Applied Analysis and Optimization
Volume 8, Number 3, 2024, 329–354

Yokohama Publishers
ISSN 2189-1664 Online Journal

RELAXED MANN-TYPE VISCOSITY IMPLICIT METHOD FOR A SYSTEM OF VARIATIONAL INCLUSIONS WITH A FIXED POINT PROBLEM CONSTRAINT OF PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In a uniformly convex and q-uniformly smooth Banach space with $q \in (1, 2]$, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings. In this paper, we introduce a relaxed Mann-type viscosity implicit method for solving a general system of variational inclusions (GSVI) with the VI and CFPP constraints. Strong convergence of the proposed algorithm to a solution of the GSVI with the VI and CFPP constraints under some suitable conditions is established. Applications of the main result to the variational inequality, split feasibility problem and LASSO problem in Hilbert spaces are given.

1. Introduction

Let $(H, \|\cdot\|)$ be a real Hilbert space, in which the inner product is denoted by $\langle \cdot, \cdot \rangle$. Let $\emptyset \neq C \subset H$ be a closed convex set. We denote by P_C the metric projection from H onto C. Given a mapping $A : C \to H$. Consider the classical variational inequality problem (VIP) of finding a point $x^* \in C$ s.t. $\langle Ax^*, y - x^* \rangle \geq 0 \ \forall y \in C$. We denote by $\mathrm{VI}(C, A)$ the solution set of the VIP. Up to now, Korpelevich's extragradient method [23] has been one of the most popular methods for solving the VIP. It is worth mentioning that if $\mathrm{VI}(C, A) \neq \emptyset$, this method has only weak convergence, and only requires that the mapping A is monotone and Lipschitz continuous. To the most of our knowledge, Korpelevich's extragradient method has been improved and modified in various ways so that some new iterative methods happen to solve the VIP and related optimization problems; see e.g., [1, 7–17, 19, 22, 28, 29, 33, 34, 41, 42] and references therein, to name but a few.

Assume that the operators $A : C \to H$ and $B : D(B) \subset C \to H$ are α-inverse-strongly monotone and maximal monotone, respectively. Consider the variational inclusion (VI) of finding a point $x^* \in C$ s.t. $0 \in (A + B)x^*$. In order to solve the FPP of nonexpansive mapping $S : C \to C$ and the VI for both monotone mappings

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2020 *Mathematics Subject Classification.* 49J30, 47H09, 47J20, 49M05.

Key words and phrases. Relaxed Mann-type viscosity implicit method, general system of variational inclusions, variational inclusion, common fixed point problem.

*The research of Lu-Chuan Ceng was partially supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau (20LJ2006100), the Innovation Program of Shanghai Municipal Ed- ucation Commission (15ZZ068) and the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

†Jen-Chih Yao was partially supported by the Grant MOST-113-2115-M-039 -001.

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A, B, Takahashi et al. [35] suggested a Mann-type Halpern iterative method, i.e., for any given $x_1 = x \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = \beta_j x_j + (1-\beta_j)S(\alpha_j x + (1-\alpha_j)J^B_{\lambda_j}(x_j - \lambda_j A x_j)) \quad \forall j \geq 1,$$

where $\{\lambda_j\} \subset (0, 2\alpha)$ and $\{\alpha_j\}, \{\beta_j\} \subset (0,1)$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}(0)$ under some mild conditions. In the practical life, many mathematical models have been formulated as the VI. Without question, many researchers have presented and developed a great number of iterative methods for solving the VI in various approaches; see e.g., [6, 8, 12, 15, 16, 24, 26, 34, 35] and the references therein. Due to the importance and interesting of the VI, many mathematicians are now interested in finding a common solution of the VI and FPP.

Recently, Manaka and Takahashi [26] suggested an iterative process, i.e., for any given $x_0 \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = \alpha_j x_j + (1-\alpha_j)SJ^B_{\lambda_j}(x_j - \lambda_j A x_j) \quad \forall j \geq 0,$$

where $\{\alpha_j\} \subset (0,1)$, $\{\lambda_j\} \subset (0,\infty)$, $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and $S : C \to C$ is a nonexpansive mapping. They proved weak convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}(0)$ under some suitable conditions.

For $q \in (1,2]$, let E be a uniformly convex and q-uniformly smooth Banach space with q-uniform smoothness coefficient κ_q. Suppose that $f : E \to E$ is a ρ-contraction and $S : E \to E$ is a nonexpansive mapping. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Very recently, Sunthrayuth and Cholamjiak [34] proposed a modified viscosity-type extragradient method for the FPP of S and the VI of finding $x^* \in E$ s.t. $0 \in (A+B)x^*$, i.e., for any given $x_0 \in E$, $\{x_j\}$ is the sequence generated by

$$\begin{cases} y_j = J^B_{\lambda_j}(x_j - \lambda_j A x_j), \\ z_j = J^B_{\lambda_j}(x_j - \lambda_j A y_j + r_j(y_j - x_j)), \\ x_{j+1} = \alpha_j f(x_j) + \beta_j x_j + \gamma_j S z_j \quad \forall j \geq 0, \end{cases}$$

where $J^B_{\lambda_j} = (I + \lambda_j B)^{-1}$, $\{r_j\}, \{\alpha_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0,1)$ and $\{\lambda_j\} \subset (0,\infty)$ are such that: (i) $\alpha_j + \beta_j + \gamma_j = 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\sum_{j=1}^{\infty} \alpha_j = \infty$; (iii) $\{\beta_j\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_j < \lambda_j / r_j \leq \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \leq r_j < 1$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}(0)$, which solves a certain VIP.

Furthermore, suppose that $J : E \to 2^{E^*}$ is the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{\phi \in E^* : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2\}$ $\forall x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. It is known that if E is smooth then J is single-valued. Let C be a nonempty closed convex subset of a smooth Banach space E. Let $A_1, A_2 : C \to E$ and $B_1, B_2 : C \to 2^E$ be nonlinear mappings with $B_i x \neq \emptyset$ $\forall x \in C, i = 1,2$. Consider the general system of variational inclusions (GSVI) of finding $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} 0 \in \zeta_1(A_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \zeta_2(A_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{1.1}$$

where ζ_i is a positive constant for $i = 1, 2$. It is known that problem (1.1) has been transformed into a fixed point problem in the following way.

Lemma 1.1. ([18, Lemma 2]) *Assume that $B_1, B_2 : C \to 2^E$ are both m-accretive operators and $A_1, A_2 : C \to E$ are both operators. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.1) *if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of the mapping $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$, and $y^* = J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)x^*$.*

Suppose that E is a uniformly convex and 2-uniformly smooth Banach space with 2-uniform smoothness coefficient κ_2. Let $B_1, B_2 : C \to 2^E$ be both m-accretive operators and $A_i : C \to E$ $(i = 1, 2)$ be ζ_i-inverse-strongly accretive operator. Let $f : C \to C$ be a contraction with constant $\delta \in [0, 1)$. Let $V : C \to C$ be a nonexpansive operator and $T : C \to C$ be a λ-strict pseudocontraction. Very recently, using Lemma 1.1, Ceng et al. [18] suggested a composite viscosity implicit rule for solving the GSVI (1.1) with the FPP constraint of T, i.e., for any given $x_0 \in C$, the sequence $\{x_j\}$ is generated by

$$\begin{cases} y_j = J_{\zeta_2}^{B_2}(x_j - \zeta_2 A_2 x_j), \\ x_j = \alpha_j f(x_{j-1}) + \delta_j x_{j-1} + \beta_j V x_{j-1} + \gamma_j[\mu S x_j + (1 - \mu)J_{\zeta_1}^{B_1}(y_j - \zeta_1 A_1 y_j)] \end{cases}$$

for any $j \geq 1$, where $\mu \in (0, 1)$, $S := (1-\alpha)I+\alpha T$ with $0 < \alpha < \min\{1, \frac{2\lambda}{\kappa_2}\}$, and the sequences $\{\alpha_j\}, \{\delta_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0, 1)$ are such that (i) $\alpha_j+\delta_j+\beta_j+\gamma_j = 1\ \forall j \geq 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = 0$; (iii) $\lim_{j\to\infty} \gamma_j = 1$; (iv) $\sum_{j=0}^{\infty} \alpha_j = \infty$. They proved that $\{x_j\}$ converges strongly to a point of $\mathrm{Fix}(G) \cap \mathrm{Fix}(T)$, which solves a certain VIP.

