Applied Analysis and Optimization
Volume 8, Number 3, 2024, 355–370

Yokohama Publishers
ISSN 2189-1664 Online Journal

ON THE DYNAMIC PROPERTIES OF A DISTRIBUTED DELAY DIFFERENTIAL EQUATION

A. M. A. EL-SAYED, S. M. SALMAN, AND M. A. SAAD

Abstract. The study of distributed delay Ricatti differential equations (DD-RDE) holds significant importance in various scientific and engineering fields due to their widespread applications in modeling complex dynamical systems. In this paper, we study a continuous and discrete model of $\alpha-$distributed delay of Riccati differential equation. First, we discuss the stability analysis of the continuous time model, followed by the related discrete RDE deduced by the piecewise constant arguments method. Second, the occurrence of Neimark Sacker discussed. Moreover, we apply a chaos control strategy to stabilize the system. Finally, numerical simulations are performed to verify the theoretical analysis.

1. Introduction

Studying differential equations is crucial for understanding how dynamic systems behave in various scientific fields. Delay differential equations (DDEs) are special types of equations that consider past events to predict current behavior, making them particularly useful in modeling systems where delays are an inherent feature. The study of delay equations began with Wiener in the 1940s [24], who introduced functional differential equations to describe systems with inherent delays . Since then, DDEs have been employed across a wide range of disciplines including biology, chemistry, physics, engineering, and economics.

In biology, DDEs are frequently used to model population dynamics, where the reproduction or growth rate of a population depends on its past states. For example, Banerjee and et al. [3] explored how time delays can lead to oscillations and even chaotic behavior in predator-prey models, providing a deeper understanding of population cycles and stability. Bajiya and et al. [2] extended this analysis to models of disease spread, demonstrating how delays in the immune response can impact the dynamics of SEIR epidemic model.

In chemistry, DDEs have been used to study reaction kinetics where delays represent the time it takes for intermediates to form or for reactants to diffuse. Torbensen and et al. [29] applied DDEs to model chemical oscillators, such as the Belousov-Zhabotinsky reaction, illustrating how delays can give rise to complex temporal patterns, including periodic and chaotic oscillations.

The field of physics has also seen significant contributions from DDEs, particularly in the study of laser dynamics and optical systems. Chembo and et al. [9] utilized DDEs to model the feedback mechanisms in optical resonators, where the delay accounts for the time light takes to travel through the cavity. This approach

2020 *Mathematics Subject Classification.* 37Gxx, 34Dxx, 34Kxx, 39Axx .
Key words and phrases. Distributed delay, stability, discritization, bifurcation, chaos control.

has been essential in understanding phenomena like optical bistability and chaos [21].

In engineering, DDEs are pivotal in control theory, where delays often arise due to signal transmission times in feedback loops. Researchers like Richard, Jean-Pierre [25] have explored the impact of these delays on system stability and control, proposing methods for designing controllers that can compensate for delays and maintain system performance.

In economics, DDEs are used to model dynamic systems where past decisions influence current economic variables. For example, Bianca and et al. [4] investigated the time delays' effects on the qualitative behavior of an economic growth model, while Bischi et al.[5] analyzed the stability of delayed differential game models in oligopolistic competition.

The concept of distributed delay in Delay Differential Equations (DDEs) extends the idea of a discrete delay by considering a delay that is spread over a range of times rather than occurring at a single fixed point. This approach is particularly valuable when modeling systems where the influence of past states is not instantaneous but rather distributed over a continuum of past times [27].

In recent years, there has been a growing interest in the application of distributed delay models in various fields. For example, in ecological modeling, Zhang et al. [32] explored how distributed delays can affect the stability of predator-prey systems, demonstrating that such delays can lead to complex dynamics, including oscillations and chaos. In economics, authors like Guerrini et al. [14] have studied the effects of fixed and continuously distributed delays in a monopoly model with constant price elasticity.

The objective of this paper is to investigate the dynamic properties of $\alpha-$distributed delay Riccati differential equations and provide a comprehensive analysis of their stability, bifurcation behavior, and chaotic dynamics. We aim to elucidate the impact of distributed delays on the qualitative behavior of DD-RDEs.

El-Sayed et al.[12, 13] studied the perturbed delay Riccati equation as in the form

$$\frac{dx(t)}{dt} = 1 - \rho x(t)x(t-r). \tag{1.1}$$

The delayed Riccati differential equation is given by (1.1) is studied in [12, 13]. In this paper, rather than focusing on the discrete delay in equation (1.1), we examine the following Ricatti equation with an $\alpha-$distributed delay as

$$\frac{dx(t)}{dt} = 1 - \rho \left[\int_0^t K(t-s)x(s)ds\right]^2, \tag{1.2}$$

where $\rho > 0$ and $K(t)$ is called the (Tempered delay) delay kernel [26, 30].

Consider the weak kernel

$$K(t) = e^{-\alpha t}, \qquad \alpha > 0.$$

Then (1.2) can be rewritten as:

$$\frac{dx}{dt} = 1 - \rho \left[\int_0^t e^{-\alpha(t-s)}x(s)ds\right]^2, \qquad t \in (0, T], \tag{1.3}$$

$$x(t) = x_0.$$

Assuming

$$y(t) = \int_0^t e^{-\alpha(t-s)} x(s) ds,$$

and using the linear chain trick [28], we obtain the following system

$$\begin{aligned} \frac{dx(t)}{dt} &= 1 - \rho y^2(t), \qquad x(0) = x_0, \\ \frac{dx(t)}{dt} &= x(t) - \alpha y(t), \qquad y(0) = 0. \end{aligned} \tag{1.4}$$

The structure of the paper is as follows. Section (2) presents the stability analysis of the continuous time model (1.4). We discretized the system (1.4) by applying the piecewise continuous arguments method and discuss bifurcation analysis of the discretized system in Section (3). Section (4) offers a numerical simulation to explain the theoretical analysis. In Section (5), we apply the method chaos control for the considered system.

