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CONVERGENCE OF INEXACT ITERATES OF NONEXPANSIVE MAPPINGS WITH SUMMABLE ERRORS IN METRIC SPACES WITH GRAPHS

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Abstract. In his seminal 1967 paper, A. M. Ostrowski showed that inexact iterates of a strict contraction converged in the presence of summable computational errors. This result was extended in our 2006 joint work with D. Butnariu to nonexpansive operators which are not necessarily strict contractions. In the present work we establish such a result for inexact iterates of operators on complete metric spaces with graphs.

1. Introduction

Since the publication of Banach's classical theorem [2], fixed point theory has become an interesting and important area of nonlinear analysis. The main goals of this theory are to obtain the existence of a fixed point for a given operator and to study the convergence of its iterates. Many results of this kind can be found, for instance, in [4, 5, 6, 9, 10, 11, 12, 14, 15]. Although Banach's fixed point theorem was established for strict contractions, there are by now a lot of existence results for different large classes of nonexpansive operators which are not necessarily strict contractions. Many of these results can be found in the book [12].

In his seminal paper [9], A. M. Ostrowski showed that inexact iterates of a strict contraction converged in the presence of summable computational errors. This result was extended in our 2006 joint work with D. Butnariu [4] to nonexpansive operators which are not necessarily strict contractions. Namely, in [4] we considered a nonexpansive operator acting on a complete metric space under the assumption that all its iterates converge. According to the generic approach, this convergence property holds for a typical nonexpansive operator [12]. In [4] it was established that in this case all inexact iterates of the operator also converge provided the computational errors are summable. This result is clearly a direct generalization of Ostrowski's theorem.

In the present work, we generalize the result of [4] to inexact iterates of operators on complete metric spaces with graphs. Our result is an extension of an analogous result which was obtained in [13] for strict contractions.

2. Main result

Assume that (X, ρ) is a metric space which is equipped with a complete metric ρ and a graph G. We denote by $V(G)$ the set of the vertices of G and by $E(G)$ the set of its edges. Clearly, $V(G) \subset X$ and $E(G) \subset X \times X$. We assume that $E(G)$

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is a closed subset of $X \times X$ equipped with the product topology. Let the operator $T : X \to X$ be a self-mapping of X and suppose that the following assumption holds true.

(A) For every pair of points $u, v \in X$, if $(u, v) \in E(G)$, then $(T(u), T(v)) \in E(G)$ and

$$\rho(T(u), T(v)) \leq \rho(u, v). \tag{2.1}$$

Note that operators satisfying (A) were studied in [1, 3].

For every point $\xi \in X$ and every nonempty set $C \subset X$, define

$$\rho(\xi, C) := \inf\{\rho(\xi, \eta) : \ \eta \in C\}.$$

For every point $\xi \in X$ and every number $\gamma > 0$, set

$$B(\xi, \gamma) := \{\eta \in X : \ \rho(\xi, \eta) \leq \gamma\}.$$

For every operator $S : X \to X$, set $S^0 z = z$ for all $z \in X$.

In this paper we establish the following result.

Theorem 2.1. *Let a sequence $\{x_k\}_{k=0}^{\infty} \subset X$ satisfy*

$$(T(x_k), x_{k+1}) \in E(G), \ k = 0, 1, \ldots, \tag{2.2}$$

and

$$\sum_{k=0}^{\infty} \rho(T(x_k), x_{k+1}) < \infty, \tag{2.3}$$

and let a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ of $\{x_k\}$ be given. Then the following assertions hold.

1. Assume that for every integer k, the sequence $\{T^j(x_{i_k})\}_{j=1}^{\infty}$ converges. Then there exists the limit

$$x_* = \lim_{i \to \infty} x_i,$$

$(x_*, T(x_*)) \in E(G)$, and if the operator T is continuous at x_*, then x_* is a fixed point of T.

2. Assume that there exists a nonempty set F such that for every natural number k,

$$\lim_{j \to \infty} \rho(T^j(x_{i_k}), F) = 0.$$

Then

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

3. Assume that for every integer $k \geq 1$, there is a nonempty compact set $E_k \subset X$ such that

$$\lim_{j \to \infty} \rho(T^j(x_{i_k}), E_k) = 0.$$

Then there is a nonempty compact set $E \subset X$ such that $\lim_{i \to \infty} \rho(x_i, E) = 0$.

Theorem 2.1 is established in Section 4.

