Applied Analysis and Optimization
Volume 9, Number 3, 2025, 313–326
Yokohama Publishers
ISSN 2432-1664 Online Journal

SOLVING A QUADRATIC SET-VALUED OPTIMIZATION PROBLEM BASED ON THE NULL SETS CONCEPT

PALAMANGA LOMPO AND ABDOULAYE COMPAORE

Abstract. This article addresses the study of quadratic optimization problems in which the coefficients are sets. The study is based on the concept of null sets. This concept allows for the use of a partial ordering between sets in the context of classical and Fukuhara differences. Based on this, the notions of optimal and H-optimal solutions have been defined. To determine the solution, the set-valued quadratic optimization problem is converted into a deterministic, bi-objective optimization problem using vectorization. Then, the scalarization technique is used to transform the vectorization problem into a single-objective, deterministic problem. A subsequent study of the Karush-Kuhn-Tucker optimality condition shows that the method provides optimal or H-optimal solutions.

1. Introduction

The modeling of practical problems in the real world has been widely applied to many research areas, including game theory, economics, finance, and image processing, among others. Optimization problems often arise when modeling certain aspects in all these fields. The most commonly encountered optimization problems are either single-objective [20] or multi-objective [4, 25–27]. As for the coefficients, they are often deterministic [21], stochastic [7], interval-varying [6, 28, 29, 32], or fuzzy [24]. However, there are situations in which the coefficients of the optimization problem are sets. D. Kuroiwa [14–16] and D. Kuroiwa et al. [17] were the first to address this type of optimization problem.

This work has inspired researchers to take an interest in this type of problem. We can mention, among others, [13], whose research has contributed to the proposal of numerous concepts in the theory of set comparison. When it [10] was, its work made it possible to transform a set optimization problem into a bi-objective optimization problem using the vectorization technique. The study in [11] focused on directional derivatives in set optimization with the set less than or equal to order relation. The optimality conditions, existence theorems, and non-convex scalarization in set optimization problems were studied in [1,2,8,9]. [18] has also designed an algorithm to solve set optimization problems in polyhedral convex games, and [3] studied the convergence of the solution sets for set optimization problems. The well-posed problem issue and the Karush-Kuhn-Tucker conditions in set optimization have been studied in [12] and [19], respectively. The concept of set optimization was used in [5] to model the optimization of photovoltaic power plant layout.

2020 *Mathematics Subject Classification.* 03E75, 90C20, 90C25, 90C46, 90C48, 90C99.
Key words and phrases. Minimal element, Maximal element, Hukuhara difference,Partial ordering, KKT-conditions.

The major drawback of all these works lies in manipulating the differences between sets. The elements, or coefficients, of the optimization problem are sets; therefore, the difference of an element does not necessarily equal zero. Inspired by this, Wu [30] introduced the concept of a null set in hyperspace, which consists of all nonempty subsets of a given normed space. He defined two partial orders based on algebraic and Hukuhara differences between any two elements of hyperspace using these concepts. These orders allowed him to solve set optimization problems. Conversely, he transformed set optimization problems into classical bi-objective optimization problems. To solve the bi-objective problem, he uses the scalarization technique. His work shows that the optimal solution to the scalarized problem is also the optimal solution to the original set optimization problem.

However, Wu's [30] work is limited to the linear case only. Therefore, this work proposes an extension to the nonlinear case. Specifically, we propose studying a quadratic set optimization case. First, we will transform the quadratic set optimization problem into a deterministic quadratic bi-objective optimization problem using null sets. Next, we will use a scalarization technique to convert the bi-objective problem into a deterministic mono-objective problem. Finally, we will propose an equivalence study between the solutions of the bi-objective problem and the initial problem, as well as between the solutions of the bi-objective problem and the solutions of the scalarized problem.

To better present our results, in Section 2, we will introduce the fundamental elements necessary for understanding the work. Section 3 will present our main results. Section 4 will conclude with a summary.

2. Preliminary

2.1. **Set analysis.** This part presents the concept of null sets and some of their properties.

Definition 2.1 ([28, 30, 31])**.** Let $(\mathcal{T}, ||\bullet||)$ be a normalized space and $\Xi_{cc}(\mathcal{T})$ be a collection of all compact and convex sets of $\mathcal{T}$. Let $A, B \in \Xi_{cc}(\mathcal{T})$. We have:

i. $\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A} \text{ and } b \in \mathcal{B}\}$,
ii. $\lambda \odot \mathcal{A} = \{\lambda a \mid a \in \mathcal{A}\}$ where λ is a real constant,
iii. $\mathcal{A} \ominus \mathcal{B} = A \oplus (-\mathcal{A}) = \{a - b \mid a \in \mathcal{A} \text{ and } b \in \mathcal{B}\}$,

where $\oplus$, $\ominus$, $\odot$ respectively designate the sum, the difference, and the multiplication between sets.

Definition 2.2. Let Θ be a continuous linear function on a set $\mathcal{T}$. We assume that it is increasing.

Let x_1 be the largest value of $\mathcal{T}$ and x_2 the smallest value of $\mathcal{T}$ we have:

$$\sup_{\alpha \in \mathcal{T}} \Theta(\alpha) = \Theta(x_1),$$

and

$$\inf_{\alpha \in \mathcal{T}} \Theta(\alpha) = \Theta(x_2).$$

Proposition 2.3 ([28, 30])**.** *Let $(\mathcal{T}, \|\bullet\|)$ be a normalized space and Θ be a linear continuous function assumed to be increasing on $\mathcal{T}$. Let $\mathcal{A}, \mathcal{B} \in K_{cc}(\mathcal{T})$ and $\lambda \in \mathbb{R}$.*

The following equalities are true.

i.

$$\begin{cases} \sup\limits_{\alpha\in\mathcal{A}\oplus\mathcal{B}} \Theta(\alpha) = \sup\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) + \sup\limits_{\alpha\in\mathcal{B}} \Theta(\alpha), \\ \\ \qquad\qquad and \\ \\ \inf\limits_{\alpha\in\mathcal{A}\oplus\mathcal{B}} \Theta(\alpha) = \inf\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) + \inf\limits_{\alpha\in\mathcal{B}} \Theta(\alpha). \end{cases} \tag{2.1}$$

ii.

