Omega \to R,\ \Omega \subset R^n,\ n \geq 3$ is said to belong to $P^{\log}(\Omega)$ if

$$|p(x) - p(y)| \leq \frac{const}{|\log|x-y||}$$

for all $x,\ y \in clos(\Omega)$ such that $2|x-y| < 1$.

We define a modular $\rho_{p(\cdot)}(u)$ of $p \in P^{\log}(\Omega)$ by

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)}\, dx, \tag{2.1}$$

and the variable exponent Lebesgue-Luxembourg space norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0\ :\quad \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}. \tag{2.2}$$

For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, where $q(x) = \frac{p(x)}{p(x)-1}, dp_m > 1$ we have the following integral inequality

$$\left|\int_\Omega uvdx\right| \le \left(\frac{1}{p_m} + \frac{1}{q_m}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)} \tag{2.3}$$

which will be called the generalized Holder inequality.

For all $u \in L^{p(\cdot)}(\Omega)$, we obtain

$$\min\left(\|u\|^{p_m}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_s}_{L^{p(\cdot)}(\Omega)}\right) \le \rho_{p(\cdot)}(u) \le \max\left(\|u\|^{p_m}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_s}_{L^{p(\cdot)}(\Omega)}\right). \tag{2.4}$$

The hypotheses on the coefficients are:

1) $a : \Omega \times R^n \to R^n$ such that $a(\cdot, \xi)$ is measurable on Ω for each $\xi \in R^n$ and $a(x, \cdot)$ is continuous on R^n for almost every x in Ω;

2) $a(x, \xi)\xi \ge \nu|\xi|^{p(x)}$ for all $\xi \in R^n$;

3) $|a(x, \xi)| \le \alpha|\xi|^{p(x)-1} + \gamma(x)$ for all $\xi \in R^n$;

$$4)\ (a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) > 0$$

for almost every x in Ω, and all $\xi_1, \xi_2 \in R^n$, $\xi_1 \ne \xi_2$, with some constants $\nu, \alpha > 0$, and nonnegative function $\gamma \in L^{q(\cdot)}(\Omega)$; 5) $d \in L^\infty(\Omega)$.

Definition 2.1. For each number $k > 0$, the truncation operator $T_k : R \to R$ is defined by

$$T_k(s) = \max\{-k, \min\{k, s\}\} \tag{2.5}$$

for all $s \in R$. Let $u : \Omega \to [-\infty, \infty]$ be a measurable function which is finite almost everywhere on Ω. Let $T_m(u) \in W^{p(\cdot)}_{1,0}(\Omega)$ for all $k > 0$. Then, a uniquely defined measurable vector function $w : \Omega \to R^n$ such that

$$\nabla T_k(u) = 1_{\{|u|<k\}} w \tag{2.6}$$

almost everywhere in Ω, for each $k > 0$, is called the generalized gradient of u and will be still denoted by ∇u.

We introduce notations: $\psi_k(s) = T_{k+1}(s) - T_k(s)$ for all $k > 0$. We obtain that $|T_k(s)| \le k$ and $|\psi_k(s)| \le 1$.

Definition 2.2. Let $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, $x \in \Omega$. We define a variable exponent elliptic capacity of a compact K of Ω by

$$Cap_{p(\cdot)}(K) = \inf\left\{\rho_{p(\cdot)}(\nabla\phi) : \phi \in C^\infty_C(\Omega), \phi \ge 1_K \textit{ a.e. in } \Omega\right\}$$

and a variable exponent elliptic capacity of an open subset B of Ω given by

$$Cap_{p(\cdot)}(B) = \sup\left\{Cap_{p(\cdot)}(K), \ K \textit{ is compact in } \Omega, \ K \subset B\right\},$$

and a variable exponent elliptic capacity of a Borelian subset E of Ω given by

$$Cap_{p(\cdot)}(E) = \inf\left\{Cap_{p(\cdot)}(B), \ B \textit{ is open in } \Omega, \ E \subset B\right\}.$$

We denote $\mathrm{M}_0(\Omega)$ the set of all measures $\mu \in \mathrm{M}_B(\Omega)$ that are absolutely continuous with the respect to the variable exponent elliptic capacity i.e., $\mu(E) = 0$ for all Borel set $E \subset \Omega$ such that $Cap_{p(\cdot)}(E) = 0$.

Lemma 2.3. *Let* $\frac{1}{p(x)}+\frac{1}{q(x)}=1, \quad x\in\Omega$. *A measure* $\mu\in \mathrm{M}_B(\Omega)$ *belongs to* $\mathrm{M}_0(\Omega)$ *if and only if there are* $F\in L^1(\Omega)$, *and* $\Theta_1\in\left(L^{q(\cdot)}(\Omega)\right)^n$ *such that*

$$\int_\Omega v d\mu=\int_\Omega vF dx+\int_\Omega \Theta\nabla v dx$$

for all $v\in W_{1,\,0}^{p(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

Let $\mathrm{M}_B(\Omega)$ *be the space of all Radon measures on* Ω *with bounded total variation. Then, each measure* $\mu\in\mathrm{M}_B(\Omega)$ *can be presented in the following form*

$$\mu=F-div(\Theta_1)+\left(\mu^+_{\sin g}-\mu^-_{\sin g}\right),$$

where $F\in L^1(\Omega)$, *and* $\Theta_1\in\left(L^{q(\cdot)}(\Omega)\right)^n$, *besides* $\mu^+_{\sin g}$, *and* $\mu^-_{\sin g}$ *are the positive and negative parts of the singular part* $\mu_{\sin g}$, *where two nonnegative measures* $\mu^+_{\sin g}$ *and* $\mu^-_{\sin g}$ *are concentrated on two disjoint subsets of zero variable exponent elliptic capacity.*

Definition 2.4. Let conditions 1) – 4) be satisfied. Then, a measurable function u is called a renormalized solution to the parabolic problem (1.1) - (2) if:

1) the function u is finite almost everywhere and $T_k(u)\in W_{1,\,0}^{p(\cdot)}(\Omega)$ for all $k>0$;

2) $|u|^{p(\cdot)-1}\in L^{q(\cdot)}(\Omega)$ for each $q(x)<\frac{n}{n-p_m}$;

3) $|\nabla u|^{p(\cdot)-1}\in L^{q(\cdot)}(\Omega)$ for each $q(x)<\frac{n}{n-1}$;

4) the equality

$$\lim_{k\to\infty}\int_{\{k\le|u|\le k+c\}} a(x,\ \nabla u)\nabla u dx=0$$

for all $c>0$;

5) the integral identity

$$\int_\Omega a(x,\ \nabla u)h'(u)\nabla\phi dx+\int_\Omega \phi a(x,\ \nabla u)h''(u)\nabla u dx+\int_\Omega \phi d(x)|u|^{p(x)-2}uh'(u)dx$$
$$=\int_\Omega F\phi h'(u)dx+\int_\Omega\Theta_1 h''(u)\phi\nabla u dx+\int_\Omega \Theta_1 h'(u)\nabla\phi dx,$$

where $F\in L^1(\Omega)$ and $\Theta_1\in\left(L^{q(\cdot)}(\Omega)\right)^n$ for all renormalizations $h\in C_C^\infty(R)$, and all $\phi\in C_C^\infty(\Omega)$.

Proposition 2.5. *For all* $u\in W_{1,0}^{p(\cdot)}(\Omega)$, *there is two positive constants* c_1 *and* c_2 *such that*

$$\|u\|_{L^{p(\cdot)}}\le c_1\|\nabla u\|_{L^{p(\cdot)}}$$

and

$$\|u\|_{L^{p^*(\cdot)}}\le c_1\|\nabla u\|_{L^{p(\cdot)}}$$

with $p^*(x)=\frac{np(x)}{n-p(x)}$ *for all* $x\in\Omega$.

