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A GENERALIZATION OF THE WARDOWSKI EXISTENCE RESULT

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Abstract. In the present paper we obtain a generalization of the Wardowski existence result which also generalizes some other results known in the literature.

1. Introduction

For nearly sixty years now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) and contractive mappings. See, for example, [3, 4, 8, 12, 13, 15–20, 23–30, 33, 34] and references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) orbits of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed points and optimization problems, which find important applications in engineering, medical and the natural sciences [5,6,10,11,33,34]. In [31] D. Wardowski introduced an interesting class of mappings which contains Banach contractions and showed the existence of fixed points for these mappings. Wardowski type contractions were studied in [7, 9, 14, 31, 32]. In the present paper we obtain a generalization of the Wardowski existence result which also generalizes some other results known in the literature.

More precisely, assume that (X, ρ) is a complete metric space, $T : X \to X$, $F : (0, \infty) \to R^1$ is a function, $\tau > 0$ and that for each $x, y \in X$ such that $x \neq y$,

$$F(\rho(T(x), T(y))) + \tau \leq F(\rho(x, y)).$$

Then T is called the Wardowski type contraction. There are some assumption which are posed on F in the literature. In particular it is assume that F is strictly monotone. Under these assumptions it was shown in [31] that the mapping T has a fixed point. There are many examples of function F in the literature but the most typical of them is the case when $F(t) = \ln(t)$, $t \in (0, \infty)$. Then T is the Wardowski type contraction if and only if for each $x, y \in X$,

$$\rho(T(x), T(y)) \leq e^{-\tau} \rho(x, y),$$

where τ is a positive constant which means that T is a strict contraction.

In [32] Wardowski generalized his result of [31] and showed that a mapping $T : X \to X$ which satisfies

$$\phi(\rho(x, y)) + F(\rho(T(x), T(y))) \leq F(\rho(x, y))$$

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for each $x, y \in X$ such that $x \neq y$, where $F : (0, \infty) \to (0, \infty)$ is a strictly increasing function and $\phi : (0, \infty) \to (0, \infty)$ satisfies

$$\liminf_{t \to s^+} \phi(t) > 0 \text{ for each } s > 0,$$

has a fixed point.

Another generalization of the fixed point results of [31, 32] was obtained in [14] where it was shown that a mapping $T : X \to X$ has a fixed point if it satisfies

$$\phi(\rho(x, y)) + F_2(\rho(T(x), T(y))) \leq F_1(\rho(x, y))$$

for each $x, y \in X$ such that $x \neq y$, where $F_1 : (0, \infty) \to (0, \infty)$ is an increasing function, $F_2 : (0, \infty) \to (0, \infty)$ is a continuous function, $F_1(x) \leq F_2(x)$ for each $x \in (0, \infty)$ and the function $\phi : (0, \infty) \to (0, \infty)$ is as above. In the present paper we obtain a generalization of all the results mentioned above. Namely, in our case the mapping T satisfies all the assumptions posed in [32] but the function F is merely increasing.

2. The first result

Assume that (X, ρ) is a complete metric space endowed with the metric ρ. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

For each $x \in X$ and each set $A \subset X$ set

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

Assume that $T : X \to X$, $F, \phi : (0, \infty) \to (0, \infty)$, the function F is increasing,

$$\liminf_{t \to s^+} \phi(t) > 0 \text{ for each } s > 0 \tag{2.1}$$

and that for each $x, y \in X$ satisfying $x \neq y$, we have

$$\phi(\rho(x, y)) + F(\rho(T(x), T(y))) \leq F(\rho(x, y)). \tag{2.2}$$

Equation (2.2) implies that for each $x, y \in X$,

$$\rho(T(x), T(y)) \leq \rho(x, y). \tag{2.3}$$

In this section we show that the mapping T has a fixed point. In contrast to the results known in the literature in our study the function F is merely increasing.