$$\sup_{\alpha\in\lambda\mathcal{A}} \Theta(\alpha) = \begin{cases} \lambda. \sup\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) \quad if \quad \lambda \geqslant 0, \\ \\ \lambda. \inf\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) \quad if \quad \lambda < 0. \end{cases} \tag{2.2}$$

iii.

$$\inf_{\alpha\in\lambda\mathcal{A}} \Theta(\alpha) = \begin{cases} \lambda. \inf\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) \ \ if \ \ \lambda \geqslant 0, \\ \\ \lambda. \sup\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) \ \ if \ \ \lambda < 0. \end{cases} \tag{2.3}$$

iv.

$$\begin{cases} \sup\limits_{\alpha\in\mathcal{A}\ominus\mathcal{A}} \Theta(\alpha) = \sup\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) - \inf\limits_{\alpha\in\mathcal{A}} \Theta(\alpha), \\ \\ \qquad\qquad and \\ \\ \inf\limits_{\alpha\in\mathcal{A}\ominus\mathcal{A}} \Theta(\alpha) = \inf\limits_{\alpha\in\mathcal{A}} \Theta(\alpha) - \sup\limits_{\alpha\in\mathcal{A}} \Theta(\alpha). \end{cases} \tag{2.4}$$

v.

$$\sup_{\alpha\in\lambda\mathcal{A}} \Theta(\alpha) + \inf_{\alpha\in\lambda\mathcal{A}} \Theta(\alpha) = \lambda(\sup_{\alpha\in A} \Theta(\alpha) + \inf_{\alpha\in\mathcal{A}} \Theta(\alpha)). \tag{2.5}$$

Remark 2.4. $\Xi_{cc}(\mathcal{T})$ is not a vector space because for all $\mathcal{A} \in \Xi_{cc}(\mathcal{T})$ and $\mathcal{B} \in \Xi_{cc}(\mathcal{T})$ we have $\mathcal{A} \oplus \mathcal{B} \notin \Xi_{cc}(\mathcal{T})$ and, for all real λ, we have $\lambda\mathcal{A} \notin \Xi_{cc}(\mathcal{T})$.

Let $\theta_{\mathcal{T}}$ be the zero element of the normalized space $\mathcal{T}$; it can be considered as the zero element of $\Xi_{cc}(\mathcal{T})$, then that $\mathcal{A} \oplus \{\theta_{\mathcal{T}}\} = \mathcal{A}$. In other words, since $\mathcal{A} \ominus \mathcal{A} \neq \{\theta_{\mathcal{T}}\}$, this means that $\mathcal{A} \ominus \mathcal{A}$ is not the zero element of $\Xi_{cc}(\mathcal{T})$. In other words, the addition of the inverse elements of $\mathcal{A}$ in $\Xi_{cc}(\mathcal{T})$ does not exist.

Definition 2.5 ([28, 30])**.** The null set of $\Xi_{cc}(\mathcal{T})$ is defined by:

$$\Omega = \{\mathcal{A} \ominus \mathcal{A} \mid \mathcal{A} \in \Xi_{cc}(\mathcal{T})\}. \tag{2.6}$$

It is considered as $\Xi_{cc}(\mathcal{T})$ the zero element.

Definition 2.6 ([24, 28, 30])**.** Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ of $\Xi_{cc}(\mathcal{T})$ in a vector space V.

i. $\mathcal{L}$ is additive if and only if $\mathcal{L}(\mathcal{A} \oplus \mathcal{A}) = \mathcal{L}(\mathcal{A}) + \mathcal{L}(\mathcal{A})$.
ii. $\mathcal{L}$ is homogeneous if and only if $\mathcal{L}(\lambda\mathcal{A}) = \lambda.\mathcal{L}(\mathcal{A})$.
iii. $\mathcal{L}$ is positively homogeneous if and only if $\mathcal{L}(\lambda\mathcal{A}) = \lambda.\mathcal{L}(\mathcal{A})$ if $\lambda \geq 0$.
iv. $\mathcal{L}$ is linear if and only if it is additive and homogeneous.

Definition 2.7. Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ of $\Xi_{cc}(\mathcal{T})$ in a vector space V. $\mathcal{L}(\lambda^k \mathcal{A}) = \lambda^k.\mathcal{L}(\mathcal{A})$ with $\lambda \geqslant 0$ and $k > 0$, $\mathcal{L}$ is homogeneous of degree k.

Proposition 2.8 ([28,30])**.** *Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ of $\Xi_{cc}(\mathcal{T})$ in a vector space V.*

i. *Suppose $-\mathcal{L}(\omega) = \mathcal{L}(-\omega)$ for all $\omega \in \Omega$. Then $\mathcal{L}(\omega) = \theta_V$ for all $\omega \in \Omega$ with θ_V is the zero element of the vector space V.*
ii. *Suppose $\mathcal{L}(\omega) = \theta_V$ for all $\omega \in \Omega$ and $\mathcal{L}$ is additive. Then $\mathcal{L}(\mathcal{A} \ominus \mathcal{B}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}(\mathcal{B})$ for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$.*
iii. *Assume that $\mathcal{L}$ is additive and that the Hukuhara difference $\mathcal{A} \ominus_H \mathcal{B}$ exists for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$. Then $\mathcal{L}(\mathcal{A} \ominus_H \mathcal{B}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}(\mathcal{B})$.*

Definition 2.9 ([28,30])**.** Let $\mathcal{C}$ be a subset of $\Xi_{cc}(\mathcal{T})$.

$$\mathcal{C} = \left\{ C \in \Xi_{cc}(\mathcal{T}) : \sup_{\alpha \in C} \Theta(\alpha) + \inf_{\alpha \in C} \Theta(\alpha) \geq 0 \right\}. \tag{2.7}$$

$\mathcal{C}$ is convex if and only if $\lambda \mathcal{A} \oplus (1-\lambda)\mathcal{B} \in \mathcal{C}$ for all $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ and $\lambda \in [0,1]$.

$\mathcal{C}$ is a cone if and only if $\lambda A \in \mathcal{C}$ for $\mathcal{A} \in \mathcal{C}$ and $\lambda > 0$.

A cone $\mathcal{C}$ is called a convex cone if and only if it is also convex.