3. A priory estimate of weak solutions to (1.1)-(1.2)

In this section, we assume the existence of a weak solution $u \in W_{1,\,0}^{p(\cdot)}(\Omega)$ to the elliptic boundary problem and apply conditions 1)-5), we obtain that this solution belongs to $L^{\infty}(\Omega)$.

Theorem 3.1. ***Assume that 1)-5) are satisfied and assume that*** $F \in L^1(\Omega)$***. Let*** $p \in P^{\log}(\Omega)$ ***and*** $p^*(x) = \frac{np(x)}{n-p(x)}$ ***such that*** $\frac{2n-1}{n} < p_m \le p_S < n$***. Let*** $u \in W_{1,\,0}^{p(\cdot)}(\Omega)$, $a(x,\ \nabla u) \in L^{q(\cdot)}(\Omega)$ ***and the integral identity***

$$\int_{\Omega} a(x,\ \nabla u)\nabla v dx + \int_{\Omega} d(x)|u|^{p(x)-2} u v dx = \int_{\Omega} F v dx + \int_{\Omega} \Theta_1 \nabla v dx.$$

hold for all $v \in W_{1,\,0}^{p(\cdot)}(\Omega)$***. Then, the following inequality***

$$\|u\|_{L^{\infty}(\Omega)} \le M,$$

where a positive constant M ***depending only on*** $p(\cdot)$, n, $\|F\|_{L^1(\Omega)}$ ***and*** $\|\Theta_1\|_{L^{q(\cdot)}(\Omega)}$.

Proof. The main steps of the proof of this theorem are essentially similar to the steps described in [17, Section 4], and the statement of the theorem is analogous to Theorem 4.4 of [17]. We will use notations analogous to [17]. We denote $A_k = \{x \in \Omega \ :\ |u(x)| \ge k\}$ and $\Xi_k = meas(A_k)$.

In the definition of a weak solution, we take $w_k = u - T_k(u)$ as a test function and obtain

$$\int_{\Omega} a(x,\ \nabla u)\nabla w_k dx + \int_{\Omega} d(x)|u|^{p(x)-2} u w_k dx = \int_{\Omega} F w_k dx + \int_{\Omega} \Theta_1 \nabla w_k dx$$

so

$$\int_{\Omega} a(x,\ \nabla u)\nabla w_k dx \le \int_{\Omega} F w_k dx + \int_{\Omega} \Theta_1 \nabla w_k dx$$

We estimate

$$\int_{\Omega} 1_{\{|u|\ge k\}} a(x,\ \nabla u)\nabla u dx \ge \nu \int_{\Omega} 1_{\{|u|\ge k\}} |\nabla u|^{p(x)} dx \ge \nu \int_{\Omega} |\nabla w_k|^{p(x)} dx.$$

By general Young inequality, we estimate

$$\int_{\Omega} \Theta_1 \nabla w_k dx \le c_1 \int_{\{x\in\Omega\,:\,|u(x)|\ge k\}} |\Theta_1|^{q(x)} dx + c_2 \int_{\Omega} |\nabla w_k|^{p(x)} dx,$$

where $c_2 < \frac{\nu}{3}$. Thus, we have

$$\frac{\nu}{2}\int_{\Omega} |\nabla w_k|^{p(x)} dx \le c_1 \int_{\{x\in\Omega\,:\,|u(x)|\ge k\}} |\Theta_1|^{q(x)} dx + \int_{\Omega} |F|\,|\nabla w_k|\, dx$$

$$\le c_1 \int_{\{x\in\Omega\,:\,|u(x)|\ge k\}} |\Theta_1|^{q(x)} dx + c_3 \|F 1_{A_k}\|_{L^{q^*(\cdot)}} \|\nabla w_k\|_{L^{p^*(\cdot)}}$$

$$\le c_1 \int_{\{x\in\Omega\,:\,|u(x)|\ge k\}} |\Theta_1|^{q(x)} dx + c_3 \|F 1_{A_k}\|_{L^{q^*(\cdot)}} \|\nabla w_k\|_{L^{p(\cdot)}}$$

and applying the Young inequality, we obtain

$$\frac{\nu}{2}\int_{\Omega}|\nabla w_k|^{p(x)}\,dx \le c_1\int_{\{x\in\Omega\,:\,|u(x)|\ge k\}}|\Theta_1|^{q(x)}\,dx \\ + \tilde{c}_3\,\|F1_{A_k}\|^{\alpha_1}_{L^{q^*(\cdot)}} + \frac{\nu}{4}\int_{\Omega}|\nabla w_k|^{p(x)}\,dx,$$

where we denote $\alpha_i,\ i=1,\dots,6$ conjugation of $\beta_i,\ i=1,\dots,6$ in sense $\alpha_i+\beta_i=\alpha_i\beta_i,\ i=1,\dots,6$ and β_1 given by

$$\beta_1=\begin{cases} p_m & if \quad \|\nabla w_k\|_{L^{p(\cdot)}}\ge 1_{A_k} \\ p_S & if \quad \|\nabla w_k\|_{L^{p(\cdot)}}< 1_{A_k}.\end{cases}$$

We have

$$\frac{\nu}{4}\int_{\Omega}|\nabla w_k|^{p(x)}\,dx \le c_1\int_{\{x\in\Omega\,:\,|u(x)|\ge k\}}|\Theta_1|^{q(x)}\,dx \\ + \tilde{c}_3\left(\int_{\{x\in\Omega\,:\,|u(x)|\ge k\}}|F|^{q^*(x)}\,dx\right)^{\frac{\alpha_1}{\beta_2}} \\ \le c_1\int_{\{x\in\Omega\,:\,|u(x)|\ge k\}}|\Theta_1|^{q(x)}\,dx \\ + \tilde{c}_3\left\||F|^{q^*(x)}\right\|^{\frac{\alpha_1}{\beta_2}}_{L^{\frac{r_2(\cdot)}{q^*(\cdot)}}}\|1_{A_k}\|^{\frac{\alpha_1}{\beta_2}}_{L^{\frac{r_2(\cdot)}{r_2(\cdot)-q^*(\cdot)}}} \\ \le c_1\left\||\Theta_1|^{q(x)}\right\|_{L^{\frac{r_1(\cdot)}{q(\cdot)}}}(\Xi_k)^{\frac{1}{\beta_6}}+\tilde{c}_3\,(\Xi_k)^{\frac{\alpha_1}{\beta_2\beta_5}},$$

where $r_1(x)>q(x)$, $r_2(x)>1$, and

$$\beta_2=\begin{cases} p^*_m & if \quad \|F1_{A_k}\|_{L^{q^*(\cdot)}}\ge 1 \\ p^*_S & if \quad \|F1_{A_k}\|_{L^{q^*(\cdot)}}< 1\end{cases},$$

$$\beta_3=\begin{cases} p^*_m & if \quad \|w_k\|_{L^{p^*(\cdot)}}\ge 1 \\ p^*_S & if \quad \|w_k\|_{L^{p^*(\cdot)}}< 1\end{cases},$$

$$\beta_4=\begin{cases} p_m & if \quad \|\nabla w_k\|_{L^{p(\cdot)}}\ge 1 \\ p_S & if \quad \|\nabla w_k\|_{L^{p(\cdot)}}< 1\end{cases},$$

$$\beta_5=\begin{cases} \left(\frac{r_2(\cdot)}{r_2(\cdot)-q^*(\cdot)}\right)_m & if \quad \|1_{A_k}\|_{L^{\frac{r_2(\cdot)}{r_2(\cdot)-q^*(\cdot)}}}\ge 1 \\ \left(\frac{r_2(\cdot)}{r_2(\cdot)-q^*(\cdot)}\right)_S & if \quad \|1_{A_k}\|_{L^{\frac{r_2(\cdot)}{r_2(\cdot)-q^*(\cdot)}}}< 1\end{cases},$$

$$\beta_6=\begin{cases} \left(\frac{r_1(\cdot)}{r_1(\cdot)-q(\cdot)}\right)_m & if \quad \|1_{A_k}\|_{L^{\frac{r_1(\cdot)}{r_1(\cdot)-q(\cdot)}}}\ge 1 \\ \left(\frac{r_1(\cdot)}{r_1(\cdot)-q(\cdot)}\right)_S & if \quad \|1_{A_k}\|_{L^{\frac{r_1(\cdot)}{r_1(\cdot)-q(\cdot)}}}< 1.\end{cases}$$