In addition, assume that $\{\mu_j\} \subset (0, \frac{1}{L})$, $\{\lambda_j\} \subset (0, 2\alpha]$ and $\{\alpha_j\}, \{\hat{\alpha}_j\} \subset (0, 1]$ with $\alpha_j + \hat{\alpha}_j \leq 1$. Ceng et al. [8] introduced a Mann-type hybrid extragradient algorithm, i.e., for any initial $u_0 = u \in C$, $\{u_j\}$ is the sequence generated by

$$\begin{cases} y_j = P_C(u_j - \mu_j \mathcal{A} u_j), \\ v_j = P_C(u_j - \mu_j \mathcal{A} y_j), \\ \hat{v}_j = J_{\lambda_j}^{B}(v_j - \lambda_j A v_j), \\ z_j = (1 - \alpha_j - \hat{\alpha}_j)u_j + \alpha_j \hat{v}_j + \hat{\alpha}_j S \hat{v}_j, \\ u_{j+1} = P_{C_j \cap Q_j} u \quad \forall j \geq 0, \end{cases}$$

where $C_j = \{x \in C : \|z_j - x\| \leq \|u_j - x\|\}$, $Q_j = \{x \in C : \langle u_j - x, u - u_j\rangle \geq 0\}$, $J_{\lambda_j}^{B} = (I + \lambda_j B)^{-1}$, $\mathcal{A} : C \to H$ is a monotone and L-Lipschitzian mapping, $A : C \to H$ is an α-inverse-strongly monotone mapping, B is a maximal monotone mapping with $D(B) = C$ and $S : C \to C$ is a nonexpansive mapping. They proved strong convergence of $\{u_j\}$ to the point $P_{\Omega} u$ in $\Omega = \mathrm{Fix}(S)\cap(A+B)^{-1}(0)\cap\mathrm{VI}(C, \mathcal{A})$ under some mild conditions.

In a uniformly convex and q-uniformly smooth Banach space with $q \in (1, 2]$, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings. In this paper, we introduce a relaxed Mann-type viscosity implicit method for solving the GSVI (1.1) with the VI and CFPP constraints. We then prove the strong convergence of the suggested algorithm to a solution of the GSVI (1.1) with the VI and CFPP constraints under some suitable conditions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces. Our results improve and extend the corresponding results in Manaka and Takahashi [26], Sunthrayuth and Cholamjiak [34], and Ceng et al. [18] to a certain extent.

2. Preliminaries

Let E be a real Banach space with the dual E^*, and $\emptyset \neq C \subset E$ be a closed convex set. For convenience, we shall use the following symbols: $x_n \to x$ (resp., $x_n \rightharpoonup x$) indicates the strong (resp., weak) convergence of the sequence $\{x_n\}$ to x. Given a self-mapping T on C. We use the symbols $\mathbf{R}$ and $\mathrm{Fix}(T)$ to denote the set of all real numbers and the fixed point set of T, respectively. Recall that T is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\| \; \forall x, y \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \varrho \in [0, 1)$ s.t. $\|f(x) - f(y)\| \leq \varrho\|x - y\| \; \forall x, y \in C$. Also, recall that the normalized duality mapping J defined by

$$J(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\} \quad \forall x \in E,$$

is the one from E into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of E^*, satisfying $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in E, \; \|x\| = \|y\| = 1, \; \|x - y\| \geq \epsilon\right\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in E, \; \|x\| = \|y\| = 1\right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0 \; \forall \epsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. Also, it is said to be q-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \leq ct^q \; \forall t > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth and if E is uniformly convex, then E is also reflexive and strictly convex. It is known that Hilbert space H is 2-uniformly smooth. Further, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [38].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q, \; \|\phi\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of E. It is known that $J_q(x) = \|x\|^{q-2}J(x) \; \forall x \neq 0$ and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued. Furthermore, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is q-uniformly smooth for $q > 2$; see [36] for more details. Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then the following inequality is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$:

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x+y)\rangle \quad \forall x,y\in E,\ j_q(x+y)\in J_q(x+y). \tag{2.1}$$

Proposition 2.1 ([38])**.** *Let $q \in (1,2]$ be a fixed real number and let E be q-uniformly smooth. Then $\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q$ $\forall x,y\in E$, where κ_q is the q-uniform smoothness coefficient of E.*

Recall that a mapping $T: C\to C$ is called pseudocontractive if for each $x,y\in C$, there exists $j(x-y)\in J(x-y)$ such that $\langle Tx-Ty, j(x-y)\rangle \le \|x-y\|^2$. Also, it is called strongly pseudocontractive if for each $x,y\in C$, there exists $j(x-y)\in J(x-y)$ such that $\langle Tx-Ty, j(x-y)\rangle \le \alpha\|x-y\|^2$ for some $\alpha\in(0,1)$. We will use the following concept in the sequel.

Definition 2.2. Let $\{S_n\}_{n=0}^\infty$ be a sequence of continuous pseudocontractive self-mappings on C. Then $\{S_n\}_{n=0}^\infty$ is said to be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell>0$ such that each S_n is ℓ-Lipschitz continuous.

Lemma 2.3 ([3])**.** *Let $\{S_n\}_{n=0}^\infty$ be a sequence of self-mappings on C such that $\sum_{n=1}^\infty \sup_{x\in C}\|S_nx - S_{n-1}x\| < \infty$. Then for each $y\in C$, $\{S_ny\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty} S_ny$ $\forall y\in C$. Then $\lim_{n\to\infty}\sup_{x\in C}\|S_nx - Sx\| = 0$.*

The following lemma can be obtained from the result in [38].

Lemma 2.4. *Let $q>1$ and $r>0$ be two fixed real numbers and let E be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g,h: \mathbf{R}_+\to\mathbf{R}_+$ with $g(0)=0$ and $h(0)=0$ such that*

(i) $\|\mu x+(1-\mu)y\|^q \le \mu\|x\|^q + (1-\mu)\|y\|^q - \mu(1-\mu)g(\|x-y\|)$ with $\mu\in[0,1]$;
(ii) $h(\|x-y\|) \le \|x\|^q - q\langle x, j_q(y)\rangle + (q-1)\|y\|^q$
for all $x,y\in B_r$ and $j_q(y)\in J_q(y)$, where $B_r := \{x\in E: \|x\|\le r\}$.

The following lemma is an analogue of Lemma 2.4(a).

Lemma 2.5. *Let $q>1$ and $r>0$ be two fixed real numbers and let E be uniformly convex. Then there exists a strictly increasing, continuous and convex function $g: \mathbf{R}_+\to\mathbf{R}_+$ with $g(0)=0$ such that $\|\lambda x+\mu y+\nu z\|^q \le \lambda\|x\|^q + \mu\|y\|^q + \nu\|z\|^q - \lambda\mu g(\|x-y\|)$ for all $x,y,z\in B_r$ and $\lambda,\mu,\nu\in[0,1]$ with $\lambda+\mu+\nu=1$.*

Proposition 2.6 ([21])**.** *Let $\emptyset\ne C\subset E$ be a closed convex set. If $T: C\to C$ is a continuous and strong pseudocontractive, then T has a unique fixed point in C.*

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x)+t(x-\Pi(x))] = \Pi(x)$, whenever $\Pi(x)+t(x-\Pi(x))\in C$ for $x\in C$ and $t\ge 0$. A mapping Π of C into itself is called a retraction if $\Pi^2=\Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z)=z$ for each $z\in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. In terms of [30], we know that if E is smooth and Π is a retraction of C onto D, then the following statements are equivalent:

(i) Π is sunny and nonexpansive;
(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle \ \forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0 \ \forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \neq \emptyset \ \forall x \in C$. Let $q > 1$. An operator B is said to be accretive if for each $x, y \in C$, $\exists j_q(x - y) \in J_q(x - y)$ s.t. $\langle u - v, j_q(x - y)\rangle \geq 0 \ \forall u \in Bx, v \in By$. An accretive operator B is said to be α-inverse-strongly accretive of order q if for each $x, y \in C$, $\exists j_q(x - y) \in J_q(x - y)$ s.t. $\langle u - v, j_q(x - y)\rangle \geq \alpha\|u - v\|^q \ \forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ is a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J^B_\lambda : (I + \lambda B)C \to C$ by $J^B_\lambda = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J^B_λ is called the resolvent of B for $\lambda > 0$.

Lemma 2.7 ([16, 24])**.** *Let $B : C \to 2^E$ be an m-accretive operator. Then the following statements hold:*

(i) *the resolvent identity: $J^B_\lambda x = J^B_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J^B_\lambda x) \ \forall \lambda, \mu > 0, \ x \in E$;*
(ii) *if J^B_λ is a resolvent of B for $\lambda > 0$, then J^B_λ is a firmly nonexpansive mapping with $\mathrm{Fix}(J^B_\lambda) = B^{-1}(0)$, where $B^{-1}(0) = \{x \in C : 0 \in Bx\}$;*
(iii) *if $E = H$ is a Hilbert space, then B is maximal monotone.*

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J^B_\lambda(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A) \ \forall \lambda > 0$.