Proposition 2.10 ([28,30])**.** *Consider the function $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ of $\Xi_{cc}(\mathcal{T})$ in a vector space V.*

Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. If $\mathcal{L}$ is additive and positively homogeneous, then the set $\mathcal{L}(\mathcal{C}) = \{\mathcal{L}(\mathcal{A}) | \mathcal{A} \in \mathcal{C}\}$ is a convex cone in a vector space V.

Proof. Assume that $\mathcal{C}$ is a convex cone; that is, if $\mathcal{A}, \mathcal{B} \in \mathcal{C}$, $\lambda \mathcal{A} \oplus \mu \mathcal{B} \in \mathcal{C}$ for $\lambda > 0, \mu > 0$. Consider the positively homogeneous additive function $\mathcal{L}$ of degree k. We have $\mathcal{L}(\lambda \mathcal{A} \oplus \mu \mathcal{B}) \in \mathcal{L}(\mathcal{C})$ then, $\lambda \mathcal{L}(\mathcal{A}) + \mu \mathcal{L}(\mathcal{B}) \in \mathcal{L}(\mathcal{C})$.

Therefore, $\mathcal{L}(\mathcal{C})$ is a convex cone in the vector space V. □

Proposition 2.11 ([30])**.** *Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. Then $\mathcal{A} \oplus \mathcal{B} \in \mathcal{C}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{C}$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathcal{C}$. We have $\mathcal{A} \in \mathcal{C}$, implying that:

$$\sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}} \Theta(\alpha), \quad (*)$$

and for $\mathcal{B} \in \mathcal{C}$, we have:

$$\sup_{\alpha \in \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}} \Theta(\alpha). \quad (**)$$

$(*)$ and $(**)$ give:

$$\begin{aligned} 0 &\leqslant \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}} \Theta(\alpha) + \sup_{\alpha \in \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}} \Theta(\alpha) \\ &= \sup_{\alpha \in \mathcal{A}} \Theta(\alpha) + \sup_{\alpha \in \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}} \Theta(\alpha) \\ &= \sup_{\alpha \in \mathcal{A} \oplus \mathcal{B}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A} \oplus B} \Theta(\alpha). \end{aligned}$$

So $\mathcal{A} \oplus \mathcal{B} \in \mathcal{C}$. □

2.2. **Preference order.**

Definition 2.12 ([28, 30])**.** Let Γ be a subset of $\Xi_{cc}(\mathcal{T})$. For all $A \in \Xi_{cc}(\mathcal{T})$, we write $\mathcal{A} \in^{\Omega} \Gamma$ if and only if $\mathcal{A} \overset{\Omega}{=} F$ for all $F \in \Gamma$.

Definition 2.13. Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$. For $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$, we define two binary relations on $\Xi_{cc}(\mathcal{T})$ as follows:

i. $\mathcal{A} \preceq \mathcal{B}$ if and only if $\mathcal{B} \ominus \mathcal{A} \in \mathcal{C}$.
ii. $\mathcal{A} \preceq_H \mathcal{B}$ if and only if $\mathcal{B} \ominus_H \mathcal{A}$ exist and, $(\mathcal{B} \ominus_H \mathcal{A}) \oplus \omega \in \mathcal{C}$ for all $\omega \in \Omega$.

Proposition 2.14 ([23, 28, 30])**.** *Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$.*

i. *Suppose $\Omega \subseteq \mathcal{C}$, then the binary relation $\preceq$ is reflexive.*
ii. *The relation $\preceq$ is transitive.*
iii. *Let $\mathcal{A}, \mathcal{B} \in \Xi_{cc}(\mathcal{T})$ and $\lambda > 0$, if $\mathcal{A} \preceq \mathcal{B}$ then $\lambda\mathcal{A} \preceq \lambda\mathcal{B}$ i.e. the binary relation $\preceq$ is compatible with scalar multiplication.*
iv. *Let $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E} \in \Xi_{cc}(\mathcal{T})$, if $\mathcal{A} \preceq \mathcal{B}$ et $\mathcal{D} \preceq \mathcal{E}$ then $\mathcal{A} \oplus \mathcal{D} \preceq \mathcal{B} \oplus \mathcal{E}$ that is, the binary relation is compatible with set addition.*

Proposition 2.15. *Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function of $\Xi_{cc}(\mathcal{T})$ in a vector space V, let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$, then we have the following propositions:*

i. *Suppose $\{\theta_X\} \in \mathcal{C}$, then the binary relation $\preceq_H$ is reflexive in $\Xi_{cc}(\mathcal{T})$.*
ii. *The binary relation $\preceq_H$ is transitive.*
iii. *The binary relation $\preceq_H$ is compatible with multiplication by a scalar in $\Xi_{cc}(\mathcal{T})$.*
iv. *The binary relation $\preceq_H$ is compatible with set addition in $\Xi_{cc}(\mathcal{T})$.*

Definition 2.16 ([28, 30])**.** Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be a function of $\Xi_{cc}(\mathcal{T})$ in a vector space V. The kernel of $\mathcal{L}$ is defined by:

$$\ker \mathcal{L} = \{\mathcal{A} : \mathcal{L}(\mathcal{A}) = \theta_V\},$$

where θ_V is the zero element of the vector space. It is obvious that

$$\mathcal{L}(\omega) = \theta_V \text{ for all } \omega \in \Omega \text{ if and only if } \Omega \subseteq Ker\mathcal{L}.$$

Let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$, and $\mathcal{L}$ be an additive and positively homogeneous function, then $\mathcal{L}(\mathcal{C})$ is a convex cone in a vector space V.

So we can define two binary relations $\preceq$ and $\preceq_H$ on $\mathcal{L}(\Xi_{cc}(\mathcal{T})) \subseteq V$ as follows:

i. $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$ if and only if $\mathcal{L}(\mathcal{B}) - \mathcal{L}(\mathcal{A}) \in \mathcal{L}(\mathcal{C})$.
ii. $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ if and only if $\mathcal{L}(\mathcal{B}) - \mathcal{L}(\mathcal{A}) \in \mathcal{L}(\mathcal{C})$ and $\mathcal{B} \ominus_H \mathcal{A}$ exist.

It is obvious that $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ implies $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$.