By the Sobolev inequality, we estimate

$$\int_{\Omega}|\nabla w_k|^{p(x)}\,dx \ge \tilde{c}_4\left(\int_{\Omega}|\nabla w_k|^{p^*(x)}\,dx\right)^{\frac{\beta_4}{\beta_3}}.$$

We have

$$\int_\Omega |\nabla w_k|^{p^*(x)}\,dx \le c\max\left(\Xi_k{}^{\frac{\alpha_1\beta_3}{\beta_2\beta_4\beta_5}},\ \Xi_k{}^{\frac{\beta_3}{\beta_4\beta_6}}\right).$$

We select τ such that $\tau - k > 1$ so $\tau - k < w_k$. Therefore, we have

$$\Xi_\tau \le \frac{const}{(\tau-k)^{p^*{}_m}}\max\left(\Xi_k{}^{\frac{\alpha_1\beta_3}{\beta_2\beta_4\beta_5}},\ \Xi_k{}^{\frac{\beta_3}{\beta_4\beta_6}}\right).$$

If $p_S < \min_{x\in clos(\Omega)} p^*(x)$ then $\frac{\beta_3}{\beta_4} > 1$, $\frac{\alpha_1}{\beta_2} > 1$. Also we have $\frac{\alpha_1\beta_3}{\beta_2\beta_4\beta_5} > 1$, $\frac{\beta_3}{\beta_4\beta_6} > 1$ for suitable functions r_1 and r_2. Then, we conclude $\|F\|_{L^\infty(\Omega)} \le M$ by [17, Section 3].

If $p(x) < p^*(x) = \frac{np(x)}{n-p(x)}$ then there constants $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\max_{y\in B(x,\,\delta_1)\cap\Omega} p(y) < \min_{y\in B(x,\,\delta_1)\cap\Omega} \frac{np(y)}{n-p(y)}$$

for all $x \in clos(\Omega)$ and

$$\max_{y\in B(x,\,\delta_2)\cap\Omega} p(y) < \inf_{y\in B(x,\,\delta_2)\cap\Omega} \left(1+\frac{1}{n}\right)p(y)$$

for all $x \in clos(\Omega)$. Since $clos(\Omega)$ is a compact set, there exists a finite cover $\{B_j,\ j=1,\dots,k\}$ of open balls B_j. There is a positive constant $\gamma > 0$ such that

$$\min(\delta_1,\ \delta_2) > meas(\Omega\cap B_j) > \gamma$$

for all $j = 1,\dots,k$. We have

$$\int_{\Omega\cap B_j} |\nabla w_k|^{(p^*{}_j)_m}\,dx \le c_4\max\left(\Xi_{k,\,j}{}^{\frac{(\alpha_1)_j(\beta_3)_j}{(\beta_2)_j(\beta_4)_j(\beta_5)_j}},\ \Xi_{k,\,j}{}^{\frac{(\beta_3)_j}{(\beta_4)_j(\beta_6)_j}}\right)$$

for all $j = 1,\dots,k$, where $(p^*{}_j)_m = \min_{\Omega\cap B_j} p^*(x)$ and $\Xi_{k,\,j} = meas(\{x\in\Omega\cap B_j : |u| > k\})$, and $(\beta_i)_j$ is the restriction of β_i on $\Omega\cap B_j$ for $i = 1,\dots,6$ and $j = 1,\dots,k$. Since $\tau - k > 1$ and $\tau - k < w_k$, we obtain

$$\Xi_\tau \le \frac{const}{(\tau-k)^{(p^*{}_j)_m}}\max\left(\Xi_{k,\,j}{}^{\frac{(\alpha_1)_j(\beta_3)_j}{(\beta_2)_j(\beta_4)_j(\beta_5)_j}},\ \Xi_{k,\,j}{}^{\frac{(\beta_3)_j}{(\beta_4)_j(\beta_6)_j}}\right)$$

for all $j = 1,\dots,k$. We have $\frac{(\beta_3)_j}{(\beta_4)_j} > 1$, $\frac{(\alpha_1)_j}{(\beta_2)_j} > 1$, and $\frac{(\alpha_1)_j(\beta_3)_j}{(\beta_2)_j(\beta_4)_j} > 1$, $\frac{(\alpha_1)_j(\beta_3)_j}{(\beta_2)_j(\beta_4)_j} > 1$ for all $x \in clos(\Omega)$ and all $j = 1,\dots,k$. The application of the Section 4 of [28] concludes the proof of the theorem.

4. The existence of renormalized solutions to the elliptic problem

The main result of the paper can be formulated in the form of the following existence theorem of the existence of renormalized solutions.

Theorem 4.1 (existence of a renormalized solution). ***Let*** $p \in P^{\log}(\Omega)$ ***such that*** $\frac{2n-1}{n} < p_m \le p_S < n$ **.** ***Let*** $\mu \in \mathrm{M}_0(\Omega)$ ***be presented in the form of decomposition***

$$\int_\Omega v\,d\mu = \int_\Omega vF\,dx + \int_\Omega \Theta_1\nabla v\,dx$$

where $F \in L^1(\Omega)$ ***and*** $\Theta_1 \in \left(L^{q(\cdot)}(\Omega)\right)^n$ ***for all*** $v \in W_{1,0}^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$***. Assume that conditions 1) – 5) hold. Then, there exists a renormalized solution to the elliptic bounded problem*** (1.1) – (1.2)***.***

In order to prove the theorem of the existence of a renormalized solution, we have to prove preparatory lemmas.

Lemma 4.2. ***Let function*** u ***be a renormalized solution to the problem*** (1.1) – (1.2)***. Then, for all*** $k > 0$***, the inequality***

$$\rho_{p(\cdot)}(\nabla T_k(u)) \le kc$$

holds with a positive constant c ***depending on initial data and the dimension*** n***.***

Proof. We assume that $\zeta \in C^\infty(R)$ is an arbitrary positive function such that $\zeta(t) = 1$ for $|t| \le 1$, $\zeta(t) = 0$ for $|t| \ge 2$, and $\zeta(t) \in [0,\ 1]$ for all others $t \in R$. For all $l \ge 2$, we define a function $H_l(\cdot)$ by

$$H_l(\tau) = \int_{[0,\ s]} \zeta_l(t)\, dt,$$

where we denote $\zeta_l(t) = 1$ for $|t| \le l-1$ and $\zeta_l(t) = \zeta(t - (l-1)\, sign(t))$ for $|t| > l-1$, here $sign(t)$ means the sign of t. For all $l \ge 2$, the function $H_l(\cdot)$ has the following properties: $H_l(s) = H_l(T_{l+1}(s))$, $\|H_l'\|_{L^\infty(R)} \le \|\zeta\|_{L^\infty(R)}$, $\sup p(H_l') \subset [-l-1,\ l+1]$ and $\sup p(H_l'') \subset [-l-1,\ -l] \cup [l,\ l+1]$.

