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TWO STRONG CONVERGENCE THEOREMS FOR ITERATIVE METHODS FOR SOLVING THE SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT-SETS IN HILBERT SPACES

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Abstract. In this paper, a slightly generalized split feasibility problem with multiple output-sets is presented in Hilbert spaces. Two iterative algorithms are proposed to solve the problem. The strong convergence and the bounded perturbation resilience for both the algorithms are established. Preliminary numerical simulations are provided to show the validity of the algorithms.

1. Introduction

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let A^* denote the adjoint of A. The split feasibility problem (SFP) is formulated as follows: Find an element $x^* \in C$ such that

$$Ax^* \in Q. \tag{1.1}$$

The SFP was first proposed by Censor and Elfving [4] for modeling certain inverse problems. It has turned out to play an key role in many application fields, such as, medical image reconstruction and signal processing (see[2]). Byrne's CQ algorithm (see[1]) is a famous method for solving the SFP, which is formulated as follows.

$$x_{n+1} = P_C(x_n - \alpha_n A^*(I - P_Q)Ax_n), \tag{1.2}$$

where the step size $\alpha_n \in (0, \frac{2}{\|A\|^2})$, and P_C and P_Q stand for the metric projection onto C and Q, respectively.

Several generalizations of the SFP have also been investigated. We mention, for instance, the multiple-set split feasibility problem (MSSFP) (see [5, 12]), the split common fixed point problem (SCFPP) (see [7, 16, 17]), the split variational inequality problem (SVIP) (see [6, 13]) and the split common null point problem (SCNPP) (see [3, 19, 18, 25]).

In 2005, the multiple-set split feasibility problem (MSSFP) was first introduced, in finite-dimensional Hilbert spaces, by Censor et al.[5]. The problem is to find an

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element x^* such that

$$x^* \in C = \bigcap_{i=1}^{t} C_i, \quad Ax^* \in Q = \bigcap_{j=1}^{r} Q_j, \tag{1.3}$$

where $C_i \subset H_1, i = 1, 2, \ldots, t$, and $Q_j \subset H_2, j = 1, 2, \ldots, r$ are nonempty closed and convex sets. When $t = r = 1$, the MSSFP (1.3) becomes the SFP.

In 2020, Reich and Tuyen [20] proposed the following split feasibility problem with multiple output sets in Hilbert spaces. Let $H, H_j, j = 1, 2, \ldots, r$ be real Hilbert spaces. Let $A_j : H \to H_j, j = 1, 2, \ldots, r$, be bounded linear operators. In addition, let C and Q_j be nonempty, closed,and convex subsets of H and $H_j, j = 1, 2, \ldots, r$, respectively. The problem is to find an element x^* such that

$$x^* \in S = C \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)) \neq \emptyset. \tag{1.4}$$

In other words, the aim is to find an $x^* \in C$ such that $A_j x^* \in Q_j$ for all $j = 1, 2, \ldots, r$. When $r = 1$, the problem (1.4) becomes the SFP. For more information about this problem, please refer to references [23, 21, 14, 22].

In order to solve problem (1.4), Reich and Tuyen [20] introduced the following iterative methods: For any $x_0, y_0 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the two sequences generated by

$$x_{n+1} = P_C[x_n - \gamma_n \sum_{j=1}^{r} A_j^*(I - P_{Q_j})A_j x_n], \tag{1.5}$$

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)P_C[y_n - \gamma_n \sum_{j=1}^{r} A_j^*(I - P_{Q_j})A_j y_n]. \tag{1.6}$$

where $f : C \to C$ is a strict contraction with the contraction coefficient $c \in [0, 1), \{\gamma_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. They established the weak and strong convergence of iterative methods (1.5) and (1.6) under appropriate conditions, respectively.

Motivated by the above results, in this paper, we slightly generalize the problem (1.4) and propose two algorithms with strong convergence. The problem is formulated as follows: Let H, $H_j, j = 1, 2, \ldots, r$, be real Hilbert spaces and let $A_j : H \to H_j, j = 1, 2, \ldots, r$, be bounded linear operators. Let C_i and Q_j be nonempty, closed and convex subsets of H and $H_j, j = 1, 2, \ldots, r$, respectively. Find an element x^*, such that

$$x^* \in S = \bigcap_{i=1}^{t} C_i \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)). \tag{1.7}$$

That is to say, $x^* \in C_i$ and $A_j x^* \in Q_j$ for all $i = 1, 2, \ldots, t, j = 1, 2 \ldots, r$.

Remark 1.1. (i) When $t = 1, r = 1$, the problem (1.7) becomes the SFP.
(ii) When $t = 1$, the problem (1.7) becomes the split feasibility problem with multiple output sets (1.4).
(iii) When $A_j \equiv A, H_j \equiv H_1$, the problem (1.7) becomes MSSFP (1.3).

It is already known that the MSSFP (1.3) is equivalent to the following minimization problem:

$$\min f(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|Ax - P_{Q_j}(Ax)\|^2, \tag{1.8}$$

where P_{C_i} and P_{Q_j} are the metric projections onto C_i and Q_j, respectively, and $l_i, i = 1, \ldots, t$ and $\lambda_j, j = 1, \ldots, r$ are all positive constants such that $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$. Similarly, the problem (1.7) is equivalent to the following minimization problem:

$$\min f(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|A_j x - P_{Q_j}(A_j x)\|^2. \tag{1.9}$$

So the gradient algorithm yields the following algorithm

$$x_{n+1} = x_n - \tau_n \nabla f(x_n), \tag{1.10}$$

where

$$\nabla f(x) = \sum_{i=1}^{t} l_i (x - P_{C_i}(x)) + \sum_{j=1}^{r} \lambda_j A_j^* (I - P_{Q_j}) A_j x \tag{1.11}$$

is $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j ||A_j||^2$ - Lipschitz continuous, and the step size $\tau_n \geq 0$. One can prove the weak convergence of this algorithm under certain conditions on τ_n.

For numerical calculations, if the sets C_i and Q_j are relatively simple sets, such as ball or half spaces, etc, P_{C_i} and P_{Q_j} have explicit expressions. However P_{C_i} and P_{Q_j} have no explicit expressions in general. So many authors adopted the relaxed projection which is proposed by Yang [29] for solving SFP in finite dimensional Hilbert spaces. The algorithm as follows:

$$x_{n+1} = P_{C^n}(x_n - \alpha_n A^*(I - P_{Q^n}) A x_n), \tag{1.12}$$

where C^n and Q^n, $n \geq 1$, are closed half spaces containing C and Q, respectively:

$$C^n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \quad Q^n = \{y \in H_2 : q(Ax_n) \leq \langle \zeta_n, Ax_n - y \rangle\}, \tag{1.13}$$

where $\xi_n \in \partial c(x_n)$ and $\zeta \in \partial q(Ax_n)$, and $c(\cdot)$ and $q(\cdot)$ are weakly lower semi-continuous and convex functions such that

$$C = \{x \in H_1 : c(x) \leq 0\}, \quad Q = \{y \in H_2 : q(y) \leq 0\}. \tag{1.14}$$

We adopt the same idea to iteration scheme (1.10). Set

$$C_i = \{x \in H : c_i(x) \leq 0\}, \quad Q_j = \{y \in H_j : q_j(y) \leq 0\}, \tag{1.15}$$

where $c_i(x), i = 1, 2, \ldots, t$ and $q_j(y), j = 1, 2, \ldots, r$ are weakly lower semi-continuous and convex functions. Define a series of closed half spaces C_i^n and Q_j^n, $n \geq 1$, by

$$\begin{aligned} C_i^n &= \{x \in H : c_i(x_n) \leq \langle \xi_i^n, x_n - x \rangle\}, \\ Q_j^n &= \{y \in H_j : q_j(A_j x_n) \leq \langle \zeta_j^n, A_j x_n - y \rangle\}, \end{aligned} \tag{1.16}$$

where $\xi_i^n \in \partial c_i(x_n)$, $i = 1, 2, \ldots, t$ and $\zeta_j^n \in \partial q_j(A_j x_n), j = 1, 2, \ldots r$.