Proposition 2.17 ([28, 30])**.** *Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function of $\Xi_{cc}(\mathcal{T})$ in a vector space V and let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$, then we have the following propositions:*

i. *Suppose $\{\theta_X\} \in \mathcal{C}$, then the binary relation $\preceq_H$ is reflexive in $\Xi_{cc}(\mathcal{T})$.*
ii. *The binary relation $\preceq_H$ is transitive.*
iii. *The binary relation $\preceq_H$ is compatible with multiplication by a scalar in $\Xi_{cc}(\mathcal{T})$.*
iv. *The binary relation $\preceq_H$ is compatible with set addition in $\Xi_{cc}(\mathcal{T})$.*

Proposition 2.18 ([22, 28, 30]). *Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function of $\Xi_{cc}(\mathcal{T})$ in a vector space V, and let $\mathcal{C}$ be a convex cone in $\Xi_{cc}(\mathcal{T})$.*

Assume that $\Omega \subseteq ker\mathcal{L}$ then we have the following results:

i. *$\mathcal{A} \preceq \mathcal{B}$ implies that $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$, and $\mathcal{A} \preceq_H \mathcal{B}$ implies t $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$.*

ii. *Assume that $ker\mathcal{L} \subseteq \mathcal{C}$ then $\mathcal{L}(\mathcal{A}) \preceq \mathcal{L}(\mathcal{B})$ implies $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{L}(\mathcal{A}) \preceq_H \mathcal{L}(\mathcal{B})$ implies that $\mathcal{A} \preceq_H \mathcal{B}$.*

2.3. **Set optimization.** Let U be a vector space and $f : U \to \Xi_{cc}(\mathcal{T})$ be a set-valued quadratic function.

$$\begin{cases} \min f(x) \\ x \in G \end{cases} \tag{2.8}$$

where G is the space of feasible solutions of a subset of U.

Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function of $\Xi_{cc}(\mathcal{T})$ in a vector space V. Then, we consider the compound function $\mathcal{L}of : U \to V$, the function of $\mathcal{L}$ and f.

$$\begin{cases} \min(\mathcal{L}of)(x) \\ x \in G. \end{cases} \tag{2.9}$$

The problem (2.9) is a deterministic bi-objective problem.

Let ϕ be a linear function defined on $\mathbb{R}^2$. It transforms the problem (2.9) into a single-objective problem.

$$\begin{cases} \min \phi(\mathcal{L}of)(x) \\ x \in G. \end{cases} \tag{2.10}$$

3. Main results

3.1. **Mathematical formulation of the problem.** This section is devoted to set-valued quadratic optimization with set-valued linear inequality constraints.

Let $f : U \to \Xi_{cc}(\mathcal{T})$ be a set-valued quadratic function. Let $\mathcal{A}_{ij}$ and $\mathcal{B}_i$ be non-empty subsets of a normalized space $(X, ||\bullet||)$ for $i = 1, 2 \cdots, n$ and $j = 1, 2 \cdots, n$.

Then, the following quadratic function is defined:

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \mathcal{A}_{ij} \oplus \sum_{i=1}^{n} x_i \mathcal{B}_i.$$

Thus, a set-valued quadratic optimization problem can be reformulated as follows:

$$\begin{cases} \min f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \mathcal{A}_{ij} \oplus \sum_{i=1}^{n} x_i \mathcal{B}_i \\ g_k(x) \preceq 0, k = 1, 2, \cdots, m \\ x \in \mathbb{R}^n_+ \end{cases} \tag{3.1}$$

and

$$\text{(3.2)} \qquad \begin{cases} \min f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \mathcal{A}_{ij} \oplus \sum_{i=1}^{n} x_i \mathcal{B}_i \\ g_k(x) \preceq_H 0, k = 1, 2, \cdots, m \\ x \in \mathbb{R}_+^n. \end{cases}$$

Definition 3.1. The decision space, or the feasible set of solutions, is defined as follows:

$$G = \left\{ x \in \mathbb{R}_+^n : \ \ g_k(x) \preceq_H 0, k = 1, 2, \cdots, m \right\}.$$

Let

$$\mathcal{F} = f(G) = \{f(x) : \ \ x \in G\} \subseteq \Xi_{cc}(\mathcal{T})$$

be the objective space and, $\mathcal{C}$ be a convex cone in $\Xi_{cc}\mathcal{T})$ defined in Definition 2.9.

Inspired by the solutions proposed by Wu [30], we observe that:

- $MIN_{\mathcal{C}}(\mathcal{F}) = \{f(x^*), \ \ \nexists\, x \in G : \ \ f(x) \preceq f(x^*)\}$.
- $H - MIN_{\mathcal{C}}(\mathcal{F}) = \{f(x^*), \ \ \nexists\, x \in G : \ \ f(x) \preceq_H f(x^*)\}$.

Definition 3.2 ([30])**.**

(1) x^* is an optimal solution of (3.1) if and only if $f(x^*) \in MIN_{\mathcal{C}}(\mathcal{F})$.
(2) x^* is an $H-$optimal solution of (3.2) if and only if $f(x^*) \in H - MIN_{\mathcal{C}}(\mathcal{F})$.

3.2. **Vectorization.** This technique transforms a set-valued quadratic optimization problem into a deterministic bi-objective quadratic optimization problem.

Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function of $\Xi_{cc}(\mathcal{T})$ in a vector space V. Then we consider the composite function $\mathcal{L}of : U \to V$ of $\mathcal{L}$ and f.

$$\text{(3.3)} \qquad \begin{cases} \min(\mathcal{L}of)(x) \\ (\mathcal{L}og_k)(x) \leqslant \mathcal{L}(0), k = 1, 2, \cdots, m \\ x \in \mathbb{R}_+^n \end{cases}$$

and

$$\text{(3.4)} \qquad \begin{cases} \min(\mathcal{L}of)(x) \\ (\mathcal{L}og_k)(x) \leqslant_H \mathcal{L}(0), k = 1, 2, \cdots, m \\ x \in \mathbb{R}_+^n, \end{cases}$$

where $\mathcal{L}of$ is the function composed with $\mathcal{L}of)(x) = (f_1(x), f_2(x))$ and f_1, f_2 are two naturally conflicting quadratic functions.