For all $k > 0$, in the definition of the renormalized solution, we take $h = H_l$ and $\phi = T_k(u)$, and obtain

$$\begin{aligned}
&\int_\Omega a(x,\ \nabla u)\, H_l{}'(u)\, \nabla T_k(u)\, dx + \int_\Omega T_k(u)\, a(x,\ \nabla u)\, H_l{}''(u)\, \nabla u dx \\
&\quad + \int_\Omega T_k(u)\, d(x)\, |u|^{p(x)-2} u H_l{}'(u)\, dx = \int_\Omega F T_k(u)\, H_l{}'(u)\, dx \\
&\quad + \int_\Omega T_k(u)\, \Theta_1 H_l{}''(u)\, \nabla u dx + \int_\Omega \Theta_1 H_l{}'(u)\, \nabla T_k(u)\, dx.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_\Omega a(x,\ \nabla u)\, H_l{}'(u)\, \nabla T_k(u)\, dx + \int_\Omega T_k(u)\, a(x,\ \nabla u)\, H_l{}''(u)\, \nabla u dx \\
&\quad + \int_\Omega T_k(u)\, d(x)\, |u|^{p(x)-2} u H_l{}'(u)\, dx \\
&\le k\, \|F\|_{L^1(\Omega)} + \int_\Omega \Theta_1 \nabla\left(H_l{}'(u)\, T_k(u)\right) dx \\
&\le \tilde{c}k\, \|F\|_{L^1(\Omega)} + \tilde{c}k\, \||\Theta_1|\|_{L^{q(\cdot)}(\Omega)}\, \|1\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

Since $a(x,\ \xi)\, \xi \ge \nu\, |\xi|^{p(x)}$ we obtain

$$\nu \int_\Omega |\nabla T_k(u)|^{p(x)}\, dx \le \tilde{c}k\, \|F\|_{L^1(\Omega)} + \tilde{c}k\, \||\Theta_1|\|_{L^{q(\cdot)}(\Omega)},$$

thus, we conclude that $\int_\Omega |\nabla T_k(u)|^{p(x)}\, dx \le kc$ with a constant depending on initial data.

Lemma 4.3. ***Let function*** u ***be a renormalized solution to the problem*** $(1.1)-(1.2)$***. Then, for all*** $k>0$***, the inequality***

$$\sup_{k>0}\int_{\{k\le|u|\le k+1\}}|\nabla u|^{p(x)}\,dx\le c$$

holds with a positive constant c ***depending on initial data and the dimension*** n***.***

Proof. For all $k>0$, in the definition of the renormalized solution, we choose take $h=H_l$ and $\phi_k=\varpi_k\left(T_m\left(u\right)\right)$ where $\varpi_k\left(s\right)=T_{k+1}\left(s\right)-T_k\left(s\right)$, and we calculate

$$\begin{aligned}&\int_\Omega a\left(x,\ \nabla u\right)H_l^{'}\left(u\right)\nabla\varpi_k\left(T_m\left(u\right)\right)dx\\&+\int_\Omega\varpi_k\left(T_m\left(u\right)\right)a\left(x,\ \nabla u\right)H_l^{''}\left(u\right)\nabla u dx\\&+\int_\Omega\varpi_k\left(T_m\left(u\right)\right)d\left(x\right)|u|^{p(x)-2}\,uH_l^{'}\left(u\right)dx=\int_\Omega F\varpi_k\left(T_m\left(u\right)\right)H_l^{'}\left(u\right)dx\\&\quad+\int_\Omega\varpi_k\left(T_m\left(u\right)\right)\Theta_1H_l^{''}\left(u\right)\nabla u dx\\&\quad+\int_\Omega\Theta_1H_l^{'}\left(u\right)\nabla\varpi_k\left(T_m\left(u\right)\right)dx,\end{aligned}$$

therefore, we have

$$\nu\int_{\{k\le|T_m(u)|\le k+1\}}|\nabla T_m\left(u\right)|^{p(x)}\,dx\le\|F\|_{L^1(\Omega)}+c_1\left\|\Theta_1\right\|_{L^{q(\cdot)}(\Omega)}$$

for all $m>0$. Thus, we have $\|T_m\left(u\right)\|_{W_{1,\,0}^{r(\cdot)}(\Omega)}\le const$ with $1\le r\left(\cdot\right)<\frac{n(p(\cdot)-1)}{n-1}$ for all $m>0$.

Lemma 4.4. ***Let*** $\{u_m,\ m\in N\}$ ***and*** u ***be measurable functions such that for all*** $k>0$ ***there are limits***

$$T_k\left(u_m\right)\overset{Weakly\ W_{1,\,0}^{p(\cdot)}}{\underset{m\to\infty}{\longrightarrow}}T_k\left(u\right)$$

and

$$\lim_{m\to\infty}\int_\Omega\left(\left(a\left(x,\ \nabla T_k\left(u_m\right)\right)-a\left(x,\ \nabla T_k\left(u\right)\right)\right)\left(\nabla T_k\left(u_m\right)-\nabla T_k\left(u\right)\right)\right)dx=0$$

then

$$\nabla T_k\left(u_m\right)\overset{L^{p(\cdot)}}{\underset{m\to\infty}{\longrightarrow}}\nabla T_k\left(u\right).$$

Proof. From the second lemma condition, we obtain that there is a subsequence $\{u_m,\ m\in N\}$, still denoted by $\{u_m,\ m\in N\}$, such that

$$u_m\overset{a.e.}{\underset{m\to\infty}{\longrightarrow}}u$$

and

$$\lim_{m\to\infty}\sum_{i=1,\dots,n}\left(a_i\left(x,\ \nabla T_k\left(u_m\right)\right)-a_i\left(x,\ \nabla T_k\left(u\right)\right)\right)\left(\nabla_iT_k\left(u_m\right)-\nabla_iT_k\left(u\right)\right)\overset{a.e.}{\underset{m\to\infty}{\longrightarrow}}0.$$

There is a negligible subset $D \subset \Omega$ such that for all $x \in \Omega \backslash D$, we have

$$|u(x)| < \infty,$$
$$|\nabla u(x)| < \infty,$$
$$|\gamma(x)| < \infty,$$

and

$$u_m(x) \underset{m\to\infty}{\longrightarrow} u(x),$$

$$\lim_{m\to\infty} \sum_{i=1,\ldots,n} \Big(a_i\big(x,\ \nabla T_k(u_m(x))\big) - a_i\big(x,\ \nabla T_k(u(x))\big)\Big) \big(\nabla_i T_k(u_m(x)) - \nabla_i T_k(u(x))\big) \underset{m\to\infty}{\longrightarrow} 0.$$

Omitting sing of summing in notations, we obtain

$$\begin{aligned}&(a(x,\ \nabla T_k(u_m(x))) - a(x,\ \nabla T_k(u(x))))(\nabla T_k(u_m(x)) - \nabla T_k(u(x)))\\ &\geq \nu |\nabla T_k(u_m(x))|^{p(x)} - c(x)\left(1 + |\nabla T_k(u_m(x))|^{p(x)} + |\nabla T_k(u_m(x))|\right)\end{aligned}$$

with a constant $c(x)$ that is a function depending on x but not depending on m. Therefore, $|\nabla T_k(u_m(x))|$ is uniformly bounded with respect to m hence

$$(a(x,\ \nabla T_k(u_m(x))) - a(x,\ \nabla T_k(u(x))))(\nabla T_k(u_m(x)) - \nabla T_k(u(x))) \overset{m\to\infty}{\longrightarrow} 0.$$

We assume that $\tilde{\xi}$ is an accumulation point of $\{\nabla T_k(u_m(x)),\ m \in N\}$ and $\left|\tilde{\xi}\right| < \infty$. Employing the monotony condition, we obtain that $\tilde{\xi} = \nabla T_k(u(x))$.