2. The Continuous-time model of (1.4)

Stability analysis aims to determine under what conditions the solutions of the Riccati equation converge or diverge. Stability criteria involve examining the eigenvalues of the associated characteristic equation, which is derived from linearizing the system around its equilibrium points.

2.1. **The fixed points and stability analysis.**

In this section, we investigate the local stability analysis of a continuous model denoted as (1.4). The equilibrium points of problem (1.4) can be obtained by solving the following equations:

$$\begin{aligned} 1 - \rho y^2 &= 0, \\ x - \alpha y &= 0. \end{aligned}$$

The system (4) possesses two equilibrium points, namely $(x_1^*, y_1^*) = (\frac{\alpha}{\sqrt{\rho}}, \frac{1}{\sqrt{\rho}})$ and $(x_2^*, y_2^*) = (-\frac{\alpha}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}})$. Next, we discuss the local stability of these equilibrium points by linearizing the system around them. This process involves approximating the nonlinear system with a linear one in the vicinity of the equilibrium point. The determination of stability behavior relies on the eigenvalues of the Jacobian matrix associated with the system.

To analyze the stability of dynamic continuous systems, the trace-determined method [18, 22] is applied. This method serves as a simple yet powerful tool for assessing the stability of such systems. To determine the stability of these points, we linearize the system around the equilibrium points, the Jacobian matrix of the system is given by

$$J(x^*, y^*) = \begin{bmatrix} 0 & -2\rho y^* \\ 1 & -\alpha \end{bmatrix}. \tag{2.1}$$

The characteristic polynomial of the Jacobian matrix has trace $\tau = -\alpha$ and detereminant $d = 2\rho y^*$.

The eigenvalues of (2.1) are

$$\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\Delta}),$$

where $\Delta = \tau^2 - 4d$.

Lemma 2.1 ([23]). *The fixed points* (x^*, y^*)

(1) *If* $\Delta > 0$, $d > 0$ *and* $\tau < 0$ *the fixed point is stable node.*
(2) *If* $\Delta > 0$, $d > 0$ *and* $\tau > 0$ *the fixed point is unstable node.*
(3) *If* $\Delta < 0$, $d > 0$ *and* $\tau < 0$ *the fixed point is stable spiral.*
(4) *If* $\Delta < 0$, $d > 0$ *and* $\tau > 0$ *the fixed point is unstable.*

For the first equilibrium point (x_1^*, y_1^*) , we calculate the Jacobian at the fixed point:

$$J(x^*, y^*) = \begin{bmatrix} 0 & -2\sqrt{\rho} \\ 1 & -\alpha \end{bmatrix}. \tag{2.2}$$

$$\tau = -\alpha, d = 2\sqrt{rho}$$

Proposition 2.2. (1) *If* $\alpha^2 < 8\sqrt{rho}$ *the eigenvalues for the first equilibrium point are complex conjugates with negative real parts, indicating it is a stable spiral (stable focus).*
(2) *If* $\alpha^2 > 8\sqrt{rho}$ *the eigenvalues for the first equilibrium point are negative real indicating, it is a stable node.*
(3) *if* $\alpha^2 = 8\sqrt{rho}$ *the eigenvalues are real and equal for the first equilibrium point, suggesting a degenerate node (either stable or unstable depending on the direction of perturbation).*

For the second equilibrium point (x_2^*, y_2^*), the trace and determinant of the Jacobian given by

$$\tau = -\alpha, d = -2\sqrt{\rho}$$

The fixed point (x_2^*, y_2^*) is unstable saddle node.

The phase diagram will show trajectories in the x-y plane. In Figure (1) shown that the precise dynamics can be visualized effectively with the phase diagram, which helps in understanding the flow of trajectories in the system's state space.

3. The discrete time model of (1.4)

We can convert the system (1.4) to discrete time system using piecewise constant arguments method [1, 11, 12] as follows:

$$\begin{aligned} \frac{dx(t)}{dt} &= (1 - \rho(y(r[\frac{t}{r}]))^2), \qquad t \in (0, T] \\ \frac{dy(t)}{dt} &= x(r[\frac{t}{r}]) - \alpha y(r[\frac{t}{r}]) \\ x(0) &= x_0, \quad y(0) = y_0, \end{aligned} \tag{3.1}$$

where [.] denotes the greatest integer function and r is a constant argument.

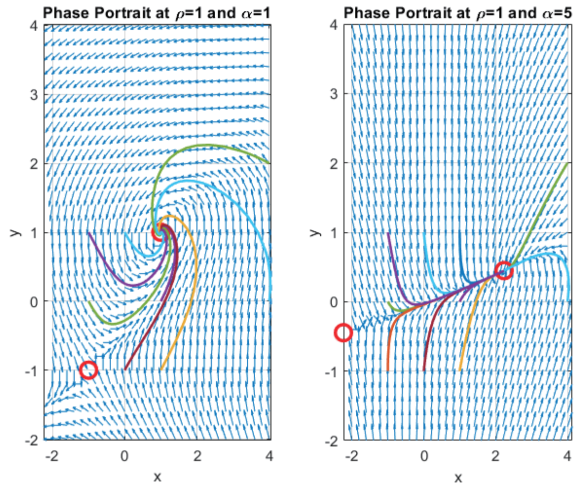


FIGURE 1. The trajectories about the fixed points at $\alpha = 1$ and $\alpha = 5$.