Inspired by the concept of the solutions used by Wu [30], let us note:

- $MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F})) = \{(\mathcal{L}of)(x^*), \ \ \nexists\, x \in G : \ \ (\mathcal{L}of)(x) \preceq (\mathcal{L}of)(x^*)\}$.
- $H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F})) = \{(\mathcal{L}of)(x^*), \ \ \nexists\, x \in G : \ \ (\mathcal{L}of)(x) \preceq_H (\mathcal{L}of)(x^*)\}$.

3.2.1. *$f_1(x), f_2(x)$ determination.*

Definition 3.3.
Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to \mathbb{R}^2$ with:

$$\mathcal{L}(A) = \left(-\sup_{\alpha \in A} \Theta(\alpha) - \inf_{\alpha \in A} \Theta(\alpha), \sup_{\alpha \in A} \Theta(\alpha) + \inf_{\alpha \in A} \Theta(\alpha) \right).$$

Consider the following set-valued quadratic function:

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \mathcal{A}_{ij} \oplus \sum_{i=1}^{n} x_i \mathcal{B}_i \equiv C(x).$$

We have

$$(\mathcal{L} o f)(x) = \left(-\sup_{\alpha \in C(x)} \Theta(\alpha) - \inf_{\alpha \in C(x)} \Theta(\alpha), \sup_{\alpha \in C(x)} \Theta(\alpha) + \inf_{\alpha \in C(x)} \Theta(\alpha) \right),$$

with

$$\sup_{\alpha \in C(x)} \Theta(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sup_{\alpha \in \mathcal{A}_{ij}} \Theta(\alpha) \right) x_i x_j + \sum_{i=1}^{n} \left(\sup_{\alpha \in \mathcal{B}_i} \Theta(\alpha) \right) x_i$$

and

$$\inf_{\alpha \in C(x)} \Theta(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\inf_{\alpha \in \mathcal{A}_{ij}} \Theta(\alpha) \right) x_i x_j + \sum_{i=1}^{n} \left(\inf_{\alpha \in \mathcal{B}_i} \Theta(\alpha) \right) x_i.$$

So

$$\begin{aligned} f_2(x) &= \sup_{x \in C(x)} \Theta(\alpha) + \inf_{x \in C(x)} \Theta(\alpha) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sup_{\alpha \in \mathcal{A}_{ij}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}_{ij}} \Theta(\alpha) \right) x_i x_j + \sum_{i=1}^{n} \left(\sup_{\alpha \in \mathcal{B}_i} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}_i} \Theta(\alpha) \right) x_i, \end{aligned}$$

and

$$\begin{aligned} f_1(x) &= -\sup_{x \in C(x)} \Theta(\alpha) - \inf_{x \in C(x)} \Theta(\alpha) \\ &= -\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sup_{\alpha \in \mathcal{A}_{ij}} \Theta(\alpha) + \inf_{\alpha \in \mathcal{A}_{ij}} \Theta(\alpha) \right) x_i x_j - \sum_{i=1}^{n} \left(\sup_{\alpha \in \mathcal{B}_i} \Theta(\alpha) + \inf_{\alpha \in \mathcal{B}_i} \Theta(\alpha) \right) x_i. \end{aligned}$$

3.2.2. *Constraints vectorization.* $(\mathcal{L} o g)(x) = \left(g_k^1(x), g_k^2(x)\right)$ and set $g_k(x) = D(x)$. There is

$$g_k^1(x) = -\sup_{\alpha \in D(x)} \Theta(\alpha) - \inf_{\alpha \in D(x)} \Theta(\alpha) \text{ and } g_k^2(x) = \sup_{\alpha \in D(x)} \Theta(\alpha) + \inf_{\alpha \in D(x)} \Theta(\alpha)$$

with $\mathcal{L}(0) = (0, 0)$.
We know that:

$$\begin{aligned} g_k(x) \preceq 0 &\Rightarrow \mathcal{L}(g_k(x)) \leqslant \mathcal{L}(0), \\ &\Rightarrow \left(g_k^1(x), g_k^2(x)\right) \leqslant (0, 0), \end{aligned}$$

$$\Rightarrow g_k^1(x) \leqslant 0 \ \ and \ \ g_k^2(x) \leqslant 0.$$

As $g_k^1(x) \leqslant g_k^2(x)$ then $g_k^1(x) \leqslant g_k^2(x) \leqslant 0$ so, we can consider the constraints: $g_k^2(x) \leqslant 0$ with $k = 1, 2, \cdots, m$.

We proceed in the same way for the constraints $g_k(x) \preceq_H 0$.

The nonlinear deterministic quadratic optimization problem can be reformulated as follows:

$$\begin{cases} \min(\mathcal{L}of)(x) = (f_1(x), f_2(x)) \\ g_k^2(x) \leqslant 0 \qquad k = 1, 2, \cdots, m \\ x \in \mathbb{R}_+^n \end{cases} \tag{3.5}$$

and

$$\begin{cases} \min(\mathcal{L}of)(x) = (f_1(x), f_2(x)) \\ g_k^2(x) \leqslant_H 0 \qquad k = 1, 2, \cdots, m \\ x \in \mathbb{R}_+^n \end{cases} \tag{3.6}$$

Definition 3.4 ([30])**.** i. x^* is a Pareto optimal solution of (3.5) if and only if $(\mathcal{L}of)(x^*) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$.

ii. x^* is an $H-$Pareto optimal solution of (3.6) if and only if $(\mathcal{L}of)(x^*) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{C}))$.

Remark 3.5. f_1, f_2 are continuous on U and have respective partial derivatives $\dfrac{\partial f_1}{\partial x_i}$ and $\dfrac{\partial f_2}{\partial x_i}$ for $i = \overline{1, n}$. They are $\mathbf{C^1}$ classes.

Proposition 3.6. *Let f be a set-valued quadratic function defined on U. If f is continuously differentiable at x^*, then quadratic functions with deterministic values are continuously differentiable at x^*.*

Proposition 3.7 ([30])**.** *Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function of $\Xi_{cc}(\mathcal{T})$ in a vector space V and let $\mathcal{C}$ be a convex cone of $\Xi_{cc}(\mathcal{T})$. Assume that $\Omega \subseteq ker\mathcal{L} \subseteq \mathcal{C}$.*

i. *x^* is an optimal solution of (3.1) if and only if x^* is a Pareto optimal solution of (3.5).*

ii. *x^* is an $H-$optimal solution of (3.2) if and only if x^* is an $H-$Pareto optimal solution of (3.6).*

Proof. i. Assume that x^* is an optimal solution of (3.1).

