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COMBINED EXTRAGRADIENT IMPLICIT RULE FOR A SYSTEM OF VARIATIONAL INEQUALITIES WITH VARIATIONAL INCLUSION CONSTRAINT INVOLVING FIXED POINTS OF PSEUDOCONTRACTIONS

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Abstract. In a real Banach space, let the VI denote a variational inclusion for two accretive operators and let the CFPP indicate a common fixed point problem of countably many pseudocontractive mappings. In this paper, we introduce a combined extragradient implicit rule for solving a general system of variational inequalities (GSVI) with the VI and CFPP constraints. We then prove the strong convergence of the suggested method to a solution of the GSVI with the VI and CFPP constraints under some mild conditions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces.

1. Introduction

In a real Hilbert space H, let the inner product and induced norm be denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Given a nonempty closed convex subset $C \subset H$. Let P_C be the metric projection of H onto C. Given a mapping $A : C \to H$. The classical variational inequality problem (VIP) is to find a point $u^* \in C$ s.t. $\langle Au^*, v - u^*\rangle \geq 0\ \forall v \in C$. We denote by $\mathrm{VI}(C, A)$ the solution set of the VIP. In 1976, Korpelevich [24] first designed an extragradient method, i.e., for any given $u_0 \in C$, the sequence $\{u_i\}$ is generated by

$$\begin{cases} v_i = P_C(u_i - \ell A u_i), \\ u_{i+1} = P_C(u_i - \ell A v_i) \quad \forall i \geq 0, \end{cases} \tag{1.1}$$

with $\ell \in (0, \frac{1}{L})$, which has been one of the most effective methods for solving the VIP till now. In the case of $\mathrm{VI}(C, A) \neq \emptyset$, the sequence $\{u_i\}$ has only weak convergence. Indeed, the convergence of $\{u_i\}$ only requires that the mapping A is monotone and Lipschitz continuous. To the most of our knowledge, Korpelevich's extragradient method has received great attention given by many authors, who improved and modified it in various ways; see e.g., [1, 6, 8–22, 25, 30, 35–37, 42, 43] and references therein, to name but a few.

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In order to solve the variational inclusion (VI) of finding $v^* \in H$ s.t. $0 \in (A + B)v^*$, Takahashi et al. [40] suggested a Halpern-type iterative method, i.e., for any given $v_0, u \in H$, $\{v_i\}$ is the sequence generated by

$$v_{i+1} = \beta_i v_i + (1-\beta_i)(\alpha_i u + (1-\alpha_i)J^B_{\lambda_i}(v_i - \lambda_i A v_i)) \quad \forall i \geq 0, \tag{1.2}$$

where A is an α-inverse-strongly monotone operator on H and B is a maximal monotone operator on H. They proved the strong convergence of $\{v_i\}$ to a solution $v^* \in (A+B)^{-1}0$ of the VI. Subsequently, Pholasa et al. [31] extended the result in [40] to the setting of Banach spaces, and proved the strong convergence of $\{v_i\}$ to a point of $(A+B)^{-1}0$.

Meantime, Takahashi et al. [39] proposed a Mann-type Halpern iterative scheme for solving the FPP of a nonexpansive mapping $S : C \to C$ and the VI for an α-inverse-strongly monotone mapping $A : C \to H$ and a maximal monotone operator $B : D(B) \subset C \to H$, i.e., for any given $y_1 = y \in C$, $\{y_i\}$ is the sequence generated by

$$y_{i+1} = \beta_i y_i + (1-\beta_i)S(\alpha_i y + (1-\alpha_i)J^B_{\lambda_i}(y_i - \lambda_i A y_i)) \quad \forall i \geq 1, \tag{1.3}$$

where $\{\lambda_i\} \subset (0, 2\alpha)$ and $\{\alpha_i\}, \{\beta_i\} \subset (0,1)$ are such that (i) $\lim_{i\to\infty} \alpha_i = 0$, $\sum_{i=1}^\infty \alpha_i = \infty$; (ii) $0 < a \leq \lambda_i \leq b < 2\alpha$, $\lim_{i\to\infty}(\lambda_i - \lambda_{i+1}) = 0$; and (iii) $0 < c \leq \beta_i \leq d < 1$. They proved the strong convergence of $\{y_i\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$.