To show uniqueness, we recall that

$$a(x,\ \nabla T_k(u(x)))\nabla T_k(u(x)) \geq \nu |\nabla T_k(u(x))|^{p(x)} \geq 0$$

and

$$a(x,\ \nabla T_k(u_m))\nabla T_k(u_m) \underset{m\to\infty}{\overset{a.e.}{\longrightarrow}} a(x,\ \nabla T_k(u))\nabla T_k(u).$$

From the Vitali theorem and $a(x,\ \nabla T_k(u_m)) \longrightarrow_{m\to\infty}^{\left(L^{q(\cdot)}\right)^n} a(x,\ \nabla T_k(u))$, we deduce that

$$\int_\Omega a(x,\ \nabla T_k(u_m))\nabla T_k(u_m)\,dx \overset{m\to\infty}{\longrightarrow} \int_\Omega a(x,\ \nabla T_k(u))\nabla T_k(u)\,dx.$$

By the Lebesgue dominated convergence theorem, we have

$$a(x,\ \nabla T_k(u_m))\nabla T_k(u_m) \underset{m\to\infty}{\overset{L^1}{\longrightarrow}} a(x,\ \nabla T_k(u))\nabla T_k(u),$$

thus

$$\nabla T_k(u_m) \underset{m\to\infty}{\overset{\left(L^{p(\cdot)}\right)^n}{\longrightarrow}} \nabla T_k(u)$$

by the Vitali convergence theorem. The lemma is proven.

Now, we are going to prove the theorem of the existence of a renormalized solution.

We consider a family of approximate problems

$$-div(a(x,\ \nabla u_m)) + d(x)|u_m|^{p(x)-2}u_m = F_m - div((\Theta_1)_m),$$

$$u_m|_{\partial\Omega} = 0,$$

where there are sequences $\{F_m\} \subset L^{\infty}(\Omega)$ and $\{(\Theta_1)_m\} \subset (L^{\infty}(\Omega))^n$ such that

$$F_m \underset{m\to\infty}{\overset{L^1}{\longrightarrow}} F,$$

and

$$(\Theta_1)_m \underset{m\to\infty}{\overset{\left(L^{q(\cdot)}\right)^n}{\longrightarrow}} \Theta_1.$$

For each problem, there exists a weak solution $u_m \in W^{p(\cdot)}_{1,\,0}(\Omega)$, which satisfies the identity of the definition of a weak solution

$$\int_{\Omega} a(x,\ \nabla u_m)\nabla\phi dx + \int_{\Omega} \phi d(x)|u_m|^{p(x)-2}u_m dx = \int_{\Omega} F_m\phi dx + \int_{\Omega} (\Theta_1)_m \nabla\phi dx$$

for all $\phi \in W^{p(\cdot)}_{1,\,0}(\Omega)$.

In order to show the existence of solutions to the approximating problems, we introduce an operator $A\ :\ W^{p(\cdot)}_{1,\,0}(\Omega) \to \left(W^{p(\cdot)}_{1,\,0}(\Omega)\right)^*$ defined by

$$A\ :\ u \mapsto \left(v \mapsto \int_{\Omega}\left(a(x,\ \nabla u)\nabla v + d(x)|u|^{p(x)-2}uv\right)dx\right)$$

for all $u,\ v \in W^{p(\cdot)}_{1,\,0}(\Omega)$. The operator $A\ :\ W^{p(\cdot)}_{1,\,0}(\Omega) \to \left(W^{p(\cdot)}_{1,\,0}(\Omega)\right)^*$ is correctly defined, coercive, monotone, and semicontinuous.

Denoting the duality pairing between $W^{p(\cdot)}_{1,\,0}(\Omega)$ and $\left(W^{p(\cdot)}_{1,\,0}(\Omega)\right)^*$ by $\langle\cdot,\ \cdot\rangle$, we calculate

$$\begin{aligned}\frac{\langle A(u),\ u\rangle}{\|u\|_{W^{p(\cdot)}_{1,\,0}}} &= c\frac{1}{\|\nabla u\|_{L^{p(\cdot)}}}\int_{\Omega}\left(a(x,\ \nabla u)\nabla u + d(x)|u|^{p(x)}\right)dx \\ &\geq \nu c\frac{1}{\|u\|_{L^{p(\cdot)}}}\min\left\{\|\nabla u\|^{p_S}_{L^{p(\cdot)}},\ \|\nabla u\|^{p_m}_{L^{p(\cdot)}}\right\} \overset{\|u\|_{L^{p(\cdot)}}\to\infty}{\longrightarrow} \infty\end{aligned}$$

with some constant $c > 0$. This yields coercivity. The boundedness of the operator

$$A\ :\ W^{p(\cdot)}_{1,\,0}(\Omega) \to \left(W^{p(\cdot)}_{1,\,0}(\Omega)\right)^*$$

follows from

$$\begin{aligned}&|\langle A(u),\ v\rangle| = \left|\int_{\Omega}\left(a(x,\ \nabla u)\nabla v + d(x)|u|^{p(x)-2}uv\right)dx\right| \\ &\leq \int_{\Omega}\left(\alpha|\nabla u|^{p(x)-1} + \gamma(x)\right)|\nabla v|dx + \int_{\Omega} d(x)|u|^{p(x)-1}|v|dx \\ &\leq \frac{p_S-1}{p_m}\alpha\rho_{p(\cdot)}(\nabla u) + \frac{1}{p_m}\alpha\rho_{p(\cdot)}(\nabla v) + \frac{1}{p_m}\rho_{p(\cdot)}(\nabla v) \\ &\quad + \frac{p_S-1}{p_m}\rho_{q(\cdot)}(\gamma) + \sup\{d(x)\}\left(\frac{p_S-1}{p_m}\rho_{p(\cdot)}(u) + \frac{1}{p_m}\rho_{p(\cdot)}(v)\right)\end{aligned}$$

for all $u,\ v \in W^{p(\cdot)}_{1,\,0}(\Omega)$.

In the view of the Minty theorem, we conclude that the operator A is surjective, thus the existence of the sequence $\{u_m\} \subset W^{p(\cdot)}_{1,\,0}(\Omega) \cap L^{\infty}(\Omega)$ of weak solutions to the approximating problems, for further information see [11, Chapter 2]. We formulate the following statements.