(1) let $t \in [0, r)$ then, $[\frac{t}{r}] = 0$,

$$\frac{dx(t)}{dt} = (1 - \rho y_0^2), \qquad t \in [0, r)$$

and the solution is given by

$$x(t) = x_0 + (1 - \rho y_0^2) \int_0^t 1dt$$
$$= x_0 + (1 - \rho y_0^2)t.$$

similarly,

$$y(t) = y_0 + (x_0 - \alpha y_0) \int_0^t 1dt$$
$$= y_0 + (x_0 - \alpha y_0)t.$$

when $t \to r$, $\quad x(r) = x_1$, we get

$$x_1 = x_0 + r(x_0 - \alpha y_0),$$
$$y_1 = y_0 + r(x_0 - \alpha y_0).$$

(2) let $t = [r, 2r)$ then, $[\frac{t}{r}] = 1$,

$$\frac{dx(t)}{dt} = (1 - \rho y_1^2), \qquad t \in [r, 2r).$$

and the solution of is given by

$$x(t) = x(r) + (1 - \rho y(r))^2 \int_r^t 1ds$$
$$= x(r) + (1 - \rho y(r)^2)(t - r),$$

also,

$$y(t) = y(r) + (x(r) - \alpha y(r))(t - r).$$

When $t \to 2r$ and $x(r) = x_1, y(r) = y_1$ we get

$$x_2 = x_1 + r(1 - \rho y_1^2),$$
$$y_2 = y_1 + r(x_1 - \alpha y_1).$$

repeated this procedure for n iterations to get the following discrete time system:

$$x_{n+1}(t) = x(nr) + (t - nr)(1 - \rho y(nr)^2), \qquad t \in [nr, (n+1)r),$$
$$y_{n+1}(t) = y(nr) + (t - nr)(x(nr) + \alpha y(nr)).$$

let $t \to (n+1)r$, we obtain the discretization as follows:

$$x_{n+1} = x_n + r(1 - \rho y_n^2),$$
$$y_{n+1} = y_n + r(x_n - \alpha y_n), \tag{3.2}$$

where $\alpha, \rho > 0$.

3.1. **3.1 The fixed points and stability analysis of (3.2).** In this section, we discuss the stability analysis of the discrete model. Now, we study the stability of the fixed points. The Jacobian matrix of the system (3.2) is given by

$$J(x, y) = \begin{bmatrix} 1 & -2r\rho y^* \\ r & 1 - \alpha r \end{bmatrix}. \tag{3.3}$$

The characteristic equation of the Jacobian matrix can be written as

$$F(\lambda) = |J - \lambda I| = \lambda^2 + P\lambda + Q = 0, \tag{3.4}$$

where

$$P = -tr(J) = \alpha r - 2,$$

and

$$Q = det(J) = (1 - \alpha r) + 2\rho r^2 y^*.$$

In order to study the modulus of eigenvalues of the characteristic equation (local stability), we first know the following lemma [16], which is the relations between roots and coefficients of the quadratic equation.

Lemma 3.1. [19] *let $F(\lambda) = \lambda^2 + P\lambda + Q = 0$. Suppose that $F(1) > 0$, $\lambda_{1,2}$ are two roots of $F(\lambda) = 0$, then*

- $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $Q < 1$.
- $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *or* ($|\lambda_1| < 1$ *and* $|\lambda_2| > 1$) *if and only if* $F(-1) < 0$.
- $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Q > 1$.
- $\lambda_1 = -1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $P \neq 0, 2$.
- λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $P^2 - 4Q < 0$ *and* $Q = 1$.

For the positive fixed point $(x_1^*, y_1^*) = (\frac{\alpha}{\sqrt{\rho}}, \frac{1}{\sqrt{\rho}})$. The Jacobian matrix be

$$J(x,y) = \begin{bmatrix} 1 & -2r\sqrt{\rho} \\ r & 1-\alpha r \end{bmatrix}. \tag{3.5}$$

The P and Q of the characteristic equation of the Jacobian matrix at the first fixed point can be written as

$$P = -2 + r\alpha,$$

and

$$Q = 1 - r\alpha + 2r^2\sqrt{\rho}.$$

Proposition 3.2. *The first fixed point* (x_1^*, y_1^*)

(1) *It is called sink (asymptotically stable) if* $2r\sqrt{\rho} < \alpha < \frac{2+r^2\sqrt{\rho}}{r}$.
(2) *It is called source if* $\alpha > \max(2r\sqrt{\rho}, \frac{2+r^2\sqrt{\rho}}{r})$.
(3) *It is called saddle if* $\alpha > \frac{2+r^2\sqrt{\rho}}{r}$.
(4) *It is a non hyperbolic if the one of the conditions holds:*
   - $\alpha > 8\sqrt{\rho}$ *and* $\alpha = \frac{2+r^2\sqrt{\rho}}{r}$ *where,* $\alpha \neq \frac{2}{r}, \frac{4}{r}$.
   - $\alpha < 8\sqrt{\rho}$ *and* $\alpha = 2r\sqrt{\rho}$.

Proposition 3.3. *The negative fixed point* (x_2^*, y_2^*) *is unstable.*

3.2. **The Bifurcation analysis.** A Neimark-Sacker bifurcation occurs when a stable fixed point transitions to instability at a specific critical value of the system's bifurcation parameter. This transition leads to the appearance of either an attracting closed invariant curve or the emergence of a repelling closed invariant curve as the parameter values cross this critical threshold. If the bifurcation results in the appearance of an attracting closed invariant curve, it is termed a supercritical Neimark-Sacker bifurcation; conversely, if it leads to the emergence of a repelling closed invariant curve, it is termed a subcritical Neimark-Sacker bifurcation. In both scenarios, such a bifurcation is associated with discrete systems whose eigenvalues are complex conjugates with a modulus of one.