It is easy to verify that $C_i^n \supset C_i, i = 1, 2, \ldots, t$ and $Q_j^n \supset Q_j, j = 1, 2, \ldots, r$, $n \geq 1$. The iteration scheme becomes

$$x_{n+1} = x_n - \tau \nabla f(x_n), \tag{1.17}$$

where

$$f_n(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i^n}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|A_j x - P_{Q_j^n}(A_j x)\|^2, \tag{1.18}$$

$$\nabla f_n(x) = \sum_{i=1}^{t} l_i (x - P_{C_i^n}(x)) + \sum_{j=1}^{r} \lambda_j A_j^* (I - P_{Q_j^n}) A_j x. \tag{1.19}$$

To avoid the complex work on norm estimation of $||A_j||$, several choice of the step size are proposed. For instance, Yang [30] introduced the self-adaptive step size in the CQ algorithms below

$$\alpha_n = \frac{\rho_n}{\|\nabla f(x_n)\|}, \tag{1.20}$$

where $\{\rho_n\}$ is a sequence of positive real numbers satisfying $\sum_{n=0}^{\infty} \rho_n = 0$ and $\sum_{n=0}^{\infty} \rho_n^2 < +\infty$, $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$. In 2012, López et al.[15] proposed the following self-adaptive step size in the relaxed CQ algorithms

$$\alpha_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \tag{1.21}$$

where $\rho_n \in (0, 4)$,$f_n(x) = \frac{1}{2}\|(I - P_{Q^n})Ax\|^2$. Some scholar, such as Gibali et al. ([9]) applied the Armijo line search to obtain the step size.

To achieve a faster convergence of the algorithms, many references have investigated the inertial technique. Suantai et al.[24] introduced the following inertial relaxed CQ algorithm for solving the multiple-sets split feasibility problems,

$$\begin{aligned} &x_{n+1} = \alpha_n u + (1 - \alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \\ &y_n = x_n + \beta_n (x_n - x_{n-1}), \quad n \geq 1, \end{aligned} \tag{1.22}$$

where β_n is the inertial coefficient, $f_n(y_n) = \frac{1}{2}\sum_{i=1}^{t} l_i \|y_n - P_{C_i^n}(y_n)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|A_j y_n - P_{Q_j^n}(A_j y_n)\|^2$, and $\tau_n = \frac{\rho_n f_n(y_n)}{\|\nabla f_n(y_n)\|^2}$, $n \geq 1$.

Another issue in the research of the algorithms is the bounded perturbation resilience. For example, Guo et al.[10] presented the following proximal gradient algorithm with perturbation,

$$x_{n+1} = prox_{\lambda_n g}(I - \lambda_k A \nabla f + e)(x_n), \tag{1.23}$$

and they proved that the sequence $\{x_n\}$ generated by the algorithm (1.23) converges weakly to a solution. For more information about the bounded perturbation resilience of the algorithms, please refer to references [27, 8], etc.

Motivated by the previous works, we propose the following iterative algorithm for solving problem(1.7),

$$\begin{aligned} &y_n = x_n + e_n, \quad n \geq 1, \\ &x_{n+1} = \alpha_n g(y_n) + (1 - \alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \end{aligned} \tag{1.24}$$

where e_n denotes the perturbation, $\{\alpha_n\} \subset (0,1)$, $g : H \to H$ is a strict contraction with the contraction coefficient $c \in [0,1)$, f_n is defined in (1.18), and τ_n is the step size, which has two different choices in Section 3.

The rest of the paper is arranged as follows. In Section 2, some useful concepts and lemmas for our analysis are reviewed. We present our algorithms and prove their strong convergence in Section 3. Finally, in Section 4, we exhibit several numerical examples to illustrate our results and observe the performance of our algorithms.

2. Preliminaries

In this section, we present some basic concepts and lemmas which will be used in this paper. Let H be a real Hilbert space, and its inner product and norm be expressed by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Besides, we use the symbol $x_n \to x$ ($x_n \rightharpoonup x$) to express that the sequence $\{x_n\}$ converges strongly (weakly) to x. Recall that an operator $T : H \to H$ is said to be nonexpansive if, for every $x, y \in H$,

$$\|Tx - Ty\| \leq \|x - y\|; \tag{2.1}$$

$T : H \to H$ is said to be firmly nonexpansive if, for every $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \tag{2.2}$$

where I is the identity operator. It can be proved that (2.2) is equivalent to the following inequality

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \tag{2.3}$$

for every $x, y \in H$. It is known that T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive.

Let C be a nonempty, closed and convex subset of H. The metric projection $P_C : H \to C$ is an important tool for our work in this paper which is defined by

$$P_C(x) = \arg\min_{y \in C} \|x - y\|^2, \quad x \in C. \tag{2.4}$$

Moreover, we have

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall x \in H, y \in C. \tag{2.5}$$

It is well known that P_C is a firmly nonexpansive operator.

Definition 2.1. Let $\varphi : H \to R$ be a convex function. The subdifferential of φ at x is defined as

$$\partial\varphi(x) = \{\xi \in H : \varphi(y) \geq \varphi(x) + \langle \xi, y - x\rangle, \forall y \in H\}. \tag{2.6}$$

Definition 2.2. A function $\varphi : H \to R$ is said to be weakly lower semicontinuous at x if x_n converges weakly to x implies

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n). \tag{2.7}$$

Definition 2.3. An algorithmic operator T is said to be bounded perturbations resilient if the iterations $x_{n+1} = Tx_n$ and $x_{n+1} = T(x_n + \beta_n e_n)$ all converge, where $\beta_n \geq 0$ for all $n \geq 0$, $\{e_n\}$ is a sequence in H, and $M \in \mathbb{R}$ and satisfies

$$\sum_{n=0}^{\infty} \beta_n < +\infty, \|e_n\| \leq M.$$

Lemma 2.4. [5] *Let $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be closed convex subsets of H_1 and H_2, respectively, $A : H_1 \to H_2$ is a bounded linear operator. Let $f(x)$ be the function defined as follows*

$$f(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|Ax - P_{Q_j}(Ax)\|^2, \tag{2.8}$$

Then $\nabla f(x)$ is Lipschitz continuous with Lipschitz constant $L = \sum_{i=1}^{t} l_i + \|A\|^2 \sum_{j=1}^{r} \lambda_j$.

Lemma 2.5 ([28])**.** *Let $\{a_n\}$ and $\{c_n\}$be sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + b_n + c_n, \quad n \ge 1, \tag{2.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(1) *If $b_n \le \alpha_n M$ for some $M \ge 0$, then $\{a_n\}$ is a bounded sequence.*
(2) *If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \le 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.6 ([11])**.** *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\delta_n, \quad n \ge 1, \tag{2.10}$$

$$s_{n+1} \le s_n - \eta_n + \gamma_n, \quad n \ge 1, \tag{2.11}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\delta_n\}$ and $\{\gamma_n\}$ are two sequences in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\lim_{n\to\infty} \gamma_n = 0$;
(3) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence $\{n_k\}$ of $\{n\}$.*

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

In this section, we present the two algorithms for the problem (1.7) and prove their strong convergence. Throughout this section, we assume that the following three assumptions hold.

(A1) The solution set S of (1.7) is nonempty.
(A2) The functions $c_i : \mathcal{H}_1 \to \mathbf{R}$ and $q_j : \mathcal{H}_2 \to \mathbf{R}$ defined in (1.15) are convex and weakly lower semicontinuous functions.
(A3) For any $x \in \mathcal{H}$ and $y_j \in \mathcal{H}_j$, at least one subgradient $\xi_i \in \partial c_i(x)$ and $\eta_j \in \partial q_j(y_j)$ can be calculated. The subdifferentials ∂c_i and ∂q_j are bounded on the bounded sets.