Proposition 2.8 ([24])**.** *The following statements hold:*

(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}(0) \ \forall \lambda > 0$*;*
(ii) $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ *for $0 < \lambda \leq r$ and $y \in C$.*

Proposition 2.9 ([39])**.** *Let E be uniformly smooth, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $f : C \to C$ be a fixed contraction. For each $t \in (0, 1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1 - t)Tz$ on C, i.e., $z_t = tf(z_t) + (1 - t)Tz_t$. Then $\{z_t\}$ converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which solves the VIP: $\langle (I - f)x^*, J(x^* - x)\rangle \leq 0 \ \forall x \in \mathrm{Fix}(T)$.*

Proposition 2.10 ([24])**.** *Let E be q-uniformly smooth with $q \in (1, 2]$. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \geq 0$,*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q \quad \forall x, y \in C,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 \leq \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.11 ([18])**.** *Let E be q-uniformly smooth with $q \in (1, 2]$. Let $B_1, B_2 : C \to 2^E$ be two m-accretive operators and $A_i : C \to E$ $(i = 1, 2)$ be σ_i-inverse-strongly accretive mapping of order q. Define an operator $G : C \to C$ by $G := J^{B_1}_{\zeta_1}(I - \zeta_1 A_1)J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)$. If $0 \leq \zeta_i \leq (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ $(i = 1, 2)$, then G is nonexpansive.*

Lemma 2.12 ([2]). *Let E be smooth, $A : C \to E$ be accretive and Π_C be a sunny nonexpansive retraction from E onto C. Then* $\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I - \lambda A))\ \forall \lambda > 0$, *where* $\mathrm{VI}(C, A)$ *is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z - y)\rangle \le 0\ \forall y \in C$.*

Recall that if $E = H$ is a Hilbert space, then the sunny nonexpansive retraction Π_C from E onto C coincides with the metric projection P_C from H onto C. Moreover, if E is uniformly smooth and T is a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \neq \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from E onto C [31]. By Lemma 2.12 we know that, $x^* \in \mathrm{Fix}(T)$ solves the VIP in Proposition 2.9 if and only if x^* solves the fixed point equation $x^* = \Pi_{\mathrm{Fix}(T)} f(x^*)$.

Proposition 2.13 ([3]). *Let $\{S_n\}_{n=0}^\infty$ be a sequence of self-mappings on C such that $\sum_{n=1}^\infty \sup_{x \in C} \|S_n x - S_{n-1} x\| < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup_{x \in C} \|S_n x - Sx\| = 0$.*

Lemma 2.14 ([25]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each integer $i \ge 1$. Define the sequence $\{\tau(n)\}_{n \ge n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \ge 1$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \le \tau(n_0 + 1) \le \cdots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \le \Gamma_{\tau(n)+1}\ \forall n \ge n_0$.

Lemma 2.15 ([4]). *Let E be strictly convex, and $\{S_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^\infty \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^\infty \lambda_n S_n x\ \forall x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ holds.*

Lemma 2.16 ([39]). *Let $\{a_n\}$ be a sequence in $[0, \infty)$ such that $a_{n+1} \le (1 - s_n)a_n + s_n \nu_n\ \forall n \ge 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions:* (i) $\{s_n\} \subset [0, 1]$, $\sum_{n=0}^\infty s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \le 0$ *or* $\sum_{n=0}^\infty |s_n \nu_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

Throughout this paper, suppose that C is a nonempty closed convex subset of a uniformly convex and q-uniformly smooth Banach space E with $q \in (1, 2]$. Let $B_1, B_2 : C \to 2^E$ be both m-accretive operators and $A_i : C \to E$ be σ_i-inverse-strongly accretive mapping of order q for $i = 1, 2$. Let the mapping $G : C \to C$ be defined as $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ with $0 < \zeta_i < (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$. Let $f : C \to C$ be a ϱ-contraction with constant $\varrho \in [0, 1)$, and $\{S_n\}_{n=0}^\infty$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Assume that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A + B)^{-1}(0) \neq \emptyset$.

Algorithm 3.1. Relaxed Mann-type viscosity implicit method for the GSVI (1.1) with the VI and CFPP constraints.

Initial Step. Given $\xi \in (0,1)$ and $x_0 \in C$ arbitrarily.

Iteration Steps. Given the current iterate x_n, compute x_{n+1} as follows:

Step 1. Calculate $w_n = s_n G x_n + (1-s_n)(\xi x_n + (1-\xi) S_n w_n)$;

Step 2. Calculate $y_n = \delta_n u_n + (1-\delta_n) J^B_{\lambda_n}(u_n - \lambda_n A u_n)$ with $u_n = G w_n$;

Step 3. Calculate $z_n = J^B_{\lambda_n}(u_n - \lambda_n A y_n + r_n(y_n - u_n))$;

Step 4. Calculate $x_{n+1} = \alpha_n f(x_n) + \beta_n u_n + \gamma_n G z_n$, where $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0,\infty)$. Set $n := n+1$ and go to Step 1.

Lemma 3.1. *If $\{x_n\}$ is the sequence generated by Algorithm 3.1, then it is bounded.*

Proof. Take an element $p \in \Omega := \bigcap_{k=0}^{\infty} \mathrm{Fix}(T_k) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}(0)$ arbitrarily. Then we have

$$p = Gp = S_n p = J^B_{\lambda_n}(p - \lambda_n A p) = J^B_{\lambda_n}\left((1-r_n)p + r_n\Big(p - \frac{\lambda_n}{r_n} A p\Big)\right).$$

By Proposition 2.10 and Lemma 2.11, we deduce that $I - \zeta_1 A_1$, $I - \zeta_2 A_2$ and $G := J^{B_1}_{\zeta_1}(I - \zeta_1 A_1) J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)$ are nonexpansive mappings. Moreover, it can be readily seen that for each $n \ge 0$, there is only an element $w_n \in C$ s.t.

$$w_n = s_n G x_n + (1-s_n)(\xi x_n + (1-\xi) S_n w_n). \tag{3.1}$$

In fact, consider the mapping $F_n u = s_n G x_n + (1-s_n)(\xi x_n + (1-\xi) S_n u)$ $\forall u \in C$. Note that $S_n : C \to C$ is a continuous pseudocontraction. Hence we obtain that for all $u, v \in C$,

$$\begin{aligned} \langle F_n u - F_n v, J(u-v)\rangle &= (1-s_n)(1-\xi)\langle S_n u - S_n v, J(u-v)\rangle \\ &\le (1-s_n)(1-\xi)\|u-v\|^2. \end{aligned}$$

Also, from $\{s_n\} \subset (0,1]$, we get $0 \le 1 - s_n < 1$ $\forall n \ge 0$. Thus, F_n is a continuous and strong pseudocontractive self-mapping on C. By Proposition 2.6, we deduce that for each $n \ge 0$, there is only an element $w_n \in C$, satisfying (3.1). Since each $S_n : C \to C$ is a pseudocontractive mapping, we get

$$\begin{aligned} \|w_n - p\|^q &= s_n \langle G x_n - p, J_q(w_n - p)\rangle \\ &\quad + (1-s_n)\langle \xi x_n + (1-\xi) S_n w_n - p, J_q(w_n - p)\rangle \\ &\le s_n \|x_n - p\|\|w_n - p\|^{q-1} + (1-s_n)[\xi \|x_n - p\|\|w_n - p\|^{q-1} \\ &\quad + (1-\xi)\|w_n - p\|^q] \\ &= [s_n + (1-s_n)\xi]\|x_n - p\|\|w_n - p\|^{q-1} \\ &\quad + (1-s_n)(1-\xi)\|w_n - p\|^q, \end{aligned}$$

which immediately yields $\|w_n - p\| \le \|x_n - p\|$ $\forall n \ge 0$. Using the nonexpansivity of G again, we deduce from $u_n = G w_n$ that

$$\|u_n - p\| \le \|w_n - p\| \le \|x_n - p\| \quad \forall n \ge 0. \tag{3.2}$$

Using Lemmas 2.4(i), 2.7(ii) and Proposition 2.10, we obtain that

$$\begin{aligned}\|y_n - p\|^q &= \|\delta_n(u_n - p) + (1-\delta_n)(J^B_{\lambda_n}(u_n - \lambda_n Au_n) - J^B_{\lambda_n}(p - \lambda_n Ap))\|^q \\ &\le \delta_n\|u_n - p\|^q + (1-\delta_n)\|J^B_{\lambda_n}(I - \lambda_n A)u_n - J^B_{\lambda_n}(I-\lambda_n A)p\|^q \\ &\le \delta_n\|u_n - p\|^q + (1-\delta_n)[\|u_n - p\|^q \\ &\quad - \lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ap\|^q] \\ &= \|u_n - p\|^q - (1-\delta_n)\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ap\|^q,\end{aligned} \tag{3.3}$$

which hence leads to $\|y_n - p\| \le \|u_n - p\|$. By the convexity of $\|\cdot\|^q$ and (3.3), we have

$$\begin{aligned}\|z_n - p\|^q &= \Big\|J^B_{\lambda_n}\Big((1-r_n)u_n + r_n\Big(y_n - \frac{\lambda_n}{r_n}Ay_n\Big)\Big) \\ &\qquad - J^B_{\lambda_n}\Big((1-r_n)p + r_n\Big(p - \frac{\lambda_n}{r_n}Ap\Big)\Big)\Big\|^q \\ &\le (1-r_n)\|u_n - p\|^q + r_n\Big\|\Big(I - \frac{\lambda_n}{r_n}A\Big)y_n - \Big(I - \frac{\lambda_n}{r_n}A\Big)p\Big\|^q \\ &\le (1-r_n)\|u_n - p\|^q \\ &\quad + r_n\Big[\|y_n - p\|^q - \frac{\lambda_n}{r_n}\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ap\|^q\Big] \\ &\le (1-r_n)\|u_n - p\|^q \\ &\quad + r_n\Big\{\|u_n - p\|^q - (1-\delta_n)\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ap\|^q \\ &\quad - \frac{\lambda_n}{r_n}\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ap\|^q\Big\} \\ &= \|u_n - p\|^q - r_n(1-\delta_n)\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ap\|^q \\ &\quad - \lambda_n\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ap\|^q.\end{aligned} \tag{3.4}$$