Theorem 3.4. [19] *Consider the family of* C^r *maps* ($r \geq 5$), $F_\mu : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ *such that the following conditions hold:*

- $F_\mu(0) = 0$, *i.e., the origin is a fixed point of* F_μ.
- $DF_\mu(0)$ *has two complex conjugate eigenvalues* $\lambda_{1,2}(\mu) = r(\mu)e^{\pm i\theta(\mu)}$, *where* $r(0) = 1$, $r'(0) \neq 1$, $\theta(0) = \theta_0$, *and* $|\lambda(0)| = 1$.
- $e^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4, 5, \ldots$ *(absence of strong resonances condition).*

If, in addition, $a \neq 0$ *where*

$$a = -Re\left[\frac{(1-2\lambda)\overline{\lambda}^2\zeta_{11}\zeta_{20}}{1-\lambda}\right] - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\overline{\lambda}\zeta_{21}),$$

with a *called the first Lyapunov coefficient, then for sufficiently small* μ, F_μ *there exists a unique invariant closed curve enclosing that bifurcates from the origin as* μ *passes through 0. If* $a > 0$, *we have a supercritical Neimark-Sacker bifurcation. If* $a < 0$, *we have a subcritical Neimark-Sacker bifurcation.*

After simple calculation:

$$|\lambda_{1,2}| = 1 \text{ if } \alpha^2 < 8\sqrt{\rho} \text{ at } \alpha = 2r\sqrt{\rho}.$$

$$|Q| = \lambda_1\lambda_2 = 1$$

$$\lambda_{1,2}(\alpha) = \frac{1}{2}\left(\alpha r - 2 \pm ir\sqrt{8\sqrt{\rho} - \alpha^2}\right),$$

$$\lambda_2 = \overline{\lambda_1},$$

$$|\lambda_1(\alpha)| = |\lambda_2(\alpha)| = \sqrt{Q(\alpha)} = \sqrt{1 - \alpha r + 2r^2\sqrt{\rho}},$$

$$\left|\frac{d|\lambda_1(\alpha)|}{d\alpha}\right|_{\alpha=0} = \left|\frac{d|\lambda_2(\alpha)|}{d\alpha}\right|_{\alpha=0} = -\frac{r}{2\sqrt{1+2r^2\sqrt{\rho}}} < 0.$$

Moreover, it is evident that $\lambda_{1,2}^m(0) \neq 1$ for all $m = 1, 2, 3, 4$ for $b \neq 2$. Thus, all conditions for the Neimark-Sacker bifurcation to occur are satisfied.

Define new variables u and v as perturbations from the fixed point: $u = x - x^*$ and $v = y - y^*$, and take α as a bifurcation parameter. Consider a small perturbation of the parameter δ as follows $\overline{\alpha} = \alpha - \delta$.

Substituting into the original system, we transform the fixed point (x^*, y^*) into $(0, 0)$. The system now has the form:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} 0 & -2\rho y^* \\ 1 & -\alpha + \delta \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} 2\rho v_n^2 \\ 0 \end{pmatrix}.$$

Next, define $a_1 = \frac{\alpha r - 2}{2}$ and $a_2 = \frac{r\sqrt{8\sqrt{\rho} - \alpha^2}}{2}$. These coefficients represent the real and imaginary parts of $\lambda_{1,2}$. Upon finding the eigenvectors associated with these eigenvalues, we construct the following invertible matrix:

$$T = \begin{pmatrix} 2\rho y^* & 0 \\ -a_1 & a_2 \end{pmatrix},$$

$$T^{-1} = \frac{1}{2\rho y^* a_2} \begin{pmatrix} a_2 & 0 \\ a_1 & 2\rho y^* \end{pmatrix}.$$

Consider the transformation:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = T \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ \frac{a_1^2 + 2\rho y^* + a_1(\alpha - \delta)}{a_2} & -a_1 - \alpha + \delta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F(X_n, Y_n) \\ G(X_n, Y_n) \end{pmatrix},$$

where

$$F(X_n, Y_n) = \frac{1}{y^*}(a_2 Y_n - a_1 X_n)^2,$$

$$G(X_n, Y_n) = \frac{a_1}{a_2 y^*}(a_2 Y_n - a_1 X_n)^2.$$

Next, we compute:

$$F_{XX} = \frac{2a_1^2}{y^*}, \quad F_{XXX} = 0, \quad F_{XY} = F_{YX} = -\frac{2a_1 a_2}{y^*}, \quad F_{YY} = \frac{2a_2^2}{y^*}, \quad F_{YYY} = 0,$$

$$G_{XX} = \frac{2a_1^3}{a_2 y^*}, \quad G_{XXX} = 0,$$

$$G_{XY} = -\frac{2a_1^2}{y^*}, \quad G_{YX} = \frac{2a_1a_2}{y^*}, \quad G_{YY} = \frac{2a_1^2}{y^*}, \quad G_{YYY} = 0.$$

Then, we calculate:

$$a = -\mathrm{Re}\Big[\frac{(1-2\lambda)\lambda^{\dagger}\zeta_{11}\zeta_{20}}{1-\lambda}\Big] - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + \mathrm{Re}(\lambda^{\dagger}\zeta_{21}),$$

where

$$\zeta_{20} = \frac{1}{8}(F_{XX} - F_{YY} + 2G_{XY}) + i(G_{XX} - G_{YY} - 2F_{XY}),$$

$$\zeta_{20} = \frac{1}{8\sqrt{\rho}}\left[-2a_1^2 - 2a_2^2 + i\left(\frac{2a_1^3}{a_2} + 2a_1a_2\right)\right],$$

$$\zeta_{02} = \frac{1}{8}(F_{XX} - F_{YY} - 2G_{XY}) + i(G_{XX} - G_{YY} + 2F_{XY}),$$

$$\zeta_{02} = \frac{1}{8\sqrt{\rho}}\left[6a_1^2 - 2a_2^2 + i\left(\frac{2a_1^3}{a_2} - 6a_1a_2\right)\right],$$

$$\zeta_{11} = \frac{1}{4}\left[F_{XX} + F_{YY} + i(G_{XX} + G_{YY})\right],$$

$$\zeta_{11} = \frac{1}{4\sqrt{\rho}}\left[2a_1^2 + 2a_2^2 + i\left(\frac{2a_1^3}{a_2} + 2a_1a_2\right)\right],$$

$$\zeta_{21} = 0.$$

Summarizing the above analysis, we have the following consequence.