Algorithm 1.

$$\begin{aligned} &y_n = x_n + e_n, \quad n \ge 1, \\ &x_{n+1} = \alpha_n g(y_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \end{aligned} \tag{3.1}$$

where $\{\alpha_n\} \subset (0, 1)$, ∇f_n is given in (1.19), τ_n is the step size, $g : H \to H$ is a strict contraction mapping with the contraction coefficient $c \in [0, 1)$.

Now we establish strong convergence theorems for Algorithm 1.

Theorem 3.1. *Let H and H_j be real Hilbert spaces, $C_i, i = 1, 2, \ldots, t$ and $Q_j, j = 1, 2, \ldots, r$ be nonempty, closed and convex subsets of H and H_j, respectively. Let $A_j : H \to H_j, j = 1, 2, \ldots, r$ be bounded linear operators with their adjoint denoted by A_j^*. Assume that $\{\alpha_n\}$, $\{e_n\}$ and τ_n satisfy the following conditions:*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \le \tau_n \le b < \min\{\frac{1}{t \max\limits_{1\le i\le t} l_i}, \frac{1}{r \max\limits_{1\le j\le r} \|A_j\|^2 \max\limits_{1\le j\le r} \lambda_j}\}$*;*

(C3) $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $\lim\limits_{n\to\infty} \frac{e_n}{\alpha_n} = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $z \in S$, where $S = \bigcap_{i=1}^{t} C_i \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j))$, and z is the unique solution to the variational inequality:

$$\langle (I - g)(z), y - z\rangle \ge 0, \quad \forall y \in S. \tag{3.2}$$

Proof. Note that $g : H \to H$ is contractive, so $P_S g$ is also contractive, thus $P_S g$ has a unique fixed point z, which by (2.5) is the unique solution of (3.2).

It is obvious that

$$\|y_n - \tau_n \nabla f_n(y_n) - z\|^2 = \|y_n - z\|^2 + \tau_n^2 \|\nabla f_n(y_n)\|^2 - 2\tau_n \langle \nabla f_n(y_n), y_n - z\rangle \tag{3.3}$$

and that

$$\begin{aligned}
\|\nabla f_n(y_n)\|^2 &= \|\sum_{i=1}^{t} l_i (I - P_{C_i^n}) y_n + \sum_{j=1}^{r} \lambda_j A_j^* (I - P_{Q_j^n}) A_j y_n\|^2 \\
&\le 2\|\sum_{i=1}^{t} l_i (I - P_{C_i^n}) y_n\|^2 + 2\|\sum_{j=1}^{r} \lambda_j A_j^* (I - P_{Q_j^n}) A_j y_n\|^2 \\
&\le 2t \sum_{i=1}^{t} l_i^2 \|(I - P_{C_i^n}) y_n\|^2 + 2r \sum_{j=1}^{r} \lambda_j^2 \|A_j^* (I - P_{Q_j^n}) A_j y_n\|^2.
\end{aligned} \tag{3.4}$$

The definition of f_n and the operators $I - P_{Q_j^n}$ being firmly nonexpansive ensure that

$$\begin{aligned}
&\langle \nabla f_n(y_n), y_n - z\rangle \\
&= \langle \sum_{i=1}^{t} l_i (I - P_{C_i^n}) y_n + \sum_{j=1}^{r} \lambda_j A_j^* (I - P_{Q_j^n}) A_j y_n, y_n - z\rangle \\
&= \sum_{i=1}^{t} l_i \langle (I - P_{C_i^n}) y_n, y_n - z\rangle + \sum_{j=1}^{r} \lambda_j \langle (I - P_{Q_j^n}) A_j y_n, A_j (y_n - z)\rangle \\
&\ge \sum_{i=1}^{t} l_i \|(I - P_{C_i^n}) y_n\|^2 + \sum_{j=1}^{r} \lambda_j \|(I - P_{Q_j^n}) A_j y_n\|^2.
\end{aligned} \tag{3.5}$$

Substituting the inequality (3.4) and (3.5) into (3.3), we obtain

(3.6)

$$\begin{aligned}\|y_n-\tau_n\nabla f_n(y_n)-z\|^2 \le \|y_n-z\|^2+\tau_n^2(2t\sum_{i=1}^{t}l_i^2\|(I-P_{C_i^n})y_n\|^2+2r\sum_{j=1}^{r}\lambda_j^2\|A_j^*(I-P_{Q_j^n})A_jy_n\|^2)\\ -2\tau_n(\sum_{i=1}^{t}l_i\|(I-P_{C_i^n})y_n\|^2+\sum_{j=1}^{r}\lambda_j\|(I-P_{Q_j^n})A_jy_n\|^2)\\ \le \|y_n-z\|^2-\tau_n(2-2t\tau_n\max_{1\le i\le t}l_i)\sum_{i=1}^{t}l_i\|(I-P_{C_i^n})y_n\|^2\\ -\tau_n(2-2r\tau_n\max_{1\le j\le r}\|A_j\|^2\max_{1\le j\le r}\lambda_j)\sum_{j=1}^{r}\lambda_j\|(I-P_{Q_j^n})A_jy_n\|^2.\end{aligned}$$

Thus we have from (C2) that

$$\|y_n-\tau_n\nabla f_n(y_n)-z\|\le\|y_n-z\|. \tag{3.7}$$

The iterative scheme of Algorithm 1 shows that

$$\|y_n-z\|=\|x_n+e_n-z\|\le\|x_n-z\|+\|e_n\|, \tag{3.8}$$

and hence

$$\begin{aligned}\|x_{n+1}-z\| &= \|\alpha_ng(y_n)+(1-\alpha_n)(y_n-\tau_n\nabla f_n(y_n))-z\|\\ &\le \alpha_n\|g(y_n)-z\|+(1-\alpha_n)\|y_n-\tau_n\nabla f_n(y_n)-z\|\\ &\le \alpha_nc\|y_n-z\|+\alpha_n\|g(z)-z\|+(1-\alpha_n)\|y_n-z\|\\ &= [1-\alpha_n(1-c)]\|y_n-z\|+\alpha_n\|g(z)-z\|\\ &\le [1-\alpha_n(1-c)]\|x_n-z\|+[1-\alpha_n(1-c)]\|e_n\|+\alpha_n\|g(z)-z\|\\ &= [1-\alpha_n(1-c)]\|x_n-z\|\\ &\quad+\alpha_n(1-c)\frac{\|g(z)-z\|+\frac{1-\alpha_n(1-c)}{\alpha_n}\|e_n\|}{1-c}.\end{aligned} \tag{3.9}$$

According to (C3), we see that $\mu_n=\frac{1-\alpha_n(1-c)}{\alpha_n}\|e_n\|\to 0$. Hence the sequence $\{\mu_n\}$ is bounded. Put

$$(1-c)M=\max\{\|g(z)-z\|,\sup_{n\ge1}\{\mu_n\}\}, \tag{3.10}$$

then (3.9) can be rewritten as follows:

$$\|x_{n+1}-z\|\le[1-\alpha_n(1-c)]\|x_n-z\|+\alpha_n(1-c)M. \tag{3.11}$$

Applying Lemma 2.5 (1) and (C3), we conclude that $\{x_n\}$ is bounded, and hence $\{y_n\}$ is bounded.