Proposition 4.5. ***There are positive constants*** c_1 ***and*** c_2 ***independent of*** $m \in N$ ***such that the inequality***

$$\int_{\{|u_m|\leq k\}} |\nabla u_m|^{p(x)}\, dx \leq c_1 k$$

and

$$\int_{\{k\leq |u_m|\leq k+1\}} |\nabla u_m|^{p(x)}\, dx \leq c_2$$

for all $k > 0$***.***

Proof. Similarly, to the previous lemma, we take $T_k(u_m)$ as a test function in the definition of a weak solution and obtain

$$\int_\Omega a(x,\ \nabla u_m)\nabla(T_k(u_m))\,dx + \int_\Omega T_k(u_m)\,d(x)\,|u_m|^{p(x)-2}\,u_m dx$$
$$= \int_\Omega F_m T_k(u_m)\,dx + \int_\Omega (\Theta_1)_m \nabla(T_k(u_m))\,dx$$

so

$$\nu \int_{\{|u_m|\leq k\}} |\nabla u_k(u)|^{p(x)}\, dx \leq \tilde{c}k\,\|F\|_{L^1(\Omega)} + \tilde{c}k\,\|\Theta_1\|_{L^{q(\cdot)}(\Omega)}\,.$$

Next, we choose $\phi = T_{k+1}(u_m) - T_k(u_m)$ as a test function in the definition of a weak solution, we have

$$\int_\Omega a(x,\ \nabla u_m)\nabla(T_{k+1}(u_m) - T_k(u_m))\,dx$$
$$+ \int_\Omega (T_{k+1}(u_m) - T_k(u_m))\,d(x)\,|u_m|^{p(x)-2}\,u_m dx$$
$$= \int_\Omega F_m(T_{k+1}(u_m) - T_k(u_m))\,dx$$
$$+ \int_\Omega (\Theta_1)_m \nabla(T_{k+1}(u_m) - T_k(u_m))\,dx$$

thus, we deduce

$$\nu \int_{\{k\leq |u_m|\leq k+1\}} |\nabla u_m|^{p(x)}\, dx$$
$$\leq \int_\Omega F_m(T_{k+1}(u_m) - T_k(u_m))\,dx$$
$$+ \int_\Omega (\Theta_1)_m \nabla(T_{k+1}(u_m) - T_k(u_m))\,dx$$
$$\leq \tilde{c}\,\|F\|_{L^1(\Omega)} + \tilde{c}\,\|\Theta_1\|_{L^{q(\cdot)}(\Omega)}$$

since $|F_m| \leq |F|$ and $|(\Theta_1)_m| \leq |\Theta_1|$.

From statement 1, we have that $\|u_m\|_{W^{r(\cdot)}_{1,\,0}(\Omega)} \leq const$ with $1 \leq r(\cdot) < \frac{n(p(\cdot)-1)}{n-1}$.

Proposition 4.6. ***We have***

$$\lim_{k\to\infty}\limsup_{m\to\infty}\int_{\{k\le|u|\le k+1\}}|\nabla u_m|^{p(x)}\,dx=0.$$

Proof. We select $\phi=T_{k+1}(u_m)-T_k(u_m)$ as a trial function and obtain

$$\begin{aligned}&\nu\int_{\{k\le|u|\le k+1\}}|\nabla u_m|^{p(x)}\,dx\\ &\le\int_\Omega F_m\left(T_{k+1}(u_m)-T_k(u_m)\right)dx\\ &\quad+\int_\Omega(\Theta_1)_m\nabla\left(T_{k+1}(u_m)-T_k(u_m)\right)dx\overset{k\to\infty}{\longrightarrow}0\end{aligned}$$

since $\nabla\left(T_{k+1}(u_m)-T_k(u_m)\right)=^{a.e.}1_{\{k\le|u|\le k+1\}}\nabla u_m$, and $T_{k+1}(u)-T_k(u)\overset{k\to\infty}{\longrightarrow}0$**.**

Therefore, $\{u_m\}\subset W_{1,\,0}^{p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ is a fundamental sequence in measure. So, there is a measurable function u and a subsequence $\{u_m\}\subset W_{1,\,0}^{p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ such that

$$u_m\underset{m\to\infty}{\overset{a.e.}{\longrightarrow}}u$$

and

$$T_k(u_m)\underset{m\to\infty}{\overset{Weakly\ W_{1,\,0}^{p(\cdot)}}{\longrightarrow}}T_k(u)$$

for all $k>0$.

Proposition 4.7. ***For each*** $k>0$***, for some positive constants*** c ***and*** c_1 ***, we have***

$$\limsup_{m\to\infty}\int_\Omega|\nabla T_k(u_m)|^{p(x)}\,dx\le c\int_\Omega|\nabla T_k(u)|^{p(x)}\,dx+c_1$$

and

$$T_k(u_m)\underset{m\to\infty}{\overset{W_{1,\,0}^{p(\cdot)}}{\longrightarrow}}T_k(u)\,.$$

Proof. In the definition of a renormalized solution, we take $\phi=T_k(u_m)-T_k(u)$ as a trial function and obtain

$$\begin{aligned}&\int_\Omega a(x,\ \nabla u_m)\,H_l'(u_m)\,\nabla\left(T_k(u_m)-T_k(u)\right)dx\\ &\quad+\int_\Omega\left(T_k(u_m)-T_k(u)\right)a(x,\ \nabla u_m)\,H_l''(u_m)\,\nabla u_m dx\\ &\quad+\int_\Omega\left(T_k(u_m)-T_k(u)\right)d(x)\,|u_m|^{p(x)-2}\,u_mH_l'(u_m)\,dx\\ &=\int_\Omega F\left(T_k(u_m)-T_k(u)\right)H_l'(u_m)\,dx\\ &\quad+\int_\Omega\left(T_k(u_m)-T_k(u)\right)\Theta_1H_l''(u_m)\,\nabla u_m dx\\ &\quad+\int_\Omega\Theta_1H_l'(u_m)\,\nabla\left(T_k(u_m)-T_k(u)\right)dx\end{aligned}$$

where $\sup p\left(H_l'\right) \subset [-l-1,\ l+1]$ and $\sup p\left(H_l''\right) \subset [-l-1,\ -l] \cup [l,\ l+1]$ for all $l \geq 2$.

Fixing the level k, we pass to the limit as $m \to \infty$ and $l \to \infty$, respectively.

First, we estimate

$$\left| \int_\Omega \left(T_k\left(u_m\right) - T_k\left(u\right)\right) a\left(x,\ \nabla u_m\right) H_l''\left(u_m\right) \nabla u_m dx \right|$$
$$\leq \left\| H_l'' \right\|_{L^\infty(R)} \left\| T_k\left(u_m\right) - T_k\left(u\right) \right\|_{L^\infty(R)} \int_{\{l \leq |u_m| \leq l+1\}} a\left(x,\ \nabla u_m\right) \nabla u_m dx$$

so

$$\limsup_{m\to\infty} \left| \int_\Omega \left(T_k\left(u_m\right) - T_k\left(u\right)\right) a\left(x,\ \nabla u_m\right) H_l''\left(u_m\right) \nabla u_m dx \right|$$
$$\leq const \limsup_{m\to\infty} \int_{\{k \leq |u_m| \leq k+1\}} a\left(x,\ \nabla u_m\right) \nabla u_m dx \underset{l\to\infty}{\longrightarrow} 0.$$

Second, by the Lebesgue convergence theorem, we have

$$\limsup_{m\to\infty} \int_\Omega F\left(T_k\left(u_m\right) - T_k\left(u\right)\right) H_l'\left(u_m\right) dx = 0$$

for all $l \geq 2$.

Third, we deduce

$$\limsup_{m\to\infty} \int_\Omega \left(T_k\left(u_m\right) - T_k\left(u\right)\right) d\left(x\right) \left|u_m\right|^{p(x)-2} u_m H_l'\left(u_m\right) dx = 0$$

and, by the Lebesgue convergence theorem

$$\lim_{k\to\infty} \lim_{m\to\infty} \int_\Omega \left(T_k\left(u_m\right) - T_k\left(u\right)\right) \Theta_1 H_l''\left(u_m\right) \nabla u_m dx = 0$$

for all $l \geq 2$.

Passing to the limit as $m \to \infty$ and $l \to \infty$, we conclude

$$\lim_{l\to\infty} \limsup_{m\to\infty} \int_\Omega a\left(x,\ \nabla u_m\right) H_l'\left(u_m\right) \nabla\left(T_k\left(u_m\right) - T_k\left(u\right)\right) dx \leq 0$$

for all $k > 0$.