Theorem 3.5. *The system (3.2) undergoes a Neimark–Sacker bifurcation at the positive fixed point (x_1^*, y_1^*), when the parameter α varies in a small neighborhood δ.*

4. Numerical results

Here, we use numerical simulation to show the bifurcation diagrams, phase portraits, and Lyapunov exponents of the system (3.2) in order to confirm the earlier findings and to show some more interesting complex dynamical behaviors that arise in systems. It is well known that maximum Lyapunov exponents, which are frequently used to denote chaotic behavior, quantify the exponential divergence of originally close state-space paths. We consider the initial point to be (0.2,0.2) and select the parameter r, α as a bifurcation parameter (varied parameter), taking the other parameters as fixed parameters. The bifurcation parameters are considered as:

For Case 1, the bifurcation and maximal Lyapunov exponent diagrams of the system are shown in Figure (2). From Figures (2a), (2c), and (2e), we note that varying the value of α affects the bifurcation diagrams. In these figures, the fixed point loses stability at $r = 0.115$ for Figure (2a), $r = 0.28$ for Figure (2c), and $r = 0.35$ for Figure (2e), with ρ fixed at 50. The maximal Lyapunov exponent is a useful tool for detecting chaos in the system. As observed in Figure (2b), the system is chaotic for $r > 0.17$. However, Figures (2d) and (2f) show that for $r > 0.38$, some Lyapunov exponents are greater than 0 and some are less than 0, indicating the presence of stable fixed points or stable periodic windows within the chaotic region.

In Figure 3, we discuss the bifurcation and chaotic behavior of the system with respect to changes in r, as referenced in Case 2. In Figure (3a), the system undergoes bifurcation at $r = 0.3$ and, as r increases, the bifurcation decreases and the system becomes more stable, as seen in (3c) and (3e). However, the maximal Lyapunov exponent shown in Figures (3b), (3d), and (3f) indicates that the system becomes more chaotic with increasing r.

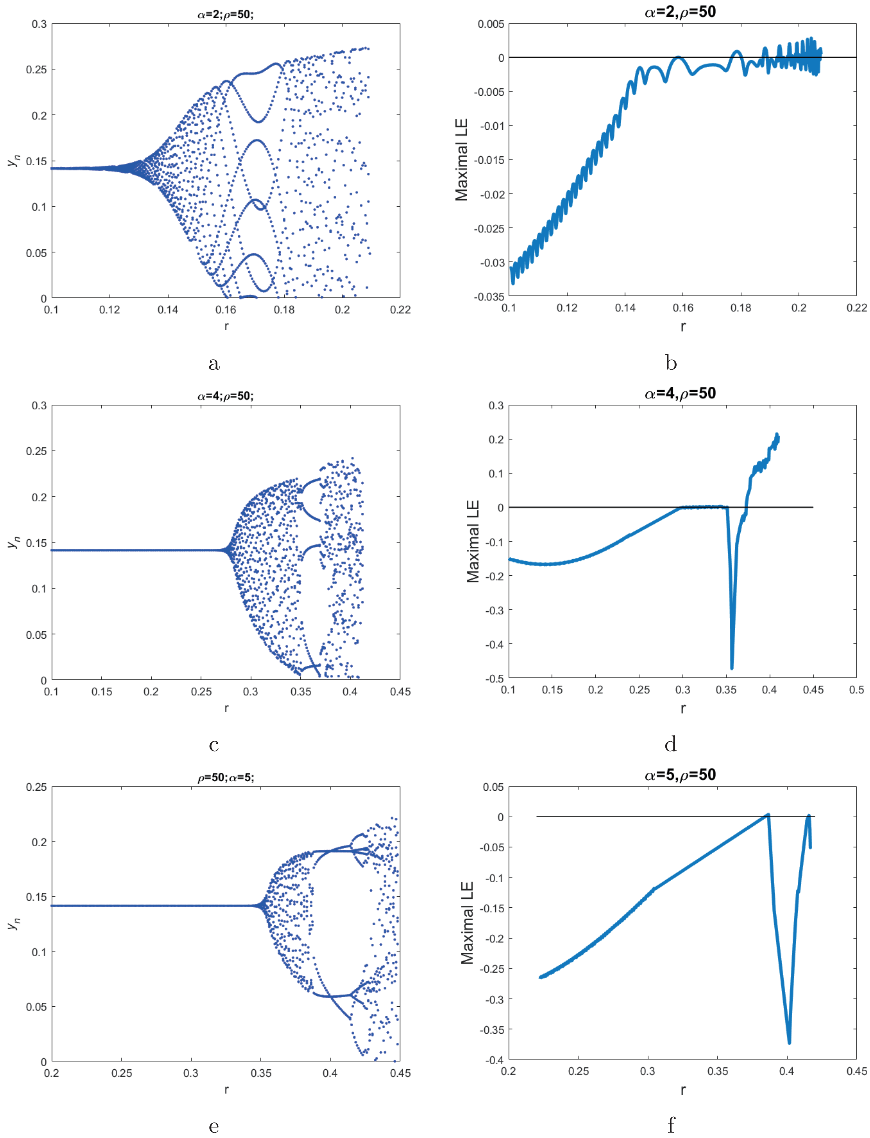


FIGURE 2. The bifurcation diagrams and maximal Lyapunov exponents of the considered systems with varying α.

The phase portrait is shown in the Figure (4), by taken $r = 0.4$ and $\rho = 50$ and the intial point is $(0.1, 0.1)$, we note that When α exceeds 5.7, there appears a trajectories tends to the fixed point and in the interval $[5.2, 5.6]$ theres appear a circular curve enclosing the fixed point (0.7636, 0.1414), and its radius becomes