Next, we calculate the following estimates:

$$\|y_n-z\|^2=\|x_n+e_n-z\|^2=\|x_n-z\|^2+2\langle x_n-z,e_n\rangle+\|e_n\|^2. \tag{3.12}$$

$$\langle x_n-z,e_n\rangle=-\frac{1}{2}\|x_n-z-e_n\|^2+\frac{1}{2}\|x_n-z\|^2+\frac{1}{2}\|e_n\|^2. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain:

$$\begin{aligned} \|y_n - z\|^2 &= \|x_n - z\|^2 + (-\|x_n - z - e_n\|^2 + \|x_n - z\|^2 + \|e_n\|^2) + \|e_n\|^2 \\ &= \|x_n - z\|^2 + (\|x_n - z\|^2 - \|x_n - z - e_n\|^2) + 2\|e_n\|^2. \end{aligned} \tag{3.14}$$

By (3.6), we obtain:

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \langle \alpha_n g(y_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n)) - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)\langle y_n - \tau_n \nabla f_n(y_n) - z, x_{n+1} - z\rangle \\ &\quad + \alpha_n \langle g(y_n) - z, x_{n+1} - z\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - \tau_n \nabla f_n(y_n) - z\|^2) \\ &\quad + \frac{\alpha_n}{2}(||g(y_n) - g(z)||^2 + ||x_{n+1} - z||^2) \\ &\quad + \alpha_n \langle g(z) - z, x_{n+1} - z\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - \tau_n \nabla f_n(y_n) - z\|^2) \\ &\quad + \frac{\alpha_n}{2}(c^2\|y_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle g(z) - z, x_{n+1} - z\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - z\|^2 \\ &\quad - \tau_n(2 - 2t\tau_n \max_{1\le i\le t} l_i)\sum_{i=1}^{t} l_i \|(I - P_{C_i^n})y_n\|^2 \\ &\quad - \tau_n(2 - 2r\tau_n \max_{1\le j\le r}\|A_j\|^2 \max_{1\le j\le r}\lambda_j)\sum_{j=1}^{r}\lambda_j\|(I - P_{Q_j^n})A_j y_n\|^2) \\ &\quad + \frac{\alpha_n}{2}(c\|y_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle g(z) - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.15}$$

Let

$$\begin{aligned} E_n = {} & \tau_n(2 - 2t\tau_n \max_{1\le i\le t} l_i)\sum_{i=1}^{t} l_i\|(I - P_{C_i^n})y_n\|^2 \\ & + \tau_n(2 - 2r\tau_n \max_{1\le j\le r}\|A_j\|^2 \max_{1\le j\le r}\lambda_j)\sum_{j=1}^{r}\lambda_j\|(I - P_{Q_j^n})A_j y_n\|^2. \end{aligned} \tag{3.16}$$

Then (3.15) can be rewritten as follows:

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le (1-\alpha_n)\|y_n - z\|^2 - (1-\alpha_n)E_n + \alpha_n c\|y_n - z\|^2 \\ &\quad + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &= [1 - \alpha_n(1-c)]\|y_n - z\|^2 - (1-\alpha_n)E_n \\ &\quad + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.17}$$

Combining (3.14) and (3.17), we obtain

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\le [1-\alpha_n(1-c)][\|x_n-z\|^2+(\|x_n-z\|^2-\|x_n-z-e_n\|^2)\\
&\quad +2\|e_n\|^2]\\
&\quad -(1-\alpha_n)E_n+2\alpha_n\langle g(z)-z, x_{n+1}-z\rangle\\
&\le [1-\alpha_n(1-c)]\|x_n-z\|^2\\
&\quad +[1-\alpha_n(1-c)]\|e_n\|(\|x_n-z\|+\|x_n-z-e_n\|)\\
&\quad +2[1-\alpha_n(1-c)]\|e_n\|^2-(1-\alpha_n)E_n\\
&\quad +2\alpha_n\langle g(z)-z, x_{n+1}-z\rangle.
\end{aligned} \tag{3.18}
$$

Set

$$
\begin{aligned}
s_n &= \|x_n-z\|^2;\\
\gamma_n &= [1-\alpha_n(1-c)]\|e_n\|(\|x_n-z\|+\|x_n-z-e_n\|)\\
&\quad +2[1-\alpha_n(1-c)]\|e_n\|^2\\
&\quad +2\alpha_n\langle g(z)-z, x_{n+1}-z\rangle;\\
\delta_n &= [1-\alpha_n(1-c)]\frac{\|e_n\|}{\alpha_n(1-c)}(\|x_n-z\|+\|x_n-z-e_n\|)\\
&\quad +2[1-\alpha_n(1-c)]\frac{\|e_n\|^2}{\alpha_n(1-c)}+\frac{2}{1-c}\langle g(z)-z, x_{n+1}-z\rangle;\\
\eta_n &= (1-\alpha_n)E_n.
\end{aligned} \tag{3.19}
$$

Then $\eta_n > 0$, $\lim_{n\to\infty}\gamma_n = 0$, and (3.18) and (3.19) can be rewritten as the following two inequalities

$$
\begin{aligned}
&s_{n+1} \le [1-\alpha_n(1-c)]s_n + \alpha_n(1-c)\delta_n, \quad n \ge 1,\\
&s_{n+1} \le s_n - \eta_n + \gamma_n, \quad n \ge 1.
\end{aligned}
$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\limsup_{k\to\infty} \eta_{n_k} \le 0,$$

i.e.,

$$\limsup_{k\to\infty} (1-\alpha_{n_k})E_{n_k} \le 0.$$

Then

$$E_{n_k} \to 0, \quad k\to\infty,$$

and hence

$$\lim_{k\to\infty} \|(I-P_{C_i^{n_k}})y_{n_k}\| = 0, \quad i=1,2,\ldots,t, \tag{3.20}$$

$$\lim_{k\to\infty} \|(I-P_{Q_j^{n_k}})A_j y_{n_k}\| = 0, \quad j=1,2,\ldots,r. \tag{3.21}$$

Since $\partial q_j, j=1,2,\ldots,r$ are bounded on bounded sets and $\{x_{n_k}\}$ is bounded, there exists a constant $\mu > 0$ such that $\|\zeta_j^{n_k}\| \le \mu, j=1,2,\ldots,r,\ k\in\mathbb{N}$, where

$\zeta_j^{n_k} \in \partial q_j(y_{n_k})$. Note that $P_{Q_j^{n_k}} A_j y_{n_k} \in P_{Q_j^{n_k}}$, (3.21) reads that

$$\begin{aligned} q_j(A_j y_{n_k}) &\leq \langle \zeta_j^{n_k}, A_j y_{n_k} - P_{Q_j^{n_k}}(A_j y_{n_k}) \rangle \\ &\leq \mu \|(I - P_{Q_j^{n_k}}) A_j y_{n_k}\| \to 0, \quad k \to \infty. \end{aligned} \tag{3.22}$$

Since $\{y_{n_k}\}$ is bounded, there exists a subsequence $\{y_{n_{k_m}}\} \subset \{y_{n_k}\}$ such that $y_{n_{k_m}} \rightharpoonup x^*$ and

$$\limsup_{k\to\infty} \langle g(z) - z, y_{n_k} - z \rangle = \lim_{m\to\infty} \langle g(z) - z, y_{n_{k_m}} - z \rangle. \tag{3.23}$$

Notice that $A_j y_{n_{k_m}} \rightharpoonup A_j x^*$ and q_j is weakly lower semi-continuous, by (3.22) we have

$$q_j(A_j x^*) \leq \liminf_{m\to\infty} q_j(A_j y_{n_{k_m}}) \leq 0. \tag{3.24}$$

Hence $A_j x^* \in Q_j$.

We next prove that $x^* \in C_i, i = 1, \ldots, t$. By the definition of $C_i^{n_k}, i = 1, \ldots, t$, the assumption (A3) and (3.20), there exists a constant $\delta > 0$ such that

$$c_i(y_{n_k}) \leq \langle \xi_i^{n_k}, y_{n_k} - P_{C_i^{n_k}}(y_{n_k}) \rangle \leq \delta \|y_{n_k} - P_{C_i^{n_k}}(y_{n_k})\| \to 0, \quad k \to \infty. \tag{3.25}$$

Then the weak lower semi-continuity of c_i and the existence of $\{y_{n_{k_m}}\}$ such that $y_{n_{k_m}} \rightharpoonup x^*$ yields that

$$c_i(x^*) \leq \liminf_{m\to\infty} c_i(y_{n_{k_m}}) \leq 0. \tag{3.26}$$

Therefore $x^* \in C_i, i = 1, \ldots, t$. So $x^* \in S$.