For all $l \geq k > 0$, we obtain

$$a\left(x,\ \nabla u_m\right) H_l'\left(u_m\right) \nabla\left(T_k\left(u_m\right) - T_k\left(u\right)\right)$$
$$= a\left(x,\ \nabla u_m\right) \nabla T_k\left(u_m\right) \geq \nu \left|\nabla T_k\left(u_m\right)\right|^{p(x)}.$$

Therefore, we have

$$\limsup_{m\to\infty} \int_\Omega \nu \left|\nabla T_k\left(u_m\right)\right|^{p(x)} dx \leq \limsup_{m\to\infty} \int_\Omega a\left(x,\ \nabla u_m\right) \nabla T_k\left(u_m\right) dx$$
$$\leq \limsup_{l\to\infty} \limsup_{m\to\infty} \int_\Omega a\left(x,\ \nabla u_m\right) H_l'\left(u_m\right) \nabla\left(T_k\left(u_m\right) - T_k\left(u\right)\right) dx.$$

Since

$$\lim_{m\to\infty} H_l'\left(u_m\right) = H_l'\left(u\right)$$

and

$$\lim_{m\to\infty} T_k(u_m) = T_k(u)$$

in the topology of $W_{1,\,0}^{p(\cdot)}(\Omega)$, we have

$$\limsup_{m\to\infty}\int_\Omega \nu\,|\nabla T_k(u_m)|^{p(x)}\,dx \le \limsup_{l\to\infty}\limsup_{m\to\infty}\int_\Omega a(x,\ \nabla u_m)\,H_l'(u_m)\,\nabla T_k(u)\,dx$$
$$= \limsup_{l\to\infty}\int_\Omega a(x,\ \nabla T_{l+1}(u))\,H_l'(u)\,\nabla T_k(u)\,dx$$
$$= \int_\Omega a(x,\ \nabla T_k(u))\,\nabla T_k(u)\,dx \le \int_\Omega \Big(\alpha\,|\nabla T_k(u)|^{p(x)} + \gamma(x)\,|\nabla T_k(u)|\Big)\,dx$$
$$\le \alpha\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx + \frac{p_S-1}{p_m}\int_\Omega (\gamma(x))^{q(x)}\,dx + \frac{1}{p_m}\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx$$
$$\le \left(\alpha+\frac{1}{p_m}\right)\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx + \frac{p_S-1}{p_m}\int_\Omega (\gamma(x))^{q(x)}\,dx.$$

Thus, we have

$$\limsup_{m\to\infty}\int_\Omega |\nabla T_k(u_m)|^{p(x)}\,dx \le \frac{1}{\nu}\left(\alpha+\frac{1}{p_m}\right)\int_\Omega |\nabla T_k(u)|^{p(x)}\,dx + \frac{p_S-1}{\nu p_m}\int_\Omega (\gamma(x))^{q(x)}\,dx.$$

Next, for all $\xi,\ \eta\in R^n$, we use the following inequality

$$2^{2-p}\,|\xi-\eta|^p \le (\xi-\eta)\Big(\xi\,|\xi|^{p-2} - \eta\,|\eta|^{p-2}\Big)$$

for all $p\ge 2$ and

$$(p-1)\,|\xi-\eta|^2 \le (\xi-\eta)\Big(\xi\,|\xi|^{p-2} - \eta\,|\eta|^{p-2}\Big)\,(|\xi|+|\eta|)^{p-2}$$

for all $1<p<2$. So, we obtain

$$2^{2-p_S}\int_{\{p(x)\ge 2\}} |\nabla T_k(u_m) - \nabla T_k(u)|^{p(x)}\,dx$$
$$\le \int_{\{p(x)\ge 2\}} \Big(|\nabla T_k(u_m)|^{p(x)-2}\,\nabla T_k(u_m) - |\nabla T_k(u)|^{p(x)-2}\,\nabla T_k(u)\Big)\cdot(\nabla T_k(u_m) - \nabla T_k(u))\,dx;$$

we denote $W = |\nabla T_k(u_m)| + |\nabla T_k(u)|$ and obtain

$$\int_{\{1<p(x)<2\}} |\nabla T_k(u_m) - \nabla T_k(u)|^{p(x)}\,dx$$
$$\le \int_{\{1<p(x)<2\}} \frac{|\nabla T_k(u_m) - \nabla T_k(u)|^{p(x)}}{W^{\frac{p(x)(2-p(x))}{2}}}\,W^{\frac{p(x)(2-p(x))}{2}}\,dx$$
$$\le 2\max\left\{\left(\int_{\{1<p(x)<2\}} \frac{|\nabla T_k(u_m) - \nabla T_k(u)|^2}{W^{2-p(x)}}\,dx\right)^{\frac{p_m}{2}},\right.$$

$$\left.\left(\int_{\{1<p(x)<2\}}\frac{\left|\nabla T_k\left(u_m\right)-\nabla T_k\left(u\right)\right|^2}{W^{2-p(x)}}dx\right)^{\frac{p_S}{2}}\right\}$$

$$\times\max\left\{\left(\int_{\{1<p(x)<2\}}W^{p(x)}dx\right)^{\frac{2-p_S}{2}},\ \left(\int_{\{1<p(x)<2\}}W^{p(x)}dx\right)^{\frac{2-p_m}{2}}\right\}.$$

Now, since the sequence $\{T_k\left(u_m\right)\}$ is bounded in $W_{1,\,0}^{p(\cdot)}\left(\Omega\right)$ and applying $T_k\left(u_m\right)\overset{Weakly\ W_{1,\,0}^{p(\cdot)}}{\longrightarrow_{m\to\infty}}T_k\left(u\right)$ for all $k>0$, we deduce that

$$\int_\Omega\left(\left|\nabla T_k\left(u_m\right)\right|^{p(x)-2}\nabla T_k\left(u_m\right)-\left|\nabla T_k\left(u\right)\right|^{p(x)-2}\nabla T_k\left(u\right)\right)$$
$$\cdot\left(\nabla T_k\left(u_m\right)-\nabla T_k\left(u\right)\right)dx\overset{m\to\infty}{\longrightarrow}0.$$

Therefore, we conclude

$$\int_\Omega\left|\nabla T_k\left(u_m\right)-\nabla T_k\left(u\right)\right|^{p(x)}dx\overset{m\to\infty}{\longrightarrow}0.$$

The statement 3 is proven.

Next, we assume that $h\in C^\infty\left(R\right)$ is such that $\sup p\left(h'\right)\subset[-k,\ k]$ and $\sup p\left(h''\right)\subset[-k,\ k]$, for some $k>0$. Then, we have

$$\int_\Omega a\left(x,\ \nabla u_m\right)h'\left(u_m\right)\nabla\phi dx+\int_\Omega\phi a\left(x,\ \nabla u_m\right)h''\left(u_m\right)\nabla u_m dx$$
$$+\int_\Omega\phi d\left(x\right)\left|u_m\right|^{p(x)-2}u_m h'\left(u_m\right)dx=\int_\Omega F\phi h'\left(u_m\right)dx$$
$$+\int_\Omega\phi\Theta_1 h''\left(u_m\right)\nabla u_m dx+\int_\Omega\Theta_1 h'\left(u_m\right)\nabla\phi dx$$

for all $\phi\in C_C^\infty\left(\Omega\right)$. We have to pass to the limit as m tends to infinity.