This ensures that $\|z_n - p\| \le \|u_n - p\|$. So it follows from (3.2) that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|f(x_n) - p\| + \beta_n\|u_n - p\| + \gamma_n\|Gz_n - p\| \\ &\le \alpha_n(\varrho\|x_n - p\| + \|p - f(p)\|) + \beta_n\|u_n - p\| + \gamma_n\|u_n - p\| \\ &\le \alpha_n(\varrho\|x_n - p\| + \|p - f(p)\|) + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= (1 - \alpha_n(1-\varrho))\|x_n - p\| + \alpha_n\|p - f(p)\| \\ &\le \max\Big\{\|x_n - p\|, \frac{\|p - f(p)\|}{1-\varrho}\Big\}.\end{aligned}$$

By induction, we get $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|p-f(p)\|}{1-\varrho}\}$ $\forall n \ge 0$. Thus, $\{x_n\}$ is bounded, and so are $\{u_n\}\{w_n\}, \{y_n\}, \{z_n\}, \{Gz_n\}, \{Au_n\}, \{Ay_n\}$. This completes the proof. □

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) $0 < \underline{\delta} \le \delta_n \le \overline{\delta} < 1$;
(C3) $0 < a \le \beta_n \le b < 1$ *and* $0 < c \le s_n \le d < 1$;
(C4) $0 < r \le r_n < 1$ *and* $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < (\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset* D *of* C. *Let* $S : C \to C$ *be a mapping defined by* $Sx = \lim_{n\to\infty} S_n x \ \forall x \in C$, *and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. *Then* $x_n \to x^* \in \Omega$, *which is the unique solution to the VIP:* $\langle (I - f)x^*, J(x^* - p)\rangle \le 0 \ \forall p \in \Omega$, *i.e., the fixed point equation* $x^* = \Pi_\Omega f(x^*)$.

Proof. First of all, let $x^* \in \Omega$ and $y^* = J_{\zeta_2}^{B_2}(x^* - \zeta_2 A_2 x^*)$. Putting $v_n := J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n)$ and using $u_n = Gw_n$, we get $u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n)$. From Proposition 2.10 we have

$$\begin{aligned}\|v_n - y^*\|^q &= \|J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n) - J_{\zeta_2}^{B_2}(x^* - \zeta_2 A_2 x^*)\|^q \\ &\le \|w_n - x^*\|^q - \zeta_2(\sigma_2 q - \kappa_q \zeta_2^{q-1})\|A_2 w_n - A_2 x^*\|^q,\end{aligned}$$

and

$$\begin{aligned}\|u_n - x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n) - J_{\zeta_1}^{B_1}(y^* - \zeta_1 A_1 y^*)\|^q \\ &\le \|v_n - y^*\|^q - \zeta_1(\sigma_1 q - \kappa_q \zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q.\end{aligned}$$

Combining the last two inequalities, we have

$$\begin{aligned}\|u_n - x^*\|^q \le \|w_n - x^*\|^q &- \zeta_2(\sigma_2 q - \kappa_q \zeta_2^{q-1})\|A_2 w_n - A_2 x^*\|^q \\ &- \zeta_1(\sigma_1 q - \kappa_q \zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q.\end{aligned}$$

Using Lemma 2.5, from (2.1), (3.2) and (3.4) we obtain that

$$\begin{aligned}&\|x_{n+1} - x^*\|^q \\ &= \|\alpha_n(f(x_n) - f(x^*)) + \beta_n(u_n - x^*) + \gamma_n(Gz_n - x^*) + \alpha_n(f(x^*) - x^*)\|^q \\ &\le \alpha_n\|f(x_n) - f(x^*)\|^q + \beta_n\|u_n - x^*\|^q + \gamma_n\|Gz_n - x^*\|^q - \beta_n\gamma_n g(\|u_n - Gz_n\|) \\ &\quad + q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle \\ &\le \alpha_n\varrho\|x_n - x^*\|^q + \beta_n\|u_n - x^*\|^q \\ &\quad + \gamma_n\{\|u_n - x^*\|^q - r_n(1 - \delta_n)\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1}) \\ &\quad \times \|Au_n - Ax^*\|^q - \lambda_n\left(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\} \\ &\quad - \beta_n\gamma_n g(\|u_n - Gz_n\|) + q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle \\ &\le \alpha_n\varrho\|x_n - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\{\|x_n - x^*\|^q \\ &\quad - \zeta_2\left(\sigma_2 q - \kappa_q\zeta_2^{q-1}\right)\|A_2 w_n - A_2 x^*\|^q - \zeta_1\left(\sigma_1 q - \kappa_q\zeta_1^{q-1}\right)\|A_1 v_n - A_1 y^*\|^q \\ &\quad - r_n(1 - \delta_n)\lambda_n\left(\sigma q - \kappa_q\lambda_n^{q-1}\right)\|Au_n - Ax^*\|^q \\ &\quad - \lambda_n\left(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\} - \beta_n\gamma_n g(\|u_n - Gz_n\|)\end{aligned}$$

(3.5)

$$
\begin{aligned}
&+ q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle \\
&= (1-\alpha_n(1-\varrho))\|x_n-x^*\|^q - \gamma_n\{\zeta_2\left(\sigma_2 q-\kappa_q\zeta_2^{q-1}\right)\|A_2w_n-A_2x^*\|^q \\
&+ \zeta_1\left(\sigma_1 q-\kappa_q\zeta_1^{q-1}\right)\times\|A_1v_n-A_1y^*\|^q \\
&+ r_n(1-\delta_n)\lambda_n\left(\sigma q-\kappa_q\lambda_n^{q-1}\right)\|Au_n-Ax^*\|^q \\
&+ \lambda_n\left(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\times\|Ay_n-Ax^*\|^q\} - \beta_n\gamma_n g(\|u_n-Gz_n\|) \\
&+ q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

For each $n \ge 0$, we set

$$
\begin{aligned}
\Gamma_n &= \|x_n-x^*\|^q, \\
\epsilon_n &= \alpha_n(1-\varrho), \\
\eta_n &= \gamma_n\Big\{\zeta_2(\sigma_2 q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q \\
&\quad + \zeta_1(\sigma_1 q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q \\
&\quad + r_n(1-\delta_n)\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q \\
&\quad + \lambda_n\Big(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n-Ax^*\|^q\Big\} \\
&\quad + \beta_n\gamma_n g(\|u_n-Gz_n\|) \\
\vartheta_n &= q\alpha_n\langle (f-I)x^*, J_q(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

Then (3.5) can be rewritten as the following formula:

$$\Gamma_{n+1} \le (1-\epsilon_n)\Gamma_n - \eta_n + \vartheta_n \quad \forall n \ge 0, \tag{3.6}$$

and hence

$$\Gamma_{n+1} \le (1-\epsilon_n)\Gamma_n + \vartheta_n \quad \forall n \ge 0. \tag{3.7}$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

Case 1. Suppose that there exists an integer $n_0 \ge 1$ such that $\{\Gamma_n\}$ is non-increasing. Then

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

From (3.6), we get

$$0 \le \eta_n \le \Gamma_n - \Gamma_{n+1} + \vartheta_n - \epsilon_n\Gamma_n.$$

Since combining $\epsilon_n \to 0$ and $\vartheta_n \to 0$ guarantees $\eta_n \to 0$, it is easy to see that $\lim_{n\to\infty} g(\|u_n-Gz_n\|) = 0$,

$$\lim_{n\to\infty}\|A_2w_n-A_2x^*\| = \lim_{n\to\infty}\|A_1v_n-A_1y^*\| = 0 \tag{3.8}$$

$$\lim_{n\to\infty}\|Au_n-Ax^*\| = \lim_{n\to\infty}\|Ay_n-Ax^*\| = 0. \tag{3.9}$$

Note that g is a strictly increasing, continuous and convex function with $g(0) = 0$. So it follows that

$$\lim_{n\to\infty} \|u_n - Gz_n\| = 0. \tag{3.10}$$

On the other hand, using Lemma 2.4(ii) and Lemma 2.7(ii), we get

$$\begin{aligned}\|v_n - y^*\|^q &= \|J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n) - J_{\zeta_2}^{B_2}(x^* - \zeta_2 A_2 x^*)\|^q \\ &\leq \langle w_n - \zeta_2 A_2 w_n - (x^* - \zeta_2 A_2 x^*), J_q(v_n - y^*)\rangle \\ &= \langle w_n - x^*, J_q(v_n - y^*)\rangle + \zeta_2 \langle A_2 x^* - A_2 w_n, J_q(v_n - y^*)\rangle \\ &\leq \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] \\ &\quad + \zeta_2 \langle A_2 x^* - A_2 w_n, J_q(v_n - y^*)\rangle,\end{aligned}$$

which hence attains

$$\|v_n - y^*\|^q \leq \|w_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2 \|A_2 x^* - A_2 w_n\| \|v_n - y^*\|^{q-1}.$$