Theorem 2.1. *Let $x \in X$. Then the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges and its limit is a fixed point of T.*

Proof. We may assume without loss of generality that

$$T^n(x) \neq T^{n+1}(x) \text{ for each integer } n \geq 0.$$

By (2.2) and (2.3), for each integer $n \geq 0$,

$$\phi(\rho(T^n(x), T^{n+1}(x))) + F(\rho(T^{n+1}(x), T^{n+2}(x))) \leq F(\rho(T^n(x), T^{n+1}(x))), \tag{2.4}$$

$$\rho(T^{n+1}(x), T^{n+2}(x)) < \rho(T^n(x), T^{n+1}(x)). \tag{2.5}$$

We show that

$$\lim_{n\to\infty} \rho(T^n(x), T^{n+1}(x)) = 0. \tag{2.6}$$

Assume the contrary. Then by (2.5),

$$\gamma := \lim_{n\to\infty} \rho(T^n(x), T^{n+1}(x)) > 0. \tag{2.7}$$

Set

$$\Delta = \lim_{s\to\gamma^+} \phi(s) > 0.$$

By (2.5), (2.7) and the relation above, there exists a natural number n_0 such that for each integer $n \geq n_0$,

$$\phi(\rho(T^n(x), T^{n+1}(x)) \geq \Delta/2. \tag{2.8}$$

In view of (2.4) and (2.8), for each integer $n \geq n_0$,

$$F(\rho(T^{n+1}(x), T^{n+2}(x))) \leq F(\rho(T^n(x), T^{n+1}(x))) - \Delta/2$$

and for each integer $n \geq n_0$,

$$F(\rho(T^n(x), T^{n+1}(x))) \leq F(\rho(T^{n_0}(x), T^{n_0+1}(x))) - 2^{-1}\Delta(n - n_0) \to -\infty.$$

On the other hand in view of (2.7), for each integer $n \geq n_0$,

$$F(\rho(T^n(x), T^{n+1}(x))) \geq F(\gamma).$$

The contradiction we have reached proves that

$$\lim_{n\to\infty} \rho(T^n(x), T^{n+1}(x))) = 0.$$

We show that $\{T^n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence. Let $\epsilon \in (0,1)$. Set

$$\Delta_0 = \liminf_{s\to\epsilon^+} \phi(s) > 0. \tag{2.9}$$

We show that for all sufficiently large natural numbers i, j,

$$\rho(T^i(x), T^j(x)) \leq \epsilon. \tag{2.10}$$

Since the function F is increasing the set of all points where F is discontinuous is countable. Therefore we may assume without loss of generality that the function F is continuous at ϵ. Choose

$$\delta \in (0, \epsilon/4)$$

such that

$$|F(\xi) - F(\epsilon)| \leq \Delta_0/8 \text{ for each } \xi \in [\epsilon - 4\delta, \epsilon + 4\delta]. \tag{2.11}$$

By (2.6), there exists a natural number n_0 such that

$$\rho(T^i(x), T^{i+1}(x)) \leq \delta \text{ for each integer } i \geq n_0. \tag{2.12}$$

We show that for all sufficiently large natural numbers i, j equation (2.10) holds. Assume the contrary. Then for each integer $k \geq 1$ there exist

$$j_k > i_k \geq k + n_0 \tag{2.13}$$

such that

$$\rho(T^{i_k}(x), T^{j_k}(x)) > \epsilon. \tag{2.14}$$

Let $k \geq 1$ be an integer. By (2.12)-(2.14),

$$j_k > i_k + 1. \tag{2.15}$$

By (2.15), we may assume without loss of generality that

$$\rho(T^{i_k}(x), T^s(x)) \leq \epsilon, \ s = i_k + 1, \dots, j_k - 1$$

and in particular

$$\rho(T^{i_k}(x), T^{j_k - 1}(x)) \leq \epsilon. \tag{2.16}$$

It follows from (2.6), (2.13) and (2.16) that

$$\rho(T^{i_k}(x), T^{j_k}(x)) \leq \rho(T^{i_k}(x), T^{j_k-1}(x)) + \rho(T^{j_k-1}(x), T^{j_k}(x))$$
$$\leq \epsilon + \rho(T^{j_k-1}(x), T^{j_k}(x))$$

and

$$\lim_{k \to \infty} \rho(T^{i_k}(x), T^{j_k}(x)) = \epsilon. \tag{2.17}$$

It follows from (2.1), (2.9), (2.14) and (2.17) that there exists an integer $k_1 \geq 1$ such that for each integer $k > k_1$,

$$\rho(T^{i_k}(x), T^{j_k}(x)) \in (\epsilon, \epsilon + \delta), \tag{2.18}$$

$$\phi(\rho(T^{i_k}(x), T^{j_k}(x))) \geq \Delta_0/2. \tag{2.19}$$

Let $k \geq k_1$ be an integer. By (2.2),

$$\phi(\rho(T^{i_k}(x), T^{j_k}(x))) + F(\rho(T^{i_k+1}(x), T^{j_k+1}(x))) \leq F(\rho(T^{i_k}(x), T^{j_k}(x))). \tag{2.20}$$