From (2.5) and (3.23) we obtain:

$$\begin{aligned} \limsup_{k\to\infty} \langle g(z) - z, y_{n_k} - z \rangle &= \lim_{m\to\infty} \langle g(z) - z, y_{n_{k_m}} - z \rangle \\ &= \langle g(z) - z, x^* - z \rangle \leq 0. \end{aligned} \tag{3.27}$$

On the other hand, by $(C3)$, we have

$$\|y_n - x_n\| = \|e_n\| \to 0, \quad n \to \infty. \tag{3.28}$$

A direct estimation gives that

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|\alpha_n(g(y_n) - x_n) + (1 - \alpha_n)(y_n - \tau_n \nabla f_n(y_n) - x_n)\| \\ &\leq \alpha_n \|g(y_n) - x_n\| + (1 - \alpha_n)\|y_n - x_n\| + (1 - \alpha_n)\tau_n \|\nabla f_n(y_n)\|. \end{aligned}$$

Thus

$$\|x_{n_k+1} - x_{n_k}\| \to 0, \quad k \to \infty. \tag{3.29}$$

From (3.27), (3.28) and (3.29), we derive that

$$\limsup_{k\to\infty} \langle g(z) - z, x_{n_k+1} - z \rangle \leq 0. \tag{3.30}$$

Then $(C3)$ and (3.30) implies that

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0. \tag{3.31}$$

Using Lemma 2.6, we conclude that $x_n \to z$. The proof is complete. □

Next, we present the algorithm with the self-adaptive step size.

Algorithm 2.

$$\begin{aligned} &y_n = x_n + e_n, \\ &x_{n+1} = \alpha_n g(y_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \quad n \geq 1, \end{aligned} \tag{3.32}$$

where $\{\alpha_n\} \subset (0,1)$, ∇f_n is given in (1.19), $\tau_n = \frac{\rho_n f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n}$, $\{\varepsilon_n\}$ is a sequence of positive numbers, and $g : H \to H$ is a strict contraction mapping H into itself with the contraction coefficient $c \in [0,1)$.

The strong convergence of Algorithm 2 is established in the following theorem.

Theorem 3.2. *Let H and H_j be real Hilbert spaces, $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ be nonempty, closed and convex subsets of H and H_j. Let $A_j : H \to H_j$ be a bounded linear operator with its adjoint A_j^*. Assume that $\{\alpha_n\}$, $\{e_n\}$, $\{\rho_n\}$ satisfy the following conditions:*

- $(C4)$ $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- $(C5)$ $\inf\limits_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$;
- $(C6)$ $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $\lim\limits_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$;
- $(C7)$ $\{\varepsilon_n\}$ *is bounded.*

Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z \in S$, where $S = \bigcap_{i=1}^t C_i \cap (\bigcap_{j=1}^r A_j^{-1}(Q_j))$, and z is the unique solution to the variational inequality:

$$\langle (I-g)(z), y - z \rangle \geq 0 \quad \forall y \in S. \tag{3.33}$$

Proof. Similar with the proof of Theorem 3.1, the variational inequality(3.33) has a unique solution, denoted by z, such that $z = P_S g(z)$.

Since $I - P_{C_i^n}, i = 1,2,\ldots,t$ and $I - P_{Q_j^n}, j = 1,2,\ldots,r$ are firmly nonexpansive and $\nabla f_n(z) = 0$ for every $n \in \mathbb{N}$, we have

$$\begin{aligned} &\langle \nabla f_n(y_n), y_n - z \rangle \\ &= \langle \sum_{i=1}^t l_i(y_n - P_{C_i^n}(y_n)) + \sum_{j=1}^r \lambda_j A_j^*(I - P_{Q_j^n})A_j y_n, y_n - z \rangle \\ &= \sum_{i=1}^t l_i \langle (I - P_{C_i^n})y_n, y_n - z \rangle + \sum_{j=1}^r \lambda_j \langle (I - P_{Q_j^n})A_j y_n, A_j y_n - A_j z \rangle \\ &\geq \sum_{i=1}^t l_i \|(I - P_{C_i^n})y_n\|^2 + \sum_{j=1}^r \lambda_j \|(I - P_{Q_j^n})A_j y_n\|^2 \\ &= 2 f_n(y_n). \end{aligned} \tag{3.34}$$

So we obtain that

$$
\begin{aligned}
&\|y_n - \tau_n \nabla f_n(y_n) - z\|^2 \\
&= \|y_n - z\|^2 + \tau_n^2 \|\nabla f_n(y_n)\|^2 - 2\tau_n \langle \nabla f_n(y_n), y_n - z\rangle \\
&\le \|y_n - z\|^2 + \tau_n^2 \|\nabla f_n(y_n)\|^2 - 4\tau_n f_n(y_n) \\
&= \|y_n - z\|^2 + \rho_n^2 \frac{f_n^2(y_n)}{(\|\nabla f_n(y_n)\|^2 + \varepsilon_n)^2} \|\nabla f_n(y_n)\|^2 - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n} \\
&\le \|y_n - z\|^2 + \rho_n^2 \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n} - 4\rho_n \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n} \\
&= \|y_n - z\|^2 - \rho_n(4 - \rho_n) \frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n}.
\end{aligned} \tag{3.35}
$$

Since $\rho_n \in (0, 4)$, $\forall n \in \mathbb{N}$, it follows that

$$\|y_n - \tau_n \nabla f_n(y_n) - z\| \le \|y_n - z\|. \tag{3.36}$$

Similar with the proof of Theorem 3.1, it holds that

$$\|x_{n+1} - z\| \le [1 - \alpha_n(1 - c)]\|x_n - z\| + \alpha_n(1 - c) \frac{\|g(z) - z\| + \frac{1 - \alpha_n(1-c)}{\alpha_n}\|e_n\|}{1 - c}, \tag{3.37}$$

and thus $\{x_n\}$ and $\{y_n\}$ are bounded.

By Lemma 2.4, we see similarly that

$$\|\nabla f_n(y_n)\| = \|\nabla f_n(y_n) - \nabla f_n(z)\| \le L\|y_n - z\|, \tag{3.38}$$

where $L = \sum_{i=1}^t l_i + \sum_{j=1}^r \lambda_j \|A_j\|^2$. That means $\{\nabla f_n(y_n)\}$ is also bounded.

Similar with (3.15), we have by (3.35) that

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= (1 - \alpha_n)\langle y_n - \tau_n \nabla f_n(y_n) - z, x_{n+1} - z\rangle + \alpha_n \langle g(y_n) - z, x_{n+1} - z\rangle \\
&\le \frac{1 - \alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - \tau_n \nabla f_n(y_n) - z\|^2) \\
&\quad + \frac{\alpha_n}{2}(\|g(y_n) - g(z)\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle g(z) - z, x_{n+1} - z\rangle \\
&\le \frac{1 - \alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - z\|^2 - \rho_n(4 - \rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n}) \\
&\quad + \frac{\alpha_n}{2}(c\|y_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle g(z) - z, x_{n+1} - z\rangle,
\end{aligned} \tag{3.39}
$$

which is rearranged to obtain

$$
\begin{aligned}
\|x_{n+1} - z\|^2 = [1 - \alpha_n(1 - c)]\|y_n - z\|^2 - (1 - \alpha_n)\rho_n(4 - \rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n} \\
+ 2\alpha_n \langle g(z) - z, x_{n+1} - z\rangle.
\end{aligned} \tag{3.40}
$$

Substituting (3.14) into (3.40) yields that

(3.41)

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le [1-\alpha_n(1-c)][\|x_n-z\|^2+(\|x_n-z\|^2-\|x_n-z-e_n\|^2)+2\|e_n\|^2]\\
&\quad -(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2+\varepsilon_n}+2\alpha_n\langle g(z)-z,x_{n+1}-z\rangle\\
&\le [1-\alpha_n(1-c)]\|x_n-z\|^2+[1-\alpha_n(1-c)]\|e_n\|(\|x_n-z\|+\|x_n-z-e_n\|)\\
&\quad +2[1-\alpha_n(1-c)]\|e_n\|^2-(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2+\varepsilon_n}\\
&\quad +2\alpha_n\langle g(z)-z,x_{n+1}-z\rangle.
\end{aligned}$$