In a similar way, we get

$$\begin{aligned}\|u_n - x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n) - J_{\zeta_1}^{B_1}(y^* - \zeta_1 A_1 y^*)\|^q \\ &\leq \langle v_n - \zeta_1 A_1 v_n - (y^* - \zeta_1 A_1 y^*), J_q(u_n - x^*)\rangle \\ &= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \zeta_1 \langle A_1 y^* - A_1 v_n, J_q(u_n - x^*)\rangle \\ &\leq \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\ &\quad + \zeta_1 \langle A_1 y^* - A_1 v_n, J_q(u_n - x^*)\rangle,\end{aligned}$$

which hence attains

$$\begin{aligned}\|u_n - x^*\|^q &\leq \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) \\ &\quad + q\zeta_1 \|A_1 y^* - A_1 v_n\| \|u_n - x^*\|^{q-1} \\ &\leq \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &\quad + q\zeta_2 \|A_2 x^* - A_2 w_n\| \|v_n - y^*\|^{q-1} \\ &\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + q\zeta_1 \|A_1 y^* - A_1 v_n\| \|u_n - x^*\|^{q-1}.\end{aligned} \tag{3.11}$$

Putting $e_n := J_{\lambda_n}^B(u_n - \lambda_n A u_n)$ and using Lemma 2.4(ii) and Lemma 2.7(ii), we get

$$\begin{aligned}\|e_n - x^*\|^q &= \|J_{\lambda_n}^B(u_n - \lambda_n A u_n) - J_{\lambda_n}^B(x^* - \lambda_n A x^*)\|^q \\ &\leq \langle (u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*), J_q(e_n - x^*)\rangle \\ &\leq \frac{1}{q}[\|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^q + (q-1)\|e_n - x^*\|^q \\ &\quad - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|)],\end{aligned}$$

which together with Proposition 2.10, implies that

$$\begin{aligned}\|e_n - x^*\|^q &\le \|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^q \\ &\quad - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|) \\ &\le \|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|).\end{aligned}$$

Consequently, we have

$$\begin{aligned}\|y_n - x^*\|^q &\le \delta_n \|u_n - x^*\|^q + (1-\delta_n)\|e_n - x^*\|^q \\ &\le \delta_n \|u_n - x^*\|^q \\ &\quad + (1-\delta_n)[\|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|)] \\ &= \|u_n - x^*\|^q - (1-\delta_n)h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|).\end{aligned}$$

This together with (3.4) and (3.11), implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le \alpha_n \|f(x_n) - x^*\|^q + \beta_n \|u_n - x^*\|^q \\ &\quad + \gamma_n \|G z_n - x^*\|^q \\ &\le \alpha_n \|f(x_n) - x^*\|^q + \beta_n \|u_n - x^*\|^q \\ &\quad + \gamma_n[(1-r_n)\|u_n - x^*\|^q + r_n\|y_n - x^*\|^q] \\ &\le \alpha_n \|f(x_n) - x^*\|^q + \beta_n \|u_n - x^*\|^q \\ &\quad + \gamma_n\{(1-r_n)\|u_n - x^*\|^q + r_n[\|u_n - x^*\|^q \\ &\quad - (1-\delta_n)h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|)]\} \\ &\le \alpha_n \|f(x_n) - x^*\|^q + \|u_n - x^*\|^q \\ &\quad - \gamma_n r_n (1-\delta_n) h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|) \\ &\le \alpha_n \|f(x_n) - x^*\|^q + \|x_n - x^*\|^q \\ &\quad - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + q\zeta_1 \|A_1 y^* - A_1 v_n\| \|u_n - x^*\|^{q-1} \\ &\quad + q\zeta_2 \|A_2 x^* - A_2 w_n\| \|v_n - y^*\|^{q-1} \\ &\quad - \gamma_n r_n (1-\delta_n) h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|),\end{aligned}$$

which immediately yields

$$\begin{aligned}&\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + \gamma_n r_n (1-\delta_n) h_1(\|u_n - \lambda_n(Au_n - Ax^*) - e_n\|) \\ &\le \alpha_n \|f(x_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1} + q\zeta_1 \|A_1 y^* - A_1 v_n\| \|u_n - x^*\|^{q-1} \\ &\quad + q\zeta_2 \|A_2 x^* - A_2 w_n\| \|v_n - y^*\|^{q-1}.\end{aligned}$$

Since $\tilde{h}_1, \tilde{h}_2$ and h_1 are strictly increasing, continuous and convex functions with $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$, we conclude from (3.8) and (3.9) that $\|w_n - v_n - x^* + y^*\| \to 0$, $\|v_n - u_n + x^* - y^*\| \to 0$ and $\|u_n - e_n\| \to 0$ as $n \to \infty$. This immediately implies that

$$\lim_{n\to\infty} \|w_n - u_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.12}$$

Furthermore, noticing $w_n = s_n Gx_n + (1 - s_n)(\xi x_n + (1 - \xi)S_n w_n)$, we obtain that

$$\begin{aligned}\|w_n - x^*\|^q &= \langle s_n Gx_n + (1 - s_n)(\xi x_n + (1 - \xi)S_n w_n) - x^*, J_q(w_n - x^*)\rangle \\ &= s_n\langle Gx_n - x^*, J_q(w_n - x^*)\rangle + (1 - s_n)[\xi\langle x_n - x^*, J_q(w_n - x^*)\rangle \\ &\quad + (1 - \xi)\langle S_n w_n - x^*, J_q(w_n - x^*)\rangle] \\ &\le s_n\langle Gx_n - x^*, J_q(w_n - x^*)\rangle + (1 - s_n)[\xi\langle x_n - x^*, J_q(w_n - x^*)\rangle \\ &\quad + (1 - \xi)\|w_n - x^*\|^q],\end{aligned}$$

which together with Lemma 2.4(ii), yields

$$\begin{aligned}&\|w_n - x^*\|^q \\ &\le \frac{1}{s_n + (1 - s_n)\xi}(1 - \xi)[s_n\langle Gx_n - x^*, J_q(w_n - x^*)\rangle \\ &\qquad + (1 - s_n)\xi\langle x_n - x^*, J_q(w_n - x^*)\rangle] \\ &\le \frac{s_n}{s_n + (1 - s_n)\xi}(1 - \xi)\frac{1}{q}[\|Gx_n - x^*\|^q + (q - 1)\|w_n - x^*\|^q - h_3(\|Gx_n - w_n\|)] \\ &\quad + \frac{(1 - s_n)\xi}{s_n + (1 - s_n)\xi}\frac{1}{q}[\|x_n - x^*\|^q + (q - 1)\|w_n - x^*\|^q - \tilde{h}_3(\|x_n - w_n\|)] \\ &\le \frac{1}{q}[\|x_n - x^*\|^q + (q - 1)\|w_n - x^*\|^q] \\ &\quad - \left[\frac{s_n}{q(s_n + (1 - s_n)\xi)}h_3(\|Gx_n - w_n\|)\right. \\ &\qquad \left. + \frac{(1 - s_n)\xi}{q(s_n + (1 - s_n)\xi)}\tilde{h}_3(\|x_n - w_n\|)\right].\end{aligned}$$

This together with (3.2) implies that

$$\begin{aligned}\|u_n - x^*\|^q &\le \|w_n - x^*\|^q \\ &\le \|x_n - x^*\|^q - \left[\frac{s_n}{s_n + (1 - s_n)\xi}h_3(\|Gx_n - w_n\|)\right. \\ &\qquad \left. + \frac{(1 - s_n)\xi}{s_n + (1 - s_n)\xi}\tilde{h}_3(\|x_n - w_n\|)\right].\end{aligned} \tag{3.13}$$

In a similar way, we have

$$\begin{aligned}\|z_n - x^*\|^q &= \|J^B_{\lambda_n}(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\ &\le \langle (u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*), J_q(z_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*)\|^q + (q - 1)\|z_n - x^*\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)],\end{aligned}$$

which together with (3.4), implies that

$$\begin{aligned}\|z_n - x^*\|^q &\le \|(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*)\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|) \\ &\le \|u_n - x^*\|^q - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|).\end{aligned}$$

This together with (3.13), ensures that

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le \alpha_n\|f(x_n)-x^*\|^q+\beta_n\|u_n-x^*\|^q+\gamma_n\|Gz_n-x^*\|^q\\ &\le \alpha_n\|f(x_n)-x^*\|^q+\beta_n\|u_n-x^*\|^q\\ &\quad+\gamma_n[\|u_n-x^*\|^q-h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)]\\ &\le \alpha_n\|f(x_n)-x^*\|^q+\|u_n-x^*\|^q\\ &-\gamma_n h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)]\\ &\le \alpha_n\|f(x_n)-x^*\|^q+\|x_n-x^*\|^q\\ &-\left[\frac{s_n}{s_n+(1-s_n)\xi}h_3(\|Gx_n-w_n\|)+\frac{(1-s_n)\xi}{s_n+(1-s_n)\xi}\tilde{h}_3(\|x_n-w_n\|)\right]\\ &\quad-\gamma_n h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)],\end{aligned}$$

which immediately leads to

$$\begin{aligned}&\frac{s_n}{s_n+(1-s_n)\xi}h_3(\|Gx_n-w_n\|)+\frac{(1-s_n)\xi}{s_n+(1-s_n)\xi}\tilde{h}_3(\|x_n-w_n\|)\\ &\quad+\gamma_n h_2(\|u_n+r_n(y_n-u_n)-\lambda_n(Ay_n-Ax^*)-z_n\|)\\ &\qquad\le\ \alpha_n\|f(x_n)-x^*\|^q+\Gamma_n-\Gamma_{n+1}.\end{aligned}$$