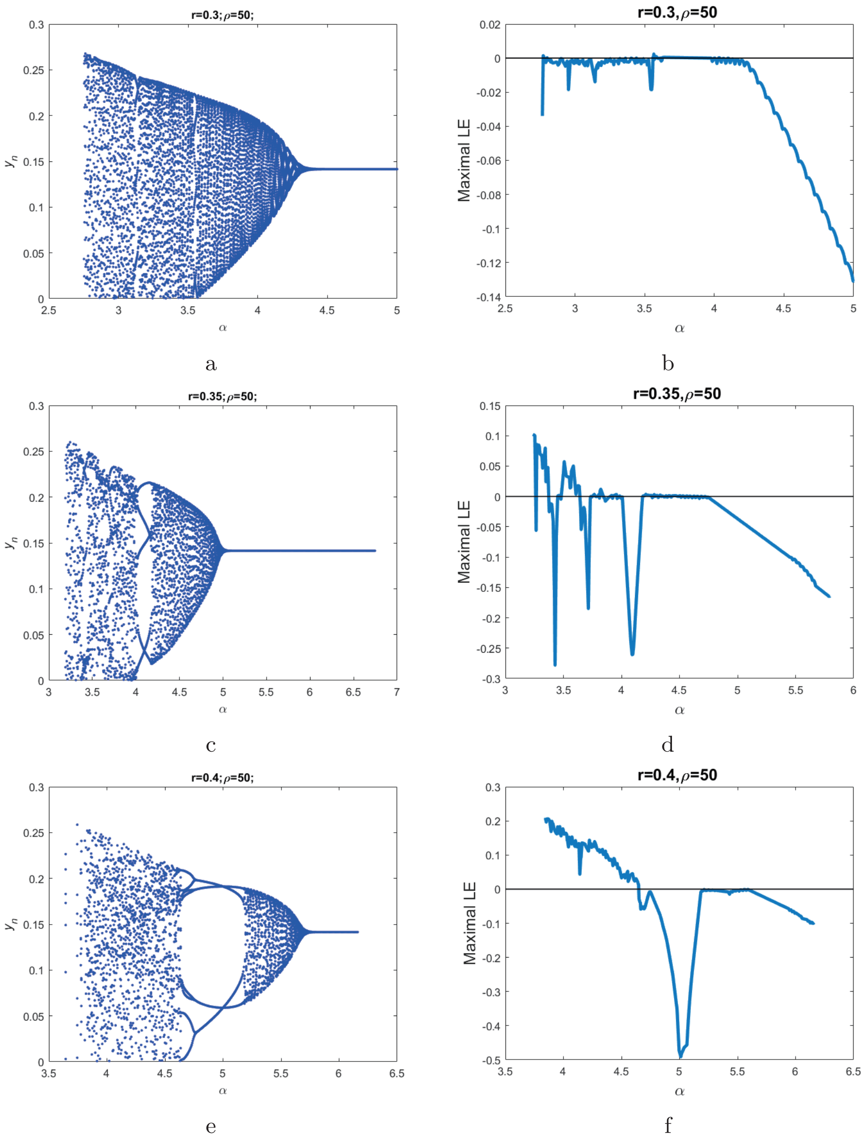


FIGURE 3. The bifurcation diagrams and maximal Lyapunov exponents of the considered systems with varying α

larger with respect to the growth of α. when $\alpha > 5.7$ the system be more stable as in Figure (3).

In the next section, we study the chaos control to get a more stable fixed point.

5. Chaos control

In this section we discuss the chaos control method [8, 16, 23] for the feedback control (3.2), to stabilize chaotic of an unstable fixed point of (3.2). Consider the

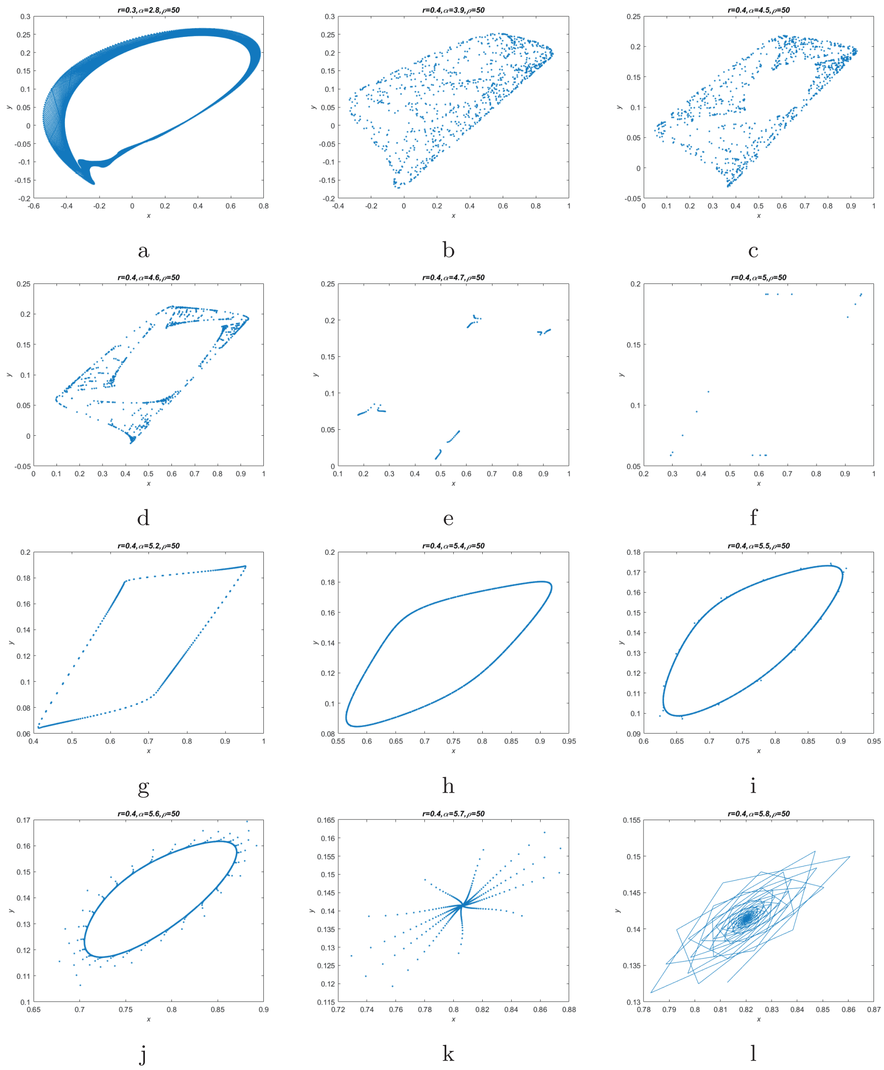


FIGURE 4. The phase diagram of system (3.2) with varying α.