Set

(3.42)

$$\begin{aligned}
s_n &= \|x_n-z\|^2;\\
\gamma_n &= [1-\alpha_n(1-c)]\|e_n\|(\|x_n-z\|+\|x_n-z-e_n\|)+2[1-\alpha_n(1-c)]\|e_n\|^2\\
&\quad +2\alpha_n\langle g(z)-z,x_{n+1}-z\rangle;\\
\delta_n &= [1-\alpha_n(1-c)]\frac{\|e_n\|}{\alpha_n(1-c)}(\|x_n-z\|+\|x_n-z-e_n\|)\\
&\quad +2[1-\alpha_n(1-c)]\frac{\|e_n\|^2}{\alpha_n(1-c)}+\frac{2}{1-c}\langle g(z)-z,x_{n+1}-z\rangle;\\
\eta_n &= (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|\nabla f_n(y_n)\|^2+\varepsilon_n}.
\end{aligned}$$

Then (3.42) can be rewritten as follows:

$$\begin{aligned}
&s_{n+1}\le[1-\alpha_n(1-c)]s_n+\alpha_n(1-c)\delta_n,\quad n\ge 1,\\
&s_{n+1}\le s_n-\eta_n+\gamma_n,\quad n\ge 1.
\end{aligned}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\limsup_{k\to\infty}\eta_{n_k}\le 0.$$

That is

$$\limsup_{k\to\infty}(1-\alpha_{n_k})\rho_{n_k}(4-\rho_{n_k})\frac{f_{n_k}^2(y_{n_k})}{\|\nabla f_{n_k}(y_{n_k})\|^2+\varepsilon_n}\le 0,$$

which by conditions (C4) and (C5) implies

$$\lim_{k\to\infty}\frac{f_{n_k}^2(y_{n_k})}{\|\nabla f_{n_k}(y_{n_k})\|^2+\varepsilon_n}=0.$$

Since $\{\|\nabla f_{n_k}(y_{n_k})\|^2+\varepsilon_n\}$ is bounded, then $f_{n_k}(y_{n_k})\to 0, k\to\infty$, which indicates that

$$\lim_{k\to\infty}\|(I-P_{C_i^{n_k}})y_{n_k}\|=0,\quad i=1,2,\dots,t, \tag{3.43}$$

$$\lim_{k\to\infty}\|(I-P_{Q_j^{n_k}})A_jy_{n_k}\|=0,\quad j=1,2,\dots,r. \tag{3.44}$$

Similar proof as Theorem 3.1 insures that any weak cluster x^* of $\{y_{n_k}\}$ satisfies $x^* \in C_i, i = 1, 2, \ldots, t$, $A_j x^* \in Q_j, j = 1, 2, \ldots, r$, and hence $x^* \in S$. It follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle g(z) - z, y_{n_k} - z\rangle &= \lim_{m\to\infty}\langle g(z) - z, y_{n_{k_m}} - z\rangle \\ &= \langle g(z) - z, x^* - z\rangle \leq 0.\end{aligned} \tag{3.45}$$

On the other hand, the condition (C6) reads that

$$\|y_n - x_n\| = \|e_n\| \to 0, \quad n \to \infty. \tag{3.46}$$

Hence, we have

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &= \|\alpha_n(g(y_n) - x_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n) - x_n)\| \\ &\leq \alpha_n\|g(y_n) - x_n\| + (1-\alpha_n)\|y_n - x_n\| + (1-\alpha_n)\tau_n\|\nabla f_n(y_n)\| \\ &= \alpha_n\|g(y_n) - x_n\| + (1-\alpha_n)\|e_n\| + (1-\alpha_n)\rho_n \frac{f_n(y_n)}{\|\nabla f_n(y_n)\|^2 + \varepsilon_n}\|\nabla f_n(y_n)\|,\end{aligned} \tag{3.47}$$

which indicates that

$$\|x_{n_k+1} - x_{n_k}\| \to 0, \quad k \to \infty. \tag{3.48}$$

Combining (3.45), (3.46) and (3.48) leads to

$$\limsup_{k\to\infty}\langle g(z) - z, x_{n_k+1} - z\rangle \leq 0, \tag{3.49}$$

which together with the condition (C6) yields that

$$\limsup_{k\to\infty}\delta_{n_k} \leq 0. \tag{3.50}$$

Using Lemma 2.6, we conclude that $x_n \to z$. We thus complete the proof. □

Remark 3.3. The strong convergence of the two algorithms still hold when $e_n = 0$. This implies that the two algorithms are both bounded perturbation resilience.

Remark 3.4. It is easy to see that the two algorithms generalize some existed iterative schemes. If $A_j = A, j = 1, 2, \ldots, r$ and g is identity operator, Algorithms 2 is reduced to the algorithm for the MSSFP in [24]; if $i = 1$ and $e_n = 0$, the Algorithm 1 is reduced to a variant of the algorithm solving the SFP with multiple output sets in [20]; and if $i = j = 1$, the algorithms are the reduced to solve the SFP [29].

4. Corollaryies

4.1. **The split common fixed point problem for nonexpansive mappings with multiple output sets.** Let H, $H_j, j = 1, 2, \ldots, r$, be real Hilbert spaces and let $A_j : H \to H_j, j = 1, 2, \ldots, r$, be bounded linear operators. Let $B_i : H \to H, i = 1, \ldots, t$ and $T_j : H_j \to H_j, j = 1, \ldots, r$, be nonexpansive mappings. We consider the following split common fixed point problem with multiple output sets: Find an element $x^* \in H$ such that

$$x^* \in S := \bigcap_{i=1}^{t} \text{Fix}(B_i) \cap (\bigcap_{j=1}^{r} A_j^{-1}\text{Fix}(T_j)), \tag{4.1}$$

where $\text{Fix}(B)$ denotes the sets of fixed points of operator B.

Let $C_i = \text{Fix}(B_i), i = 1, 2, \ldots, t$ and $Q_j = \text{Fix}(T_j), j = 1, 2, \ldots, r$. Note that the set of fixed points of a nonexpansive operator is closed and convex, then problem (4.1) becomes Problem (1.7). Thus we obtain the following corollary for solving Problem (4.1).

Corollary 4.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 1 with $C_i = Fix(B_i), i = 1, 2, \ldots, t$ and $Q_j = Fix(T_j), j = 1, 2, \ldots, r$, respectively. If the sequences $\{\alpha_n\}$, $\{\tau_n\}$ and $\{e_n\}$ satisfy the conditions (C1)-(C3), then the sequence $\{x_n\}$ converges strongly to $z \in S$, which is the unique solution to the variational inequality*

$$\langle (I - g)(z), y - z \rangle \geq 0, \quad \forall y \in S.$$

Corollary 4.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 2 with $C_i = Fix(B_i), i = 1, 2, \ldots, t$ and $Q_j = Fix(T_j), j = 1, 2, \ldots, r$, respectively. If the sequences $\{\alpha_n\}$, $\{\rho_n\}$, $\{e_n\}$ and $\{\varepsilon_n\}$ satisfy the conditions (C4)-(C7), then the sequence $\{x_n\}$ converges strongly to $z \in S$, which is the unique solution to the variational inequality*

$$\langle (I - g)(z), y - z \rangle \geq 0, \quad \forall y \in S.$$

4.2. **Variational inequality problem with multiple output sets.** Let $A : H \to H$ be an operator from the Hilbert space H into itself and let D be a nonempty, closed and convex subset of H. The variational inequality problem (VIP) is to find an element $x^* \in D$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall\, x \in D. \tag{4.2}$$

We denote by $\text{VIP}(A, D)$ the solution set of this problem.

It is known that $\text{VIP}(A, D) = \text{Fix}(P_D(I - \lambda A))$, where $\lambda > 0$. If A is a β-inverse strongly monotone (ism for short) operator, that is, there exists a positive real number β such that

$$\langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2, \quad \forall\, x, y \in H \tag{4.3}$$

then $I - \lambda A$ is a nonexpansive mapping for $\lambda \in (0, 2\beta]$, and hence the solution set $\text{VIP}(A, D)$ is closed and convex.