Since h_2, h_3 and $\tilde{h}_3$ are strictly increasing, continuous and convex functions with $h_2(0)=h_3(0)=\tilde{h}_3(0)=0$, from (3.9) and (3.12) we have

$$\lim_{n\to\infty}\|Gx_n-w_n\|=\lim_{n\to\infty}\|x_n-w_n\|=\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.14}$$

So, it follows from (3.12) and (3.14) that

$$\begin{aligned}\|x_n-u_n\|&\le\|x_n-w_n\|+\|w_n-u_n\|\to 0 \quad (n\to\infty),\\ \|x_n-z_n\|&\le\|x_n-u_n\|+\|u_n-z_n\|\to 0 \quad (n\to\infty),\end{aligned}$$

and hence

$$\|x_n-Gx_n\|\le\|x_n-u_n\|+\|u_n-w_n\|+\|w_n-Gx_n\|\to 0 \quad (n\to\infty). \tag{3.15}$$

We now put $p_n := \xi x_n+(1-\xi)S_nw_n\ \forall n\ge 0$. Then we have $w_n=s_nGx_n+(1-s_n)p_n\ \forall n\ge 0$. So it follows from (3.14) and (3.15) that

$$\begin{aligned}\|p_n-x_n\|&=\frac{1}{1-s_n}\|w_n-x_n-s_n(Gx_n-x_n)\|\\ &\le\frac{1}{1-d}(\|w_n-x_n\|+\|Gx_n-x_n\|)\to 0 \quad (n\to\infty),\end{aligned}$$

and hence

$$\lim_{n\to\infty}\|S_nw_n-x_n\|=\frac{1}{1-\xi}\lim_{n\to\infty}\|p_n-x_n\|=0.$$

Since $\{S_n\}_{n=0}^\infty$ is ℓ-uniformly Lipschitzian on C, we deduce from (3.14) that

$$\begin{aligned}\|S_nx_n-x_n\|&\le\|S_nx_n-S_nw_n\|+\|S_nw_n-x_n\|\\ &\le\ell\|x_n-w_n\|+\|S_nw_n-x_n\|\to 0 \quad (n\to\infty).\end{aligned} \tag{3.16}$$

We next claim that $\|x_n-\overline{S}x_n\|\to 0$ as $n\to\infty$ where $\overline{S}:=(2I-S)^{-1}$. In fact, it is first clear that $S: C\to C$ is pseudocontractive and ℓ-Lipschitzian, where $Sx=\lim_{n\to\infty}S_nx\ \forall x\in C$. We claim that $\lim_{n\to\infty}\|Sx_n-x_n\|=0$. Using the boundedness of $\{x_n\}$ and setting $D = \overline{\text{conv}}\{x_n : n \geq 0\}$ (the closed convex hull of the set $\{x_n : n \geq 0\}$), by the assumption we have $\sum_{n=1}^{\infty} \sup_{x \in D} \|S_n x - S_{n-1} x\| < \infty$. Hence, by Proposition 2.13 we get $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$, which immediately arrives at

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0.$$

Thus, from (3.16) we have

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad (n \to \infty). \tag{3.17}$$

Now, let us show that if we define $\overline{S} := (2I - S)^{-1}$, then $\overline{S} : C \to C$ is nonexpansive, $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ and $\lim_{n\to\infty} \|x_n - \overline{S}x_n\| = 0$. As a matter of fact, put $\overline{S} := (2I - S)^{-1}$, where I is the identity operator of E. Then it is known that $\overline{S}$ is nonexpansive and $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ as a consequence of [27, Theorem 6]. From (3.17) it follows that

$$\begin{aligned} \|x_n - \overline{S}x_n\| &= \|\overline{S}\,\overline{S}^{-1} x_n - \overline{S}x_n\| \leq \|\overline{S}^{-1} x_n - x_n\| \\ &= \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.18}$$

For each $n \geq 0$, we put $T_{\lambda_n} := J_{\lambda_n}^B (I - \lambda_n A)$. Then from (3.12) we have

$$\begin{aligned} \|x_n - T_{\lambda_n} x_n\| &\leq \|x_n - u_n\| + \|u_n - T_{\lambda_n} u_n\| + \|T_{\lambda_n} u_n - T_{\lambda_n} x_n\| \\ &\leq 2\|x_n - u_n\| + \|u_n - e_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

Noticing $0 < \lambda \leq \lambda_n \ \forall n \geq 0$ and using Proposition 2.8(ii), we obtain

$$\|T_\lambda x_n - x_n\| \leq 2\|T_{\lambda_n} x_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.19}$$

We define the mapping $\Phi : C \to C$ by $\Phi x := \nu_1 \overline{S}x + \nu_2 Gx + (1 - \nu_1 - \nu_2) T_\lambda x \ \forall x \in C$ with $\nu_1 + \nu_2 < 1$ for constants $\nu_1, \nu_2 \in (0, 1)$. Then by Lemma 2.15 and Proposition 2.8(i), we know that Φ is nonexpansive and

$$\text{Fix}(\Phi) = \text{Fix}(\overline{S}) \cap \text{Fix}(G) \cap \text{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{Fix}(G) \cap (A + B)^{-1}(0) \ (=: \mathit{\Omega}).$$

Taking into account that

$$\|\Phi x_n - x_n\| \leq \nu_1 \|\overline{S}x_n - x_n\| + \nu_2 \|Gx_n - x_n\| + (1 - \nu_1 - \nu_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.15), (3.18) and (3.19) that

$$\lim_{n\to\infty} \|\Phi x_n - x_n\| = 0. \tag{3.20}$$

Let $z_t = tf(z_t) + (1 - t)\Phi z_t \ \forall t \in (0, 1)$. Then it follows from Proposition 2.9 that $\{z_t\}$ converges strongly to a point $x^* \in \text{Fix}(\Phi) = \mathit{\Omega}$, which solves the VIP:

$$\langle (I - f)x^*, J(x^* - p) \rangle \leq 0 \quad \forall p \in \mathit{\Omega}.$$

Also, from (2.1) we get

$$\begin{aligned}\|z_t - x_n\|^q &= \|t(f(z_t) - x_n) + (1-t)(\Phi z_t - x_n)\|^q \\ &\le (1-t)^q\|\Phi z_t - x_n\|^q + qt\langle f(z_t) - x_n, J_q(z_t - x_n)\rangle \\ &= (1-t)^q\|\Phi z_t - x_n\|^q + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle \\ &\quad + qt\langle z_t - x_n, J_q(z_t - x_n)\rangle \\ &\le (1-t)^q(\|\Phi z_t - \Phi x_n\| + \|\Phi x_n - x_n\|)^q \\ &\quad + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q \\ &\le (1-t)^q(\|z_t - x_n\| + \|\Phi x_n - x_n\|)^q \\ &\quad + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q,\end{aligned}$$

which immediately attains

$$\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \le \frac{(1-t)^q}{qt}(\|z_t - x_n\| + \|\Phi x_n - x_n\|)^q + \frac{qt-1}{qt}\|z_t - x_n\|^q.$$

From (3.20), we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \\ &\le \frac{(1-t)^q}{qt}M + \frac{qt-1}{qt}M = \left(\frac{(1-t)^q + qt - 1}{qt}\right)M,\end{aligned} \tag{3.21}$$

where M is a constant such that $\|z_t - x_n\|^q \le M$ for all $n \ge 0$ and $t \in (0,1)$. It is clear that $((1-t)^q + qt - 1)/qt \to 0$ as $t \to 0$. Since J_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_t \to x^*$, we get

$$\|J_q(x_n - z_t) - J_q(x_n - x^*)\| \to 0 \quad (t \to 0).$$

So we obtain

$$\begin{aligned}&|\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\ &\quad = |\langle f(z_t) - f(x^*), J_q(x_n - z_t)\rangle + \langle f(x^*) - x^*, J_q(x_n - z_t)\rangle \\ &\qquad + \langle x^* - z_t, J_q(x_n - z_t)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\ &\quad \le |\langle f(x^*) - x^*, J_q(x_n - z_t) - J_q(x_n - x^*)\rangle| + |\langle f(z_t) - f(x^*), J_q(x_n - z_t)\rangle| \\ &\qquad + |\langle x^* - z_t, J_q(x_n - z_t)\rangle| \\ &\le \|f(x^*) - x^*\|\|J_q(x_n - z_t) - J_q(x_n - x^*)\| + (1+\varrho)\|z_t - x^*\|\|x_n - z_t\|^{q-1}.\end{aligned}$$

Thus, for each $n \ge 0$, we have

$$\lim_{t\to 0}\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle = \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle.$$

From (3.21), as $t \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle \le 0. \tag{3.22}$$

By (C1) and (3.10), we get

$$\begin{aligned}&\|x_{n+1} - x_n\| = \|\alpha_n f(x_n) + \beta_n u_n + \gamma_n G z_n - x_n\| \\ &\le \alpha_n\|f(x_n) - x_n\| + \|u_n - x_n\| + \|Gz_n - u_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we have

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, J_q(x_{n+1}-x^*)\rangle \le 0. \tag{3.24}$$

Using Lemma 2.16 and (3.24), we can conclude that $\Gamma_n \to 0$ as $n\to\infty$. Therefore, $x_n \to x^*$ as $n\to\infty$.