Let $A^* = f(x^*)$.

Let $\mathcal{F}$ be a subset of $\Xi_{cc}(\mathcal{T})$ and $A^* \in \mathcal{F}$, as $\Omega \subseteq ker\ mathcalL \subseteq \mathcal{C}$, we have $A^* \in MIN_{\mathcal{C}}(\mathcal{F})$ which implies that $\mathcal{L}(A^*) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$ i.e. $\mathcal{L}(f(x^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using the Definition 3.4, x^* is a Pareto optimal solution of (3.5).

Conversely, x^* is a Pareto optimal solution of (3.5). By definition we have $\mathcal{L}(f(x^*)) \in MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using the Definition 3.2 we have: $f(x^*) \in MIN_{\mathcal{L}}(\mathcal{F})$ and by definition x^* is an optimal solution of (3.1).

ii. Assume that x^* is an $H-$optimal solution of (3.2).

Let $A^* = f(x^*)$.

Let $\mathcal{F}$ be a subset of $\Xi_{cc}(\mathcal{T})$ and $A^* \in \mathcal{F}$, as $\Omega \subseteq ker\ mathcalL \subseteq \mathcal{C}$ we have $A^* \in H - MIN_{\mathcal{C}}(\mathcal{F})$ which implies that $\mathcal{L}(A^*) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$ i.e. $\mathcal{L}(f(x^*)) \in H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using the Definition 3.4, x^* is a $H-$ Pareto optimal solution of (3.6).

Conversely, x^* is a $H-$Pareto optimal solution of (3.6) by definition we have $\mathcal{L}(f(x^*))\ inH - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Using the Definition 3.2 we have: $f(x^*) \in H - MIN_{\mathcal{C}}(\mathcal{F})$ and by definition x^* is an $H-$optimal solution of (3.2). □

3.3. **Scalarization.** The aggregation process transforms the problem (3.6) into a single-objective optimization problem using a scalarization function.

Definition 3.8. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$.

The scalarization function is defined as follows:

$$\phi(x, y) = \lambda_1 x + \lambda_2 y + k$$

where $k > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ are constants, with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 < \lambda_2$.

Using scalarization, we obtain:

$$\begin{aligned}\phi((\mathcal{L}of)(x)) &= \phi\left(-\sup_{x\in C(x)}\Theta(\alpha) - \inf_{x\in C(x)}\Theta(\alpha),\ \sup_{x\in C(x)}\Theta(\alpha) + \inf_{x\in C(x)}\Theta(\alpha)\right)\\ &= (\lambda_2-\lambda_1)\left[\sup_{x\in C(x)}\Theta(\alpha) + \inf_{x\in C(x)}\Theta(\alpha)\right] + k,\\ &= (\lambda_2-\lambda_1)\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sup_{\alpha\in\mathcal{A}_{ij}}\Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_{ij}}\Theta(\alpha)\right)x_ix_j + \sum_{i=1}^{n}\left(\sup_{\alpha\in\mathcal{B}_i}\Theta(\alpha) + \inf_{\alpha\in\mathcal{B}_i}\Theta(\alpha)\right)x_i\right] + k.\end{aligned}$$

As $\lambda_1 < \lambda_2$ and $k > 0$ then, $\phi((\mathcal{L}of)(x)) > 0$. Therefore, $\phi \in \mathcal{L}(\mathcal{C})$ with

$$\begin{aligned}\mathcal{L}(\mathcal{C}) &= \left\{\left(-\sup_{\alpha\in C}\Theta(\alpha) - \inf_{\alpha\in C}\Theta(\alpha), \sup_{\alpha\in C}\Theta(\alpha) + \inf_{\alpha\in C}\Theta(\alpha)\right) \in \mathbb{R}^2 : \sup_{\alpha\in C}\Theta(\alpha) + \inf_{\alpha\in C}\Theta(\alpha) \geqslant 0\right\}\\ &\subseteq \left\{(-x, x) \in \mathbb{R}^2 : x \geqslant 0\right\}.\end{aligned}$$

The following deterministic, single-objective optimization problem is obtained:

$$(3.7)\quad \begin{cases}\min \phi((\mathcal{L}of)(x))\\ \quad = (\lambda_2-\lambda_1)\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sup_{\alpha\in\mathcal{A}_{ij}}\Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_{ij}}\Theta(\alpha)\right)x_ix_j\right.\\ \qquad\qquad \left. + \sum_{i=1}^{n}\left(\sup_{\alpha\in\mathcal{B}_i}\Theta(\alpha) + \inf_{\alpha\in\mathcal{B}_i}\Theta(\alpha)\right)x_i\right] + k,\\ s.t :\\ g_k^2(x) \leqslant 0, \quad k = 1, 2, \cdots, m,\\ x \in \mathbb{R}_+^n,\end{cases}$$

where k is a constant that can be ignored.

$$(3.8)\quad \begin{cases} \min \phi((\mathcal{L}of)(x)) \\ \qquad = (\lambda_2 - \lambda_1)\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sup_{\alpha\in\mathcal{A}_{ij}}\Theta(\alpha) + \inf_{\alpha\in\mathcal{A}_{ij}}\Theta(\alpha)\right)x_i x_j \right. \\ \qquad\qquad \left. + \sum_{i=1}^{n}\left(\sup_{\alpha\in\mathcal{B}_i}\Theta(\alpha) + \inf_{\alpha\in\mathcal{B}_i}\Theta(\alpha)\right)x_i\right], \\ s.t: \\ g_k^2(x) \leqslant 0, \qquad k = 1, 2, \cdots, m, \\ x \in \mathbb{R}_+^n. \end{cases}$$

Theorem 3.9 ([30])**.** *Let $\mathcal{L} : \Xi_{cc}(\mathcal{T}) \to V$ be an additive and positively homogeneous function, and let $\mathcal{C}$ be a convex cone satisfying $\theta_X \in \mathcal{C}$. Suppose $\Omega \subseteq Ker\mathcal{L} \subseteq \mathcal{C}$. If x^* is an optimal solution to problem (3.8), then x^* is both an optimal solution and an $H-$ optimal solution to problem (3.2).*

Proof. Assume that x^* is not an optimal solution of the problem (3.6).