We have

$$a\left(x,\ \nabla u_m\right)h'\left(u_m\right)=a\left(x,\ \nabla T_k\left(u_m\right)\right)h'\left(u_m\right)$$

so

$$a\left(x,\ \nabla T_k\left(u_m\right)\right)h'\left(u_m\right)\underset{m\to\infty}{\overset{L^{q(\cdot)}}{\longrightarrow}}a\left(x,\ \nabla T_k\left(u\right)\right)h'\left(u\right)$$

and

$$a\left(x,\ \nabla T_k\left(u\right)\right)h'\left(u\right)=a\left(x,\ \nabla u\right)h'\left(u\right).$$

Next, we obtain

$$a\left(x,\ \nabla u_m\right)h''\left(u_m\right)\nabla u_m\overset{a.e.}{=}a\left(x,\ \nabla T_k\left(u_m\right)\right)h''\left(u_m\right)\nabla T_k\left(u_m\right)$$

so

$$a\left(x,\ \nabla T_k\left(u_m\right)\right)h''\left(u_m\right)\nabla T_k\left(u_m\right)\underset{m\to\infty}{\overset{L^1}{\longrightarrow}}a\left(x,\ \nabla T_k\left(u\right)\right)h''\left(u\right)\nabla T_k\left(u\right)$$

and

$$a\left(x,\ \nabla T_k\left(u\right)\right)h''\left(u\right)\nabla T_k\left(u\right)=a\left(x,\ \nabla u\right)h''\left(u\right)\nabla u.$$

Also, we obtain

$$d\left(x\right)\left|u_m\right|^{p(x)-2}u_m h'\left(u_m\right)\overset{a.e.}{=}d\left(x\right)\left|T_k\left(u_m\right)\right|^{p(x)-2}T_k\left(u_m\right)h'\left(u_m\right),$$

$$d(x)\left|T_k(u_m)\right|^{p(x)-2}T_k(u_m)h'(u_m)\underset{m\to\infty}{\overset{L^1}{\longrightarrow}}d(x)\left|T_k(u)\right|^{p(x)-2}T_k(u)h'(u)$$

and

$$d(x)\left|T_k(u)\right|^{p(x)-2}T_k(u)h'(u)=d(x)\left|u\right|^{p(x)-2}uh'(u).$$

Thus, passing to the limit as m approaches infinity, we deduce that

$$\int_\Omega a(x,\ \nabla u)h'(u)\nabla\phi dx+\int_\Omega \phi a(x,\ \nabla u)h''(u)\nabla u dx$$
$$+\int_\Omega d(x)\left|u\right|^{p(x)-2}u\phi h'(u)dx=\int_\Omega F\phi h'(u)dx$$
$$\int_\Omega \Theta_1 h''(u)\phi\nabla u dx+\int_\Omega \Theta_1 h'(u)\nabla\phi dx.$$

The existence of a renormalized solution is proven.

References

[1] G, Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) **28** (1999), 741–808.

[2] M. Ding, C. Zhang and S. Zhou, *On optimal C1,α estimates for p(x)-Laplace type equations*, Nonlinear Anal. **200** (2020): 112030,

[3] B. El Haji, M. El Moumni and K. Kouhaila, *Existence of entropy solutions for nonlinear elliptic problem having large monotonicity in weighted Orlicz-Sobolev spaces*, LE MATEMATICHE **LXXVI** (2021), 37–61.

[4] X. L. Fan, and Q. H. Zhang, *Existence of solutions for p(x)-Laplacian Dirichlet problems*, Nonlinear Anal. **52** (2003), 1843–1852.

[5] X. L. Fan and D. Zhao, *On the spaces Lp(x) and Wm,p(x)*. J. Math. Anal. Appl. **263** (2001), 424–446.

[6] M. Kim and S. C. Lee, *Supersolutions and superharmonic functions for nonlocal operators with Orlicz growth*, (2023), arXiv: 2311.01246.

[7] M. Kim, K. A. Lee and S.-C. Lee, *Wolff potential estimates and Wiener criterion for nonlocal equations with Orlicz growth*, J. Funct. Anal. **288** (2025): No. 110690.

[8] Y. C. Kim, *Nonlocal Harnack inequalities for nonlocal heat equations*, J. Differ. Equ. **267** (2019), 6691–6757.

[9] D. Li and C. Zhang, *On the solutions of a nonlocal 1-Laplacian equation with L1-data*, Discrete Contin. Dyn. Syst. **44** (2024), 1354–1375.

[10] G. Li, D. Motreanu, H. Wu and Q. Zhang, *Multiple solutions with the constant sign for a (p, q)-elliptic system Dirichlet problem with product nonlinear term*, Boundary Value Problems, **2018** (2018): 67.

[11] J. L. Lions, *Quelques Methodes de Résolution des Problèmes aux Limites Non Linéaire*, Dunod et Gauthier-Villars, Paris, 1969.

[12] P. L. Lions, *Mathematical Topics in Fluid Mechanics*, vol. 1, in: Oxford Lecture Series in Mathematics and its Applications, vol. 3, Oxford University Press, New York, 1996.

[13] D. Liu, B. Wang and P. Zhao, *On the trace regularity results of Musielak–Orlicz–Sobolev spaces in a bounded domain*, Comm. Pure Appl. Anal. **15** (2016), 1643–1659.

[14] D. Liu and P. Zhao, *Solutions for a quasilinear elliptic equation in Musielak–Sobolev spaces*, Nonlinear Anal. Real World Appl. **26** (2016), 315–329.

[15] N. Pan, B. L. Zhang and J. Cao, *Weak solutions for parabolic equations with p(x)-growth*, Electronic Journal of Differential Equations **2016**, (2016), 1–15.

[16] R. J. Perna and P. L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability*. Ann. Math. **130** (1989), 321–366.

[17] G. Stampacchia, *Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus*. In Annales de l'Institut Fourier **15** (1965), 189– 257.

[18] Q. Xiong, and Z. Zhang, *Gradient potential estimates for elliptic obstacle problems*, J. Math. Anal. Appl. **495** (2021): 124698.

[19] Q. Xiong, Z. Zhang and L. Ma, *Gradient potential estimates in elliptic obstacle problems with Orlicz growth*, Calc. Var. Partial Differential Equations **61** (2022): Paper No. 83.

[20] Q. Xiong, Z. Zhang, L. Ma, *Riesz potential estimates for problems with Orlicz growth*, J. Math. Anal. Appl. **515** (2022), no. 2, Paper No. 126448.

[21] F. Yao, *Holder regularity for the general parabolic p(x,t)-Laplacian equations*, NoDEA Nonlinear Differential Equations Appl. **22** (2015), 105–119.

[22] J. X. Yin, J. K. Li and Y. Y. Ke, *Existence of positive solutions for the p(x)-Laplacian equation.* Rocky Mt. J. Math. **42** (2012), 1675–1758.

[23] L. Ying, Y. Fengping and Zh. Shulin, *Entropy and renormalized solutions to the general nonlinear elliptic equations in Musielak–Orlicz spaces*, Nonlinear Anal. **61** (2021), 1–20.

[24] C. Yu and D. Ri, *Global L∞-estimates and Holder continuity of weak solutions to elliptic equations with the general nonstandard growth conditions*, Nonlinear Anal. **156** (2017), 144–166.

[25] Q. H. Zhang, *Existence of solutions for p(x)-Laplacian equations with singular coefficients in RN*, J. Math. Anal. Appl. **348** (2008), 38–50.

[26] X. Zhang and S. Zheng, *Besov regularity for the gradients of solutions to non-uniformly elliptic obstacle problems*, J. Math. Anal. Appl. **504** (2021): 125402.

[27] L. Zhao, and S. Zheng, *Besov regularity for a class of elliptic obstacle problems with double-phase Orlicz growth*, Journal of Mathematical Analysis and Applications **535** (2024):128119.

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