3. An auxiliary result

Lemma 3.1. *Assume that* $\{x_k\}_{k=0}^{\infty} \subset X$ *and that*

$$(T(x_k), x_{k+1}) \in E(G),\ k = 0, 1, \dots, \tag{3.1}$$

$$\sum_{k=0}^{\infty} \rho(T(x_k), x_{k+1}) < \infty, \tag{3.2}$$

and q, n *are natural numbers. Then*

$$\rho(x_{n+q}, T^n(x_q)) \leq \sum_{j=1+q}^{q+n} \rho(T(x_{j-1}), x_j).$$

Proof. In view of (3.1) and Assumption (A), for every integer $p \geq 0$ and every integer $s \geq 1$,

$$(T^{s+1}(x_p), T^s(x_{p+1})) \in E(G) \tag{3.3}$$

and

$$\rho(T^{s+1}(x_p), T^s(x_{p+1})) \leq \rho(T(x_p), x_{p+1}). \tag{3.4}$$

By (3.4),

$$\rho(x_{n+q}, T^n(x_q)) \leq \sum_{j=1}^{n} \rho(T^{n-j+1}(x_{q+j-1}), T^{n-j}(x_{q+j}))$$

$$\leq \sum_{j=1}^{n} \rho(T(x_{q+j-1}), x_{q+j}) \leq \sum_{j=1+q}^{q+n} \rho(T(x_{j-1}), x_j).$$

Lemma 3.1 is proved. □

4. Proof of Theorem 2.1

Let $\epsilon \in (0, \infty)$. Relation (2.3) implies that there is a natural number k for which

$$\sum_{j=i_k-1}^{\infty} \rho(x_{j+1}, T(x_j)) < \epsilon/4. \tag{4.1}$$

Lemma 3.1 and relations (2.2), (2.3), (3.1), (3.2) and (4.1) imply that for every $n \in \{1, 2, \dots\}$,

$$\rho(x_{n+i_k}, T^n(x_{i_k})) \leq \sum_{j=1+i_k}^{i_k+n} \rho(T(x_{j-1}), x_j) < \epsilon/4. \tag{4.2}$$

We prove the three assertions formulated in the statement of Theorem 2.1 one by one.

We first prove Assertion 1. There exists

$$y_* = \lim_{j\to\infty} T^j(x_{i_k}). \tag{4.3}$$

By (4.2) and (4.3), for all sufficiently large positive integers n,

$$\rho(x_{n+i_k}, y_*) \leq \rho(y_*, T^n(x_{i_k})) + \rho(T^n(x_{i_k}), x_{i_k+n}) < \epsilon.$$

Since ϵ is any element of the interval $(0, \infty)$, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{4.4}$$

It follows from (2.3) and (4.4) that

$$x_* = \lim_{i\to\infty} T(x_i). \tag{4.5}$$

Since the set $E(G)$ is assumed to be closed, relations (2.3), (4.4) and (4.5) imply that

$$(x_*, x_*) \in E(G).$$

If the operator T is continuous at x_*, then x_* is a fixed point of T. Assertion 1 is proved.

Next, we prove Assertion 2. By our assumptions,

$$\lim_{j\to\infty} \rho(T^j(x_{i_k}), F) = 0.$$

When combined with (4.2), this implies that for every sufficiently large natural number n,

$$\rho(x_{n+i_k}, F) \le \rho(x_{n+i_k}, T^n(x_{i_k})) + \rho(T^n(x_{i_k}), F) < \epsilon.$$

Since ϵ is any element of $(0, \infty)$, it follows that

$$\lim_{i\to\infty} \rho(x_i, F) = 0.$$

Thus Assertion 2 is also proved.

Finally, we prove Assertion 3. There exists a compact set $E_0 \subset X$ such that

$$\lim_{n\to\infty} \rho(T^n(x_k), E_0) = 0. \tag{4.6}$$

By (4.2) and (4.6), for every sufficiently large positive integer n,

$$\rho(x_{n+i_k}, E_0) \le \rho(x_{n+i_k}, T^n(x_{i_k})) + \rho(T^n(x_{i_k}), E_0) < \epsilon.$$

This implies that each subsequence of the sequence $\{x_i\}_{i=0}^{\infty}$ has a convergent subsequence. Let E be the collection of all subsequential limit points of $\{x_i\}_{i=0}^{\infty}$. It is clear that E is compact and that

$$\lim_{i\to\infty} \rho(x_i, E) = 0.$$

This completes the proof of Assertion 3 and of Theorem 2.1 itself.

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