On a $(\mathcal{L}of)(x^*) \notin H - MIN_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{F}))$. Which means that $(\mathcal{L}of)(x^*)$ is not an $H - minimal$ element of $\mathcal{L}(\mathcal{F})$ based on the binary relation $\preceq_H$.

According to the definition of the element $H - minimal$, there exists $x \in G$ such that $(\mathcal{L}of)(x) \preceq_H (\mathcal{L}of)(x^*)$ and $(\mathcal{L}of)(x^*) \not\preceq_H (\mathcal{L}of)(x)$. Since $(\mathcal{L}of)(x^*) \preceq_H (\mathcal{L}of)(x)$ by Proposition 2.17, it follows that $x \neq x^*$. We also have $(\mathcal{L}of)(x^*) - (\mathcal{L}of)(x) \in \mathcal{L}(\mathcal{C})$.

Since $\phi \in \mathcal{L}(\mathcal{C})$, we obtain:

$$\phi((\mathcal{L}of)(x^*)) - \phi((\mathcal{L}of)(x)) = \phi((\mathcal{L}of)(x^*) - (\mathcal{L}of)(x)) \geqslant 0$$

which is a contradiction.

This contradiction implies that x^* is an $H-optimal$ solution to the problem (3.6). Using Proposition 3.7, it follows that x^* is an $H-$ optimal solution to problem (3.5).

On the other hand, considering the binary relation $\preceq$ and using Proposition 3.7, we can demonstrate similarly that x^* is an optimal solution to problem (3.5).

□

3.4. **Karush-Kuhn-Tucker optimality condition.** This step uses the Karush-Kuhn-Tucker optimality condition to find an optimal or H-optimal solution to a deterministic, single-objective problem.

We only consider quadratic functions with convex set values. Since the constraints are linear, the set G is convex.

The following theorems about Karush-Kuhn-Tucker optimality conditions are sufficient.

Theorem 3.10. *Let $\Omega \subseteq Ker\mathcal{L} \subseteq \mathcal{C}$. Suppose the set-valued objective function is convex and continuously differentiable at x^*. If there are also Lagrange multipliers μ_k with $k = \overline{1, m}$ such that:*

i. $\nabla\phi((\mathcal{L}of)(x^*)) + \sum_{k=1}^{m}\mu_k\nabla g_k^2(x^*) = 0$, *for $k = \overline{1, m}$;*

ii. $\mu_k g_k^2(x^*) = 0$;

then x^* is an optimal solution.

Proof. Let us prove the theorem absurdly. Suppose that x^* is not an optimal solution. Then there exists $x \in G$ such that:

$$f(x) \preceq f(x^*) \Rightarrow \mathcal{L}(f(x)) \preceq \mathcal{L}(f(x^*))$$

$$\Rightarrow (f_1(x), f_2(x)) \preceq (f_1(x^*), f_2(x^*))$$

Let λ_1 and λ_2 be positive real numbers and let $\lambda_1 < \lambda_2$ therefore which implies that $\phi((\mathcal{L}of)(x)) < \phi((\mathcal{L}of)(x^*))$.
Let f be a convex and continuously differentiable function at x^*, then the functions f_1 and f_2 are also convex and continuously differentiable at x^* according to Remark 3.6. This implies that $\mathcal{L}of$ is convex and continuously differentiable. Moreover $x^* \in G$, we obtain the optimality condition of KKT:

i. $\nabla\phi((\mathcal{L}of)(x^*)) + \sum_{k=1}^{m} \mu_k \nabla g_k^2(x^*) = 0$, for $k = \overline{1,m}$;
ii. $\mu_k g_k^2(x^*) = 0$.

Therefore, x^* is an optimal solution. It follows that x is an optimal solution to the problem. However, this is absurd according to the initial assumption. Therefore, x^* is an optimal solution for set-valued optimization. □

Theorem 3.11. *Let $\Omega \subseteq Ker\mathcal{L} \subseteq \mathcal{C}$. Suppose the set-valued objective function is convex and continuously $H-$ differentiable at x^*. If there are also Lagrange multipliers ν_k with $k = \overline{1,m}$ such that:*

i. $\nabla\phi((\mathcal{L}of)(x^*)) + \sum_{k=1}^{m} \mu_k \nabla g_k^2(x^*) = 0$, *for* $k = \overline{1,m}$,
ii. $\mu_k g_k^2(x^*) = 0$,

then, x^* is an $H-$ optimal solution.

Proof. Let us demonstrate the theorem by absurdity. Suppose that x^* is not an $H-$ optimal solution. Then, there exists $x \in G$ such that:

$$f(x) \preceq_H f(x^*) \Rightarrow \mathcal{L}(f(x)) \preceq_H \mathcal{L}(f(x^*)),$$

$$\Rightarrow (f_1(x), f_2(x)) \preceq_H (f_1(x^*), f_2(x^*)).$$

Let λ_1 and λ_2 be positive real numbers, and let $\lambda_1 < \lambda_2$, therefore which implies that $\phi((\mathcal{L}of)(x)) <_H \phi((\mathcal{L}of)(x^*))$.

Let f be a convex and continuously $H-$ differential function at x^*, then the functions f_1 and f_2 are also convex and continuously $H-$ differential at x^* by Proposition 3.6. This implies that $\mathcal{L}of$ is convex and continuously $H-$ differential. Moreover $x^* \in G$, we obtain the optimality condition of KKT:

i. $\nabla\phi((\mathcal{L}of)(x^*)) + \sum_{k=1}^{m} \mu_k \nabla g_k^2(x^*) = 0$, for $k = \overline{1,m}$,
ii. $\mu_k g_k^2(x^*) = 0$.

So, x^* is an $H-$ optimal solution. It follows that x is an $H-$ optimal solution to the problem. Which is absurd according to the initial assumption. So, x^* is an $H-$ optimal solution to the set-valued optimization problem. □

4. Conclusion

In this paper, set-valued quadratic optimization problems have been studied. This study used a combination of two methods: the concept of null sets and the KKT optimality conditions. First, each quadratic set optimization problem was transformed into a deterministic quadratic optimization problem. Then, the KKT optimality condition was used to find an optimal solution. While the results of this work contribute to the literature on nonlinear set optimization problems, they remain largely theoretical. These results could be used to model uncertainties that are difficult to capture in optimization problems.