Case 2. Suppose that $\exists\{\Gamma_{k_i}\}\subset\{\Gamma_k\}$ s.t. $\Gamma_{k_i}<\Gamma_{k_i+1}$ $\forall i\in\mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau:\mathbf{N}\to\mathbf{N}$ by

$$\tau(k):=\max\{i\le k:\Gamma_i<\Gamma_{i+1}\}.$$

Using Lemma 2.14, we get

$$\Gamma_{\tau(k)}\le\Gamma_{\tau(k)+1}\quad\text{and}\quad\Gamma_k\le\Gamma_{\tau(k)+1}.$$

Putting $\Gamma_k=\|x_k-x^*\|^q$ $\forall k\in\mathbf{N}$ and using the same inference as in Case 1, we can obtain

$$\lim_{k\to\infty}\|x_{\tau(k)+1}-x_{\tau(k)}\|=0 \tag{3.25}$$

and

$$\limsup_{k\to\infty}\langle f(x^*)-x^*, J_q(x_{\tau(k)+1}-x^*)\rangle\le 0.$$

Since $\Gamma_{\tau(k)}\le\Gamma_{\tau(k)+1}$ and $\alpha_{\tau(k)}>0$, we conclude from (3.7) that

$$\|x_{\tau(k)}-x^*\|^q\le\frac{q}{1-\varrho}\langle f(x^*)-x^*, J_q(x_{\tau(k)+1}-x^*)\rangle$$

and hence

$$\limsup_{k\to\infty}\|x_{\tau(k)}-x^*\|^q\le 0.$$

Consequently,

$$\lim_{k\to\infty}\|x_{\tau(k)}-x^*\|^q=0.$$

Using Proposition 2.1 and (3.25), we obtain

$$\begin{aligned}\|x_{\tau(k)+1}-x^*\|^q-\|x_{\tau(k)}-x^*\|^q &\le q\langle x_{\tau(k)+1}-x_{\tau(k)}, J_q(x_{\tau(k)}-x^*)\rangle\\ &\quad+\kappa_q\|x_{\tau(k)+1}-x_{\tau(k)}\|^q\\ &\le q\|x_{\tau(k)+1}-x_{\tau(k)}\|\|x_{\tau(k)}-x^*\|^{q-1}\\ &\quad+\kappa_q\|x_{\tau(k)+1}-x_{\tau(k)}\|^q\to 0\quad(k\to\infty).\end{aligned}$$

Owing to $\Gamma_k\le\Gamma_{\tau(k)+1}$, we get

$$\begin{aligned}\|x_k-x^*\|^q\le&\|x_{\tau(k)+1}-x^*\|^q\\ &\le\|x_{\tau(k)}-x^*\|^q+q\|x_{\tau(k)+1}-x_{\tau(k)}\|\|x_{\tau(k)}-x^*\|^{q-1}\\ &\quad+\kappa_q\|x_{\tau(k)+1}-x_{\tau(k)}\|^q.\end{aligned}$$

It is easy to see from (3.25) that $x_k\to x^*$ as $k\to\infty$. This completes the proof. □

We also obtain the strong convergence result for the relaxed Mann-type viscosity implicit method in a real Hilbert space H. It is well known that $\kappa_2=1$ [38]. Hence, by Theorem 3.2 we derive the following conclusion.

Corollary 3.3. *Let $\emptyset \neq C \subset H$ be a closed convex set. Let $f : C \to C$ be a ϱ-contraction with constant $\varrho \in [0,1)$, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Suppose that $B_1, B_2 : C \to 2^H$ are both maximal monotone operators and $A_i : C \to H$ is σ_i-inverse-strongly monotone mapping for $i = 1, 2$. Define the mapping $G : C \to C$ by $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1) J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ with $0 < \zeta_i < 2\sigma_i$ for $i = 1, 2$. Let $A : C \to H$ and $B : C \to 2^H$ be a σ-inverse-strongly monotone mapping and a maximal monotone operator, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}(0) \neq \emptyset$. For any given $x_0 \in C$ and $\xi \in (0,1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} w_n = s_n G x_n + (1 - s_n)(\xi x_n + (1 - \xi) S_n w_n), \\ y_n = \delta_n G w_n + (1 - \delta_n) J_{\lambda_n}^B (I - \lambda_n A) G w_n, \\ z_n = J_{\lambda_n}^B (G w_n - \lambda_n A y_n + r_n (y_n - G w_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n G w_n + \gamma_n G z_n \quad \forall n \geq 0, \end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < \delta \leq \delta_n \leq \overline{\delta} < 1$*;*
(C3) $0 < a \leq \beta_n \leq b < 1$ *and* $0 < c \leq s_n \leq d < 1$*;*
(C4) $0 < r \leq r_n < 1$ *and* $0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < 2\sigma$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C. Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x$ $\forall x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, p - x^* \rangle \geq 0$ $\forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

Remark 3.4. Compared with the corresponding results in Manaka and Takahashi [26], Sunthrayuth and Cholamjiak [34], and Ceng et al. [18], our results improve and extend them in the following aspects.

(i) The problem of solving the VI for both monotone operators A, B with the FPP constraint of a nonexpansive mapping S in [26, Theorem 3.1] is extended to develop our problem of solving the GSVI (1.1) with the constraints of the VI for both accretive operators A, B and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of ℓ-uniformly Lipschitzian pseudocontractions. The Mann-type iterative scheme with weak convergence in [26, Theorem 3.1] is extended to develop our relaxed Mann-type viscosity implicit method with strong convergence.

(ii) The problem of solving the GSVI (1.1) with the FPP constraint of a strict pseudocontraction T in [18, Theorem 1], is extended to develop our problem of solving the GSVI (1.1) with the constraints of the VI for two accretive operators A, B and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of ℓ-uniformly Lipschitzian pseudocontractions. The composite viscosity implicit rule in [18, Theorem 1] is extended to develop our relaxed Mann-type viscosity implicit method.

(iii) The problem of solving the VI for both accretive operators A, B with the FPP constraint of a nonexpansive mapping S in [34, Theorem 3.3] is extended to develop our problem of solving the GSVI (1.1) with the constraints of the VI for both accretive operators A, B and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of ℓ-uniformly

Lipschitzian pseudocontractions. The modified viscosity-type extragradient method in [34, Theorem 3.3] is extended to develop our relaxed Mann-type viscosity implicit method.

4. Some applications

In this section, we give some applications of Corollary 3.3 to important mathematical problems in the setting of Hilbert spaces.

4.1. **Application to variational inequality problem.** Given a nonempty closed convex subset $C \subset H$ and a nonlinear monotone operator $A : C \to H$. Consider the classical VIP of finding $u^* \in C$ s.t.

$$\langle Au^*, v - u^* \rangle \geq 0 \quad \forall v \in C. \tag{4.1}$$

The solution set of problem (4.1) is denoted by $\mathrm{VI}(C, A)$. It is clear that $u^* \in C$ solves VIP (4.1) if and only if it solves the fixed point equation $u^* = P_C(u^* - \lambda Au^*)$ with $\lambda > 0$. Let i_C be the indicator function of C defined by

$$i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{if } u \notin C. \end{cases}$$

We use $N_C(u)$ to indicate the normal cone of C at $u \in H$, i.e., $N_C(u) = \{w \in H : \langle w, v - u \rangle \leq 0 \ \forall v \in C\}$. It is known that i_C is a proper, convex and lower semicontinuous function and its subdifferential ∂i_C is a maximal monotone mapping [32]. We define the resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x) \quad \forall x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{w \in H : i_C(u) + \langle w, v - u \rangle \leq i_C(v) \ \forall v \in H\} \\ &= \{w \in H : \langle w, v - u \rangle \leq 0 \ \forall v \in C\} = N_C(u) \quad \forall u \in C. \end{aligned}$$

Hence, we get

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) &\Leftrightarrow \ x - u \in \lambda N_C(u) \\ &\Leftrightarrow \ \langle x - u, v - u \rangle \leq 0 \quad \forall v \in C \\ &\Leftrightarrow \ u = P_C(x), \end{aligned}$$

where P_C is the metric projection of H onto C. Moreover, we also have $(A + \partial i_C)^{-1}(0) = \mathrm{VI}(C, A)$ [35]. Thus, putting $B = \partial i_C$ in Corollary 3.3, we obtain the following result:

Theorem 4.1. *Let f, A, A_i, B_i $(i = 1, 2)$ and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.3. Suppose that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$. For any given $x_0 \in C$ and $\xi \in (0, 1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n = s_n G x_n + (1 - s_n)(\xi x_n + (1 - \xi) S_n w_n), \\ y_n = \delta_n G w_n + (1 - \delta_n) P_C(I - \lambda_n A) G w_n, \\ z_n = P_C(G w_n - \lambda_n A y_n + r_n(y_n - G w_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n G w_n + \gamma_n G z_n \quad \forall n \geq 0, \end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1]$ *with* $\alpha_n+\beta_n+\gamma_n = 1$ *and* $\{\lambda_n\} \subset (0,\infty)$ *are such that the conditions (C1)-(C4) in Corollary 3.3 hold. Then* $x_n \to x^* \in \Omega$, *which is the unique solution to the VIP:* $\langle (I-f)x^*, p-x^*\rangle \geq 0\ \forall p \in \Omega$, *i.e., the fixed point equation* $x^* = P_\Omega f(x^*)$.