following controlled form of system (3.2):

$$\begin{aligned} x_{n+1} &= x_n + r(1 - \rho y_n^2) + u_n, \\ y_{n+1} &= y_n + r(x_n - \alpha y_n). \end{aligned} \tag{5.1}$$

where, $u_n = -k_1(x_n - x^*) - k_2(y_n - y^*)$ which is the control force, the jacobian matrix of the new feedback control (5.1) is

$$J(x^*, y^*) = \begin{bmatrix} 1 - k_1 & -2\rho r y^* - k_2 \\ r & 1 - \alpha r \end{bmatrix} = \begin{bmatrix} 1 - k_1 & -2r\sqrt{\rho} - k_2 \\ r & 1 - \alpha r \end{bmatrix}, \tag{5.2}$$

then

$$Trac(J) = \lambda_1 + \lambda_2 = 2 - r\alpha - k_1. \tag{5.3}$$

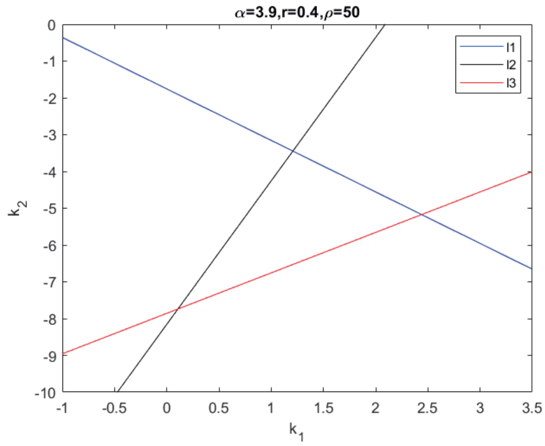


FIGURE 5. The bounded triangle for the stabilize the fixed point for $\alpha = 3.9, r = 0.4, \rho = 50$

$$\lambda_1\lambda_2 = det(J) = (1 - k_1)(1 - \alpha r) + 2r^2\sqrt{\rho} + rk_2. \tag{5.4}$$

The equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$ must be solved in order to get the lines of marginal stability. These requirements ensure that the modulus of the eigenvalues λ_1 and λ_2 is smaller than 1. the three equations as follows: let $\lambda_1\lambda_2 = 1$

$$l_1 : (1 - \alpha r)k_1 - rk_2 = 2r^2\sqrt{\rho} - \alpha r. \tag{5.5}$$

let $\lambda_1 = 1$ in (5.3) and (5.4)

$$l_2 : (\alpha r)k_1 - rk_2 = 1 + 2r^2\sqrt{\rho}. \tag{5.6}$$

let $\lambda_1 = -1$ in (5.3) and (5.4)

$$l_3 : (2 - \alpha r)k_1 - rk_2 = 4 - 2\alpha r + 2r^2\sqrt{\rho}. \tag{5.7}$$

then the stable eigenvalues lies in the triangular region bounded by l_1, l_2 and l_3 in Figure (5). To investigate how the state feedback control the unstable fixed point, we have run a few numerical simulations, where the fixed parameter values are as follows: $\alpha = 3.9, r = 0.4, \rho = 50$. The feedback gain is $k_1 = 1.5, k_2 = -6$, and the starting value is (0.4, 0.1). Figure (6) and (7) illustrates how a chaotic track is brought to a stable point (0.5515, .1414).

6. CONCLUSION

In conclusion, our investigation into the dynamic properties of the Riccati differential equation with distributed delay has yielded several important findings and implications. Through rigorous analysis and numerical simulations, we have uncovered the complex behaviors exhibited by systems governed by this equation, highlighting the intricate interplay between delay and dynamic phenomena.

Our study demonstrates that the distributed delay parameter significantly influences system stability. We observed notable instances of stability loss and the emergence of Neimark-Sacker bifurcations as the delay varied. Additionally, the

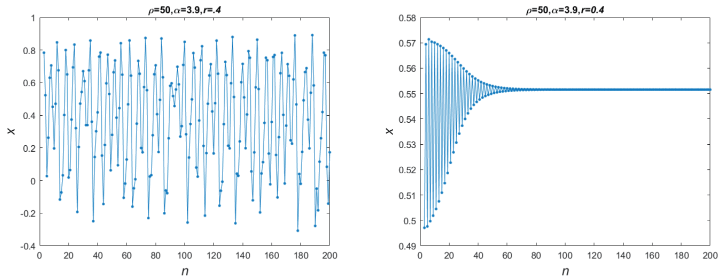
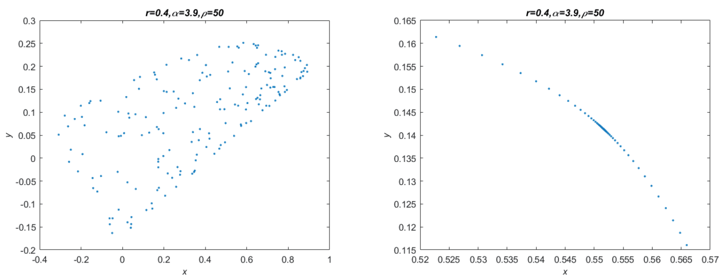


FIGURE 6. The effect of chaos control parameter at fixing parameters $\alpha = 3.9, r = 0.4, \rho = 50$.



a The phase portrait before control b The phase portrait after control

FIGURE 7. The phase diagram for the controlled system with $\alpha = 3.9, r = 0.4, \rho = 50$. for $k_1 = 1.5, k_2 = -6$

presence of distributed delay introduces a range of dynamic behaviors, including amplitude and frequency modulation, which have substantial implications for real-world applications in fields such as biological systems, population dynamics, and control theory.

This work advances the understanding of distributed delay Riccati equations by providing practical methods for discretization, stability analysis, and chaos control. The insights gained contribute to the broader field of dynamic systems and lay the groundwork for future research and application in various domains.