References

[1] M. Alonso and L. Rodriguez-Marin, *Set-relations and optimality conditions in set-valued maps,* Nonlinear Anal. **63** (2005), 1167–1179.

[2] M. Alonso and L. Rodriguez-Marin, *Optimality conditions for set-valued maps with set optimization,* Nonlinear Anal. **70** (2009), 3057–3064.

[3] Q. H. Ansari, N. Husaanin and P. K. Sharma, *Convergence of the solution sets for set optimisation problems,* J. Nonlinear Var. Anal. **6** (2022), 165–183.

[4] T. Antczak, *Parametric optimality and duality results for nondifferentiable L-univex multiobjective fractional programming problems,* Commun. Optim. Theory **2024** (2024): 21.

[5] M. Bischoff, J. Jahn and E. Kobis, *Hard uncertainties in multiobjective layout optimization of photovoltaic power plants,* Optimization, **66** (2017), 361–380.

[6] A. Dwivedi, S. K. Mishra, J. Shi and V. Laha, *On optimality and duality for approximate solutions in nonsmooth interval-valued multiobjective semi-infinite programming,* Appl. Anal. Optim. **8** (2024), 145–165.

[7] V. Hajipour, D. D. Caprio, F. J. Santos-Arteaga, A. Amirsahami and S. V. Noshafagh, *A two-stage stochastic programming model for comprehensive risk response action selection: A case study in Industry 4.0,* Expert Syst. Appl. **261** (2025): 125565.

[8] E. Hernandez and L. Rodriguez-Marin, *Existence theorems for set optimization problems,* Nonlinear Anal. **67** (2007), 1276-1736.

[9] E. Hernandez and L. Rodriguez-Marin, *Nonconvex scalarization in set optimization with set-valued maps,* J. Math. Anal. Appl. **325** (2007), 1–18.

[10] J. Jahn, *Vectorization in set optimization,* J. Optim. Theory Appl. **167** (2015), 783–795.

[11] J. Jahn, *Directional derivatives in set optimization with the set less order relation,* Taiwanese J. Math. **19** (2015), 737–757.

[12] J. Jahn, *Karush-Kuhn-Tucker conditions in set optimization,* J. Optim. Theory Appl. **712** (2017), 707–725.

[13] J. Jahn and T. X. D. Ha, *New order relations in set optimization,* J. Optim. Theory Appl. **148** (2011), 209–236.

[14] D. Kuroiwa, *Convexity for set-valued maps,* Appl. Math. Lett. **9** (1996), 97–101.

[15] D. Kuroiwa, *On set-valued optimization,* Nonlinear Anal. **47** (2001), 1395–1400.

[16] D. Kuroiwa, *Existence theorems of set optimization with set-valued maps,* J. Inf. Optim. Sci. **24** (2003), 73–84.

[17] D. Kuroiwa, T. Tanaka and T.X.D. Ha, *On cone convexity of set-valued maps,* Nonlinear Anal. **30** (1997), 1487–1496.

[18] A. Löhne, C. Schrage, *An algorithm to solve polyhedral convex set optimization problems,* Optimization **62** (2013), 131–141.

[19] X. J. Long and J. W. Peng, *AGeneralized B-well-posedness for set optimization problems,* Optimization **157** (2013), 612-623.

[20] J. Maissam, F. Smarandache and T. Fujita, *SNeutrosophic augmented Lagrange multipliers method nonlinear programming problems constrained by Inequalities,* Neutrosophic sets syst. **81** (2025), 741–752.

[21] K. Miettinen, *Nonlinear Multiobjective Optimization,* Int. Ser. Oper. Res. Manag. Sci. 1999.

[22] J. C. Sama and K. Some, *Solving Fuzzy Nonlinear Optimization Problems Using Null Set Concept,* Int. J. Fuzzy Syst. **03** (2023), 674–685.

[23] J. C. Sama and K. Some, *Null set concept for optimal solutions of fuzzy nonlinear optimization problems,* Iran. J. Fuzzy Syst. **21** (2024), 1–18.

[24] J. D. L. Sama, K. Somé and A. Compaoré, *Hybrid approach for solving fuzzy fractional linear optimization problems,* Ann. Fuzzy Math. Inform. **25** (2023), 111–123.

[25] T. H. Pham, *On approximate positively properly efficient solutions in nonsmooth semi-infinite multiobjective optimization problems with data uncertainty*, J. Appl. Numer. Optim. **6** (2024), 271–289.

[26] A. Tougma, K. Somé and A. Compaoré, *Extension of the projected grandient and Armijo's rule concepts for solving convex nonlinear multiobjective optimization problems,* Appl. Anal. Optim. **7** (2023), 263–278.

[27] N. M. Tung and N. M. Cuong, *Second-order optimality conditions for nonsmooth multiobjective optimization subjected to mixed constraints,* Appl. Set-Valued Anal. Optim. **7** (2025), 39–55.

[28] H.-C. Wu, *On interval-valued nonlinear programming problems,* J. Math. Anal. Appl. **338** (2008), 299–316.

[29] H.-C. Wu, *Wolfe Duality for Interval-Valued Optimization,* J. Optim. Theory Appl. **138** (2008), 497–509.

[30] H.-C. Wu, *Solving set optimization problems based on the concept of null set,* J. Math. Anal. Appl. **472** (2019), 1741–176.

[31] H.-C. Wu, *Informal norm in hyperspace and its topological structure,* Mathematics, **7** (2019) 1–20.

[32] J. Zhang, *Optimality condition and Wolfe duality for invex interval-valued nonlinear programming problems,* J. Appl. Math. **2013** (2013): 641345.

Manuscript received July 10 2025
revised November 30 2025

P. LOMPO
Laboratoire de Mathématiques, Informatique et Applications, Université Norbert zongo, Koudougou, Burkina Faso
E-mail address: `monfilsdieudonne@gmail.com`

A. COMPAORE
Laboratoire de Mathématiques, Informatique et Applications, Université Norbert zongo, Koudougou, Burkina Faso
E-mail address: `ab.compaore1@gmail.com`