4.2. **Application to split feasibility problem.** Let H_1 and H_2 be two real Hilbert spaces. Consider the following split feasibility problem (SFP) of finding

$$u \in C \quad \text{subject to} \quad \mathcal{T}u \in Q, \tag{4.2}$$

where C and Q are closed convex subsets of H_1 and H_2, respectively, and $\mathcal{T} : H_1 \to H_2$ is a bounded linear operator with its adjoint $\mathcal{T}^*$. The solution set of SFP is denoted by $\mho := C \cap \mathcal{T}^{-1}Q = \{u \in C : \mathcal{T}u \in Q\}$. In 1994, Censor and Elfving [20] first introduced the SFP for modelling inverse problems of radiation therapy treatment planning in a finite dimensional Hilbert space, which arise from phase retrieval and in medical image reconstruction.

It is known that $z \in C$ solves the SFP (4.2) if and only if z is a solution of the minimization problem: $\min_{y\in C} g(y) := \frac{1}{2}\|\mathcal{T}y - P_Q\mathcal{T}y\|^2$. Note that the function g is differentiable convex and has the Lipschitzian gradient defined by $\nabla g = \mathcal{T}^*(I - P_Q)\mathcal{T}$. Moreover, ∇g is $\frac{1}{\|\mathcal{T}\|^2}$-inverse-strongly monotone, where $\|\mathcal{T}\|^2$ is the spectral radius of $\mathcal{T}^*\mathcal{T}$ [5]. So, $z \in C$ solves the SFP if and only if it solves the variational inclusion problem of finding $z \in H_1$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial i_C(z) &\Leftrightarrow\ 0 \in z + \lambda\partial i_C(z) - (z - \lambda\nabla g(z)) \\ &\Leftrightarrow\ z - \lambda\nabla g(z) \in z + \lambda\partial i_C(z) \\ &\Leftrightarrow\ z = (I + \lambda\partial i_C)^{-1}(z - \lambda\nabla g(z)) \\ &\Leftrightarrow\ z = P_C(z - \lambda\nabla g(z)). \end{aligned}$$

Now, setting $A = \nabla g,\ B = \partial i_C$ and $\sigma = \frac{1}{\|\mathcal{T}\|^2}$ in Corollary 3.3, we obtain the following result:

Theorem 4.2. *Let* $f, A_i, B_i\ (i = 1,2)$ *and* $\{S_n\}_{n=0}^\infty$ *be the same as in Corollary 3.3. Assume that* $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mho \neq \emptyset$. *For any given* $x_0 \in C$ *and* $\xi \in (0,1)$, *let* $\{x_n\}_{n=0}^\infty$ *be the sequence generated by*

$$\begin{cases} w_n = s_nGx_n + (1-s_n)(\xi x_n + (1-\xi)S_nw_n), \\ y_n = \delta_nGw_n + (1-\delta_n)P_C(I - \lambda_n\mathcal{T}^*(I-P_Q)\mathcal{T})Gw_n, \\ z_n = P_C(Gw_n - \lambda_n\mathcal{T}^*(I-P_Q)\mathcal{T}y_n + r_n(y_n - Gw_n)), \\ x_{n+1} = \alpha_nf(x_n) + \beta_nGw_n + \gamma_nGz_n \quad \forall n \geq 0, \end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1]$ *with* $\alpha_n+\beta_n+\gamma_n = 1$ *and* $\{\lambda_n\} \subset (0,\infty)$ *are such that the conditions (C1)-(C4) in Corollary 3.3 hold where* $\sigma = \frac{1}{\|\mathcal{T}\|^2}$. *Then* $x_n \to x^* \in \Omega$, *which is the unique solution to the VIP:* $\langle (I-f)x^*, p-x^*\rangle \geq 0\ \forall p \in \Omega$, *i.e., the fixed point equation* $x^* = P_\Omega f(x^*)$.

4.3. **Application to LASSO problem.** In this subsection, we first recall the least absolute shrinkage and selection operator (LASSO) [37], which can be formulated as a convex constrained optimization problem:

$$\min_{y\in H} \frac{1}{2}\|\mathcal{T}y-b\|_2^2 \quad \text{subject to} \quad \|y\|_1 \le s, \tag{4.3}$$

where $\mathcal{T}: H \to H$ is a bounded operator on H, b is a fixed vector in H and $s > 0$. Let $\mho$ be the solution set of LASSO (4.3). The LASSO has received much attention because of the involvement of the ℓ_1 norm which promotes sparsity, phenomenon of many practical problems arising in statics model, image compression, compressed sensing and signal processing theory.

In terms of the optimization theory, one knows that the solution to the LASSO problem (4.3) is a minimizer of the following convex unconstrained minimization problem so-called Basis Pursuit denoising problem:

$$\min_{y\in H} g(y) + h(y),$$

where $g(y) := \frac{1}{2}\|\mathcal{T}y-b\|_2^2,\ h(y) := \lambda\|y\|_1$ and $\lambda \ge 0$ is a regularization parameter. It is known that $\nabla g(y) = \mathcal{T}^*(\mathcal{T}y-b)$ is $\frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$-inverse-strongly monotone. Hence, we have that z solves the LASSO if and only if z solves the variational inclusion problem of finding $z \in H$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial h(z) &\Leftrightarrow\ 0 \in z + \lambda\partial h(z) - (z - \lambda\nabla g(z)) \\ &\Leftrightarrow\ z - \lambda\nabla g(z) \in z + \lambda\partial h(z) \\ &\Leftrightarrow\ z = (I + \lambda\partial h)^{-1}(z - \lambda\nabla g(z)) \\ &\Leftrightarrow\ z = \mathrm{prox}_h(z - \lambda\nabla g(z)), \end{aligned}$$

where $\mathrm{prox}_h(y)$ is the proximal of $h(y) := \lambda\|y\|_1$ given by

$$\mathrm{prox}_h(y) = \mathrm{argmin}_{u\in H}\{\lambda\|u\|_1 + \frac{1}{2}\|u-y\|_2^2\} \quad \forall y \in H,$$

which is separable in indices. Then, for $y \in H$,

$$\begin{aligned} \mathrm{prox}_h(y) &= \mathrm{prox}_{\lambda\|\cdot\|_1}(y) \\ &= (\mathrm{prox}_{\lambda|\cdot|}(y_1), \mathrm{prox}_{\lambda|\cdot|}(y_2), ..., \mathrm{prox}_{\lambda|\cdot|}(y_n)), \end{aligned}$$

where $\mathrm{prox}_{\lambda|\cdot|}(y_i) = \mathrm{sgn}(y_i)\max\{|y_i| - \lambda, 0\}$ for $i = 1, 2, ..., n$.

In 2014, Xu [40] suggested the following proximal-gradient algorithm (PGA):

$$x_{k+1} = \mathrm{prox}_h(x_k - \lambda_k\mathcal{T}^*(\mathcal{T}x_k - b)).$$

He proved the weak convergence of the PGA to a solution of the LASSO problem (4.3).

Next, putting $C = H,\ A = \nabla g,\ B = \partial h$ and $\sigma = \frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$ in Corollary 3.3, we obtain the following result:

Theorem 4.3. *Let f, A_i, B_i $(i = 1, 2)$ and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.3 with $C = H$. Assume that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mho \neq \emptyset$. For any given*

$x_0 \in H$ and $\xi \in (0,1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by

$$\begin{cases} w_n = s_n G x_n + (1-s_n)(\xi x_n + (1-\xi) S_n w_n), \\ y_n = \delta_n G w_n + (1-\delta_n)\mathrm{prox}_h(G w_n - \lambda_n \mathcal{T}^*(\mathcal{T} G w_n - b)), \\ z_n = \mathrm{prox}_h(G w_n - \lambda_n \mathcal{T}^*(\mathcal{T} y_n - b) + r_n(y_n - G w_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n G w_n + \gamma_n G z_n \quad \forall n \geq 0, \end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0,\infty)$ are such that the conditions (C1)-(C4) in Corollary 3.3 hold where $\sigma = \frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I-f)x^*, p - x^* \rangle \geq 0$ $\forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

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Manuscript received June 1 2024
revised September 3 2024

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