REFERENCES

[1] M. U. Akhmet, *Stability of differential equations with piecewise constant arguments of generalized type*, Nonlinear Analysis: Theory, Methods & Applications **68** (2008), 794–803.

[2] V. P. Bajiya, J. P. Tripathi, V. Kakkar, J. Wang and G. Sun, *Global dynamics of a multi-group SEIR epidemic model with infection age*, Chinese Annals of Mathematics, Series B, **42** (2021), 833–860.

[3] M. Banerjee and L. Zhang, *Time delay can enhance spatio-temporal chaos in a prey–predator model*, Ecological Complexity **27** (2016), 17–28.

[4] C. Bianca, M. Ferrara and L. Guerrini, *The time delays' effects on the qualitative behavior of an economic growth model*, Abstract and Applied Analysis **2013** (2023):901014.

[5] G.-I. Bischi, C. Chiarella, M. Kopel, F. Szidarovszky, et al., *Nonlinear Oligopolies*, Springer, 2010.

[6] J. Carr, *Applications of Center Manifold Theory*, Heidelberg, Berlin, 1981.

[7] T. Cassidy, *Distributed delay differential equation representations of cyclic differential equations*, SIAM Journal on Applied Mathematics **81** (2021) 1742–1766.

[8] G. Chen and X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications*, vol. 24, World Scientific, 1998.

[9] Y. K. Chembo, D. Brunner, M. Jacquot, and L. Larger, *Optoelectronic oscillators with time-delayed feedback*, Reviews of Modern Physics **91** (2019): 035006.

[10] A. M. A. El-Sayed and S. M. Salman, *Chaos and bifurcation of discontinuous dynamical systems with piecewise constant arguments*, Malaya Journal of Matematik **1** (2012), 14–18.

[11] A. M. A. El-Sayed and S. M. Salman, *Chaos and bifurcation of the logistic discontinuous dynamical systems with piecewise constant arguments*, Malaya Journal of Matematik **1** (2013), 14–20.

[12] A. M. A. El-Sayed and S. M. Salman, *On a discretization process of fractional-order Riccati differential equation*, J. Fract. Calc. Appl. **4** (2013), 251–259.

[13] A. El-Sayed, S. M. Salman and A. A. F. AbdElfattah, *On the dynamics of a Riccati differential equation with perturbed delay*, Electronic Journal of Mathematical Analysis and Applications **11** (2023), 1–15.

[14] L. Guerrini, N. Pecora and M. Sodini, *Effects of fixed and continuously distributed delays in a monopoly model with constant price elasticity*, Decisions in Economics and Finance, **41** (2018), 239–257.

[15] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, vol. 99, Springer Science & Business Media, 2013.

[16] Z. He and X. Lai, *Bifurcation and chaotic behavior of a discrete-time predator–prey system*, Nonlinear Analysis: Real World Applications **12** (2011), 403–417.

[17] R. C. Hilborn, *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers*, Oxford University Press, 2000.

[18] M. W. Hirsch, S. Smale and R. L. Devaney, *Differential Equations, Dynamical Systems, and An Introduction to Chaos*, Academic Press, 2012.

[19] Y. A. Kuznetsov, Iu A. Kuznetsov and Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, vol. 112, Springer, 1998.

[20] M. R. S. Kulenovic and G. Ladas, *Dynamics of Second Order Rational Difference Equations: with Open Problems and Conjectures*, CRC Press, 2001.

[21] L. Larger, *Complexity in electro-optic delay dynamics: modelling, design and applications*, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **371** (2013); 20120464.

[22] D. Lathrop, *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*, American Institute of Physics, 2015.

[23] S. Lynch, et al., *Dynamical Systems with Applications Using Mathematica*, Springer, 2007.

[24] P. R. Masani, *Norbert Wiener 1894–1964*, vol. 5, Birkhäuser, 2012.

[25] J.-P. Richard, *Time-delay systems: an overview of some recent advances and open problems*, Automatica **39** (2003), 1667–1694.

[26] X. Shi, X. Zhou and X. Song, *Dynamical behavior for an eco-epidemiological model with discrete and distributed delay*, Journal of Applied Mathematics and Computing **33** (2010), 305–325.

[27] H. L. Smith, *Distributed delay equations and the linear chain trick*, in: An Introduction to Delay Differential Equations with Applications to the Life Sciences, Springer, 2011, pp. 119–130.

[28] H. L. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, vol. 57, Springer New York, 2011.

[29] K. Torbensen, F. Rossi, S. Ristori and A. Abou-Hassan, *Chemical communication and dynamics of droplet emulsions in networks of Belousov–Zhabotinsky micro-oscillators produced by microfluidics*, Lab on a Chip **17** (2017), 1179–1189.

[30] J. Wu, X.-S. Zhan, X.-H. Zhang and H.-L. Gao, *Stability and hopf bifurcation analysis on a numerical discretization of the distributed delay equation*, Chinese Physics Letters, **29** (2012): 050203.

[31] W. Xu, D. W. C. Ho, L. Li and J. Cao, *Event-triggered schemes on leader-following consensus of general linear multiagent systems under different topologies*, IEEE Transactions on Cybernetics **47** (2015), 212–223.

[32] C.-H. Zhang, X.-P. Yan and G.-H. Cui, *Hopf bifurcations in a predator–prey system with a discrete delay and a distributed delay*, Nonlinear Analysis: Real World Applications **11** (2010), 4141–4153.

Manuscript received August 17 2024
revised September 20 2024

A. M. A. El-Sayed
Faculty of Science, Alexandria University, Alexandria, Egypt
E-mail address: `amasayed@alexu.edu.eg`

S. M. Salman
Faculty of Education, Alexandria University, Alexandria, Egypt
E-mail address: `samastars9@gmail.com`

M. A. Saad
Faculty of Science, Alexandria University, Alexandria, Egypt
E-mail address: `Muhammed.Saad@alexu.edu.eg`