Let H, $H_j, j = 1, 2, \ldots, r$, be real Hilbert spaces and let $A_j : H \to H_j, j = 1, 2, \ldots, r$, be bounded linear operators. Let $T_i : H \to H, i = 1, \ldots, t$, be β_i-ism operators and let $B_j : H_j \to H_j, j = 1, 2, \ldots, r$, be $\tilde{\beta}_j$-ism operators. Let $D_i, i = 1, 2, \ldots, t$ be nonempty, closed and convex subsets of H, and $\tilde{D}_j, j = 1, 2, \ldots, r$ be nonempty, closed and convex subsets of H_j, respectively. We consider the following variational inequality problem with multiple output sets: Find an element $x^* \in H$ such that

$$x^* \in S := \bigcap_{i=1}^{t} \mathrm{VIP}(T_i, D_i) \cap (\bigcap_{j=1}^{r} A_j^{-1}\mathrm{VIP}(B_j, \tilde{D}_j)). \tag{4.4}$$

Let $C_i = \mathrm{VIP}(T_i, D_i), i = 1, 2, \ldots, t$ and $Q_j = \mathrm{VIP}(B_j, \tilde{D}_j), j = 1, 2, \ldots, r$. Then problem (4.4) becomes Problem (1.7). Thus we obtain the following corollary for solving Problem (4.4).

Corollary 4.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 1 with C_i = VIP$(T_i, D_i), i = 1, 2, \ldots, t$ and Q_j = VIP$(B_j, \tilde{D}_j), j = 1, 2, \ldots, r$, respectively. If the sequences $\{\alpha_n\}$, $\{\tau_n\}$ and $\{e_n\}$ satisfy the conditions (C1)-(C3), then the sequence $\{x_n\}$ converges strongly to $z \in S$, which is the unique solution to the variational inequality*

$$\langle (I - g)(z), y - z\rangle \geq 0, \quad \forall y \in S.$$

Corollary 4.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 2 with C_i = VIP$(T_i, D_i), i = 1, 2, \ldots, t$ and Q_j = VIP$(B_j, \tilde{D}_j), j = 1, 2, \ldots, r$, respectively. If the sequences $\{\alpha_n\}$, $\{\rho_n\}$, $\{e_n\}$ and $\{\varepsilon_n\}$ satisfy the conditions (C4)-(C7), then the sequence $\{x_n\}$ converges strongly to $z \in S$, which is the unique solution to the variational inequality*

$$\langle (I - g)(z), y - z\rangle \geq 0, \quad \forall y \in S.$$

5. Numerical experiments

In this section, we present some numerical simulations to show the validity of Algorithm 1 and Algorithm 2. In the experiments, we set e_n to be the inertial term, i.e., $e_n = \beta_n(x_n - x_{n-1})$, where

$$\beta_n = \begin{cases} \dfrac{\theta_n}{||x_n - x_{n-1}||}, & ||x_n - x_{n-1}|| > 1 \\ \theta_n, & ||x_n - x_{n-1}|| \leq 1. \end{cases}$$

It can be proved (see [26]) that if $\theta_n \geq 0$ and $\sum_{n=0}^{\infty} \theta_n < +\infty$, the corresponding algorithms are still strongly convergent.

The codes are written in Matlab R2018b and run on Inter(R) Core(TM) i9-12900H CPU @ 2.50 GHz , RAM 16.00 GB.

Example. Consider the following problem: find an element $x^* \in \mathbb{R}^5$ such that

$$x^* \in S = \bigcap_{i=1}^{t} C_i \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)),$$

where $C_i = \{u \in \mathbb{R}^5 : \langle a_i, u\rangle \leq b_i\}$, $Q_j = \{u \in \mathbb{R}^{5(j+1)} : \langle e_j, u\rangle \leq d_j\}$, the coordinates of $a_i,\ e_j$ are randomly generated in the closed interval $[-1, 1]$, $b_i,\ d_j$ are randomly generated in the closed interval $[0, 1]$ for all $i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, r$, and $A_j : \mathbb{R}^5 \to \mathbb{R}^{5(j+1)}$ be bounded linear operators, of which the elements are randomly generated in the closed interval $[-5, 5]$.

Define the function TOL

$$\text{TOL} = \frac{1}{t+r}\left(\sum_{i=1}^{t} \|x_n - P_{C_i}x_n\|^2 + \sum_{j=1}^{r} \|A_j x_n - P_{Q_j}A_j x_n\|^2\right) \tag{5.1}$$

for all $n \geq 1$. Note that if at the n-th step, TOL $= 0$, then $x_n \in S$, that is, x_n is a solution to this problem. Take TOL $< 10^{-6}$ as the stopping criterion.

Set $t = 20,\ r = 30,\ \rho_n = 0.3,\ \alpha_n = \frac{1}{n^{0.5}},\ \theta_n = \frac{1}{n^{1.2}},\ \varepsilon_n = 0.1,\ g(x) = 0.7x,\ l_i = \lambda_j = \frac{1}{t+r}$ and e the vector of corresponding dimension whose coordinates are all 1.

First, we examine the impact of different choices of step sizes and the inertial terms on the convergence. Denote by Alg.1.1 and Alg.2.1 the proposed Algorithm 1 and Algorithm 2; and by Alg.1.2 and Alg.2.2 the ones without inertial term. We choose different initial points and examine the convergence of the sequences $\{x_n\}$ which is generated by Alg.1.1, Alg.2.1, Alg.1.2 and Alg.2.2 The results of numerical experiments are reported in Table 1, Table 2 and Fig 1, Fig 2.

From the tables and the figures we know that the inertial perturbation can improve the convergence of the algorithms, and that self-adaptive step size is more efficient.

Table 1. The numerical results for Alg.1.1 and Alg.1.2

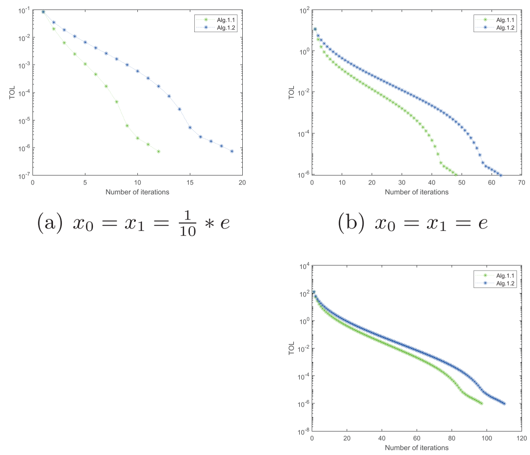
	Alg.1.1		Alg.1.2	
Initial Point	n	Time(s)	n	Time(s)
$x_0 = x_1 = \frac{1}{10} * e$	12	0.1138	19	0.1755
$x_0 = x_1 = e$	48	0.3867	63	0.5679
$x_0 = x_1 = 2 * rand(5, 1)$	72	0.6326	90	0.8166
$x_0 = x_1 = 5 * rand(5, 1)$	97	0.8948	110	0.9863

Table 2. The numerical results for Alg.2.1 and Alg.2.2

	Alg.2.1		Alg.2.2	
Initial Point	n	Time(s)	n	Time(s)
$x_0 = x_1 = \frac{1}{10}e$	8	0.0556	12	0.0769
$x_0 = x_1 = e$	22	0.1308	28	0.1589
$x_0 = x_1 = 2 * rand(5, 1)$	27	0.2000	32	0.2020
$x_0 = x_1 = 5 * rand(5, 1)$	32	0.2136	35	0.2382

Next, we consider the impact of α_n on the the convergence in Algorithm 1. For $\alpha_n = \frac{1}{n^{0.5}},\ \alpha_n = \frac{1}{n^{0.6}}$ or $\alpha_n = \frac{1}{n^{0.7}}$, with other parameters retaining the same values as above, the numerical result is Fig. 3(a). It seems that the gradient descent part dominates the convergence.