Equations (2.11) and (2.18)-(2.20) imply that

$$\Delta_0/2 + F(\rho(T^{i_k+1}(x), T^{j_k+1}(x))) \leq F(\epsilon) + \Delta/4,$$
$$F(\rho(T^{i_k+1}(x), T^{j_k+1}(x))) \leq F(\epsilon) - \Delta_0/4.$$

Equation (2.11) implies that

$$\rho(T^{i_k+1}(x), T^{j_k+1}(x)) \leq \epsilon - 4\delta.$$

Together with (2.12) this implies that

$$\rho(T^{i_k}(x), T^{j_k}(x)) \leq \rho(T^{i_k}(x), T^{i_k+1}(x)) + \rho(T^{i_k+1}(x), T^{j_k+1}(x))$$
$$+\rho(T^{j_k}(x), T^{j_k+1}(x)) \leq \epsilon - 4\delta + 2\delta.$$

This contradicts (2.14). The contradiction we have reached proves that (2.10) holds for all sufficiently large natural numbers i, j. Therefore $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{n \to \infty} T^n(x).$$

By (2.3) and the equation above,

$$d(T(x_*), T^{n+1}(x)) \leq d(x_*, T^n(x)) \to 0$$

as $n \to \infty$ and $x_* \in T(x_*)$. Theorem 2.1 is proved. □

3. The second result

Fix $\theta \in X$.

Theorem 3.1. *Assume that the function F is bounded from above on every interval $[a, b]$ such that $0 < a < b$, $x_* \in X$, $x_* = T(x_*)$, for each $t > 0$,*

$$\inf\{\phi(s) : \ s \in [t, \infty, 0\} > 0 \tag{3.1}$$

and that for each $x \in X \setminus \{x_*\}$,

$$\phi(\rho(x, x_*)) + F(\rho(T(x), x_*)) \leq F(\rho(x, x_*)). \tag{3.2}$$

Let $M > 0$, $\epsilon \in (0, 1)$. Then there exists an integer $n_0 \geq 1$ such that for each $x \in B(\theta, M)$ and each integer $n \geq n_0$, we have

$$\rho(T^n(x), x_*) \leq \epsilon.$$

Proof. By (3.2), for each $y \in X$,

$$\rho(T(y), x_*) \leq \rho(y, x_*). \tag{3.3}$$

By (3.1) and our assumptions there exist

$$\delta \in (0, \epsilon/4)$$

such that

$$\phi(t) \geq \delta, \ t \in [4^{-1}\epsilon, \infty) \tag{3.4}$$

and

$$M_1 > F(t), \ t \in [M, \epsilon/4]. \tag{3.5}$$

Choose a natural number n_0 such that

$$n_0\delta > M_1 - F(4^{-1}\epsilon).$$

Let $x \in B(\theta, M)$. We show that there exists $j \in \{0, \ldots, n_0\}$ such that

$$\rho(T^j(x), x_*) \leq \epsilon/4.$$

Assume the contrary. For each $j \in \{0, \ldots, n_0\}$,

$$\rho(T^j(x), x_*) > \epsilon/4. \tag{3.6}$$

By (3.2), (3.6) and (3.9),

$$\phi(\rho(T^j(x), x_*)) + F(\rho(T^{j+1}(x), x_*)) \leq F(\rho(T^j(x), x_*)),$$

$$\delta + F(\rho(T^{j+1}(x), x_*)) \leq F(\rho(T^{j+1}(x), x_*)),$$

$$F(\epsilon/4) \leq F(\rho(T^{n_0}(x), x_*) \leq F(\rho(T^j(x), x_*)) - n_0\delta \leq M_1 - n_0\delta.$$

This contradicts (3.6). The contradiction we have reached proves that there exists $j \in \{0, \ldots, n_0\}$ for which

$$\rho(T^j(x), x_*) \leq \epsilon/4, \ \rho(T^i(x), x_*) \leq \epsilon$$

for each integer $i \geq n_0$. Theorem 3.1 is proved. ∎

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