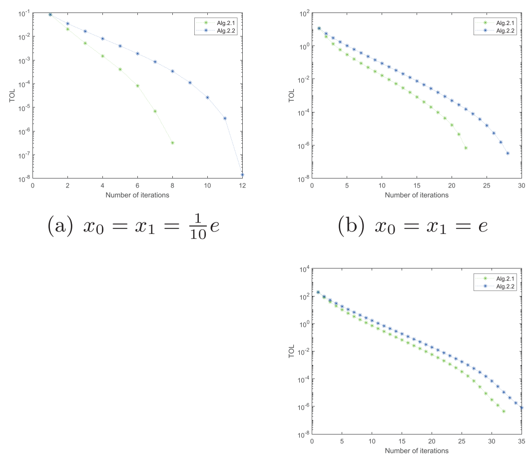
Now we consider the impact of ρ_n in Algorithm 2, which is the parameter of the self-adaptive step size. For $\rho_n = 0.3,\ \rho_n = 1,\ \rho_n = 2,\ \rho_n = 3$ or $\rho_n = 3.95$, with other parameters retaining the same values as above, the result is shown in Fig 3(b). It seems that the algorithm converges faster if the value of ρ_n is taken around the



(a) $x_0 = x_1 = \frac{1}{10} * e$ (b) $x_0 = x_1 = e$ (c) $x_0 = x_1 = 2 * rand(5,1)$

(d) $x_0 = x_1 = 5 * rand(5,1)$

FIGURE 1. Comparison of different choices of initial point.



(a) $x_0 = x_1 = \frac{1}{10} e$ (b) $x_0 = x_1 = e$ (c) $x_0 = x_1 = 2 * rand(5,1)$

(d) $x_0 = x_1 = 5 * rand(5,1)$

FIGURE 2. Comparison of different choices of initial point.

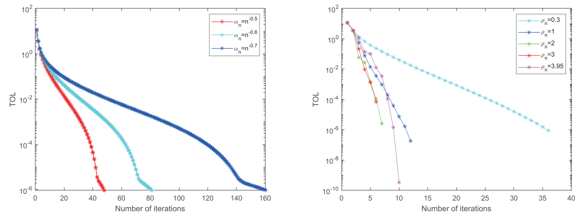
midpoint of the interval $(0,4)$, and that the error becomes smaller if ρ_n is closed to 4.

Example. Consider the following problem: find an element $x^* \in \mathbb{R}^{10}$ such that

$$x^* \in S = \bigcap_{i=1}^{t} C_i \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)),$$

where

$$C_i = \{x \in \mathbb{R}^{10} : \|x - c_i\|^2 \leq a_i^2\}$$
$$Q_j = \{A_j x \in \mathbb{R}^{10(j+1)} : \|A_j x - b_j\|^2 \leq d_j^2\}$$



(a) Different choices of α_n (b) Different choices of ρ_n

FIGURE 3. Comparison of different choices of α_n and ρ_n.

where $c_i \in \mathbb{R}^{10}$, $b_j \in \mathbb{R}^{10(j+1)}$, of which the coordinates are randomly generated in the closed interval $[-1, 1]$; $a_i, d_j \in \mathbb{R}$ are randomly generated in the closed interval $[5, 10]$ and $[10, 20]$, respectively; $A_j : \mathbb{R}^{10} \to \mathbb{R}^{10(j+1)}$, of which the elements are randomly generated in the closed interval $[-5, 5]$.

In this case, from (1.16), we can obtain C_i^n and Q_j^n as follows:

$$C_i^n = \{x \in \mathbb{R}^{10} : \|x_n - c_i\|^2 - a_i^2 \leq 2\langle x_n - c_i, x_n - x\rangle\}$$
$$Q_j^n = \{y \in \mathbb{R}^{10(j+1)} : \|A_j x_n - b_j\|^2 - d_j^2 \leq 2\langle A_j x_n - b_j, A_j x_n - y\rangle\}.$$

Set $t = 10$, $r = 20$, $\rho_n = 0.3$, $\alpha_n = \frac{1}{n^{0.7}}$, $\theta_n = \frac{1}{n^{1.2}}$, $\varepsilon_n = 0.1$, $g(x) = 0.7x$, $l_i = \lambda_j = \frac{1}{t+r}$, and e is a vector of corresponding dimension of which the coordinates are all 1.

Take TOL $< 10^{-4}$ as the stopping criterion, the fuction TOL be given in (5.1).

The comparison of the convergence of the sequences $\{x_n\}$ which is generated by Alg.1.1, Alg.2.1, Alg.1.2 and Alg.2.2 is carried out. The results of numerical experiments are reported in Table 3, Table 4, Fig 4 and Fig 5. We see again that the algorithms with inertial terms and self-adaptive step sizes has advantage over the ones without them. However, it should also be noted that the convergence can not always be obviously accelerated by the inertial terms. In this case, we might need the the alternated inertial technique or two-step inertial technique, etc, which will be discussed in future.

TABLE 3. The numerical results for Alg.1.1 and Alg.1.2

	Alg.1.1		Alg.1.2	
Initial Point	n	Time(s)	n	Time(s)
$x_0 = x_1 = rand(10, 1)$	18	0.2510	24	0.3320
$x_0 = x_1 = 5 * rand(10, 1)$	38	0.5395	43	0.5828
$x_0 = x_1 = 10 * rand(10, 1)$	48	0.6695	52	0.6908
$x_0 = x_1 = 20 * rand(10, 1)$	59	0.8556	62	0.8792

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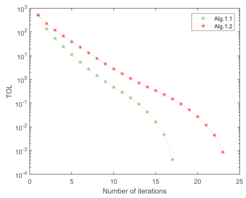
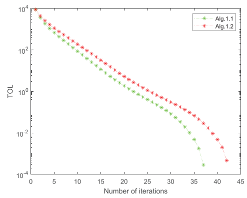
TABLE 4. The numerical results for Alg.2.1 and Alg.2.2

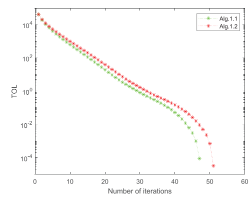
	Alg.2.1		Alg.2.2	
Initial Point	n	Time(s)	n	Time(s)
$x_0 = x_1 = rand(10, 1)$	15	0.1516	22	0.2143
$x_0 = x_1 = 5 * rand(10, 1)$	32	0.3337	37	0.3998
$x_0 = x_1 = 10 * rand(10, 1)$	45	0.4794	49	0.5112
$x_0 = x_1 = 20 * rand(10, 1)$	50	0.5218	53	0.5543



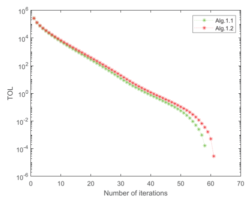
(a) $x_0 = x_1 = rand(10, 1)$



(b) $x_0 = x_1 = 5*rand(10, 1)$



(c) $x_0 = x_1 = 10 * rand(10, 1)$



(d) $x_0 = x_1 = 20 * rand(10, 1)$

FIGURE 4. Comparison of different choices of initial point.

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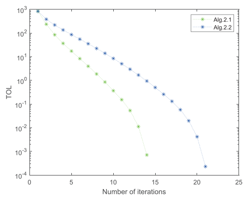
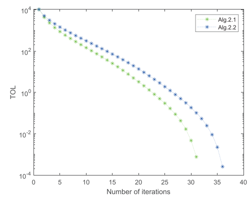
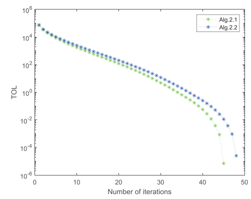


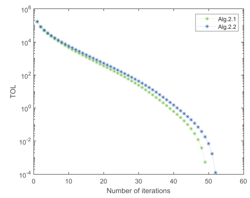
(a) $x_0 = x_1 = rand(10, 1)$



(b) $x_0 = x_1 = 5*rand(10, 1)$



(c) $x_0 = x_1 = 10 * rand(10, 1)$



(d) $x_0 = x_1 = 20 * rand(10, 1)$

FIGURE 5. Comparison of different choices of initial point.

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