Owing to the importance and interesting of the VI, many researchers have presented and developed a great number of iterative methods for solving the VI in several approaches; see e.g., [12,15,16,18,26,28,31,36,39,40] and the references therein. Moreover, they consider the FPP of finding a point $u^* \in C$ such that $u^* = Su^*$ where $S : C \to C$ is a nonlinear mapping. The solution set of the FPP is denoted by $\mathrm{Fix}(S)$. In the practical life, many mathematical models have been formulated as this problem. Many mathematicians are now interested in finding a common solution of the VI and FPP, i.e., find a point u^* s.t. $u^* \in \mathrm{Fix}(S) \cap (A+B)^{-1}0$.

Suppose that $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and $S : C \to C$ is a nonexpansive mapping. In 2011, Manaka and Takahashi [28] suggested an iterative process, i.e., for any given $u_0 \in C$, $\{u_i\}$ is the sequence generated by

$$u_{i+1} = \alpha_i u_i + (1-\alpha_i)SJ^B_{\lambda_i}(u_i - \lambda_i A u_i) \quad \forall i \geq 0, \tag{1.4}$$

where $\{\alpha_i\} \subset (0,1)$ and $\{\lambda_i\} \subset (0,\infty)$. They proved the weak convergence of $\{u_i\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some suitable conditions.

Furthermore, assume that $q \in (1,2]$ and E is a real Banach space. Let $f : E \to E$ be a ρ-contraction and $S : E \to E$ be a nonexpansive mapping. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Very recently, in order to solve the FPP of S and the VI of finding $u^* \in E$ s.t. $0 \in (A+B)u^*$, Sunthrayuth and Cholamjiak [36] suggested a modified viscosity-type extragradient method in the setting of uniformly convex and q-uniformly smooth Banach space E with q-uniform smoothness coefficient κ_q, i.e., for any given $u_0 \in E$, $\{u_i\}$ is the sequence generated by

$$\begin{cases} y_i = J^B_{\lambda_i}(u_i - \lambda_i A u_i), \\ z_i = J^B_{\lambda_i}(u_i - \lambda_i A y_i + r_i(y_i - u_i)), \\ u_{i+1} = \alpha_i f(u_i) + \beta_i u_i + \gamma_i S z_i \quad \forall i \geq 0, \end{cases} \tag{1.5}$$

where $J^B_{\lambda_i} = (I + \lambda_i B)^{-1}$, $\{r_i\}, \{\alpha_i\}, \{\beta_i\}, \{\gamma_i\} \subset (0,1)$ and $\{\lambda_i\} \subset (0,\infty)$ are such that: (i) $\alpha_i+\beta_i+\gamma_i = 1$; (ii) $\lim_{i\to\infty} \alpha_i = 0$, $\sum_{i=1}^{\infty} \alpha_i = \infty$; (iii) $\{\beta_i\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_i < \lambda_i/r_i \leq \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \leq r_i < 1$. They proved strong convergence of $\{u_i\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves a certain hierarchical variational inequality (HVI).

On the other hand, let $J : E \to 2^{E^*}$ be the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}$ $\forall x \in E$, where $\langle \cdot,\cdot\rangle$ represents the generalized duality pairing between E and E^*. Recall that if E is smooth then J is single-valued. Let $B_1, B_2 : C \to E$ be two nonlinear mappings in a smooth Banach space E. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{1.6}$$

where μ_i is a positive constant for $i = 1, 2$. In particular, if $E = H$ a real Hilbert space, it is easy to see that the GSVI (1.6) reduces to the GSVI considered in [19],

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{1.7}$$

It is known that problem (1.7) has been transformed into a fixed point problem in the following way.

Lemma 1.1 (see [19]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.7) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.*

In 2018, using Lemma 1.1, Cai et al. [6] introduced a viscosity implicit rule for solving the GSVI (1.7) with the FPP constraint of an asymptotically nonexpansive mapping T with a sequence $\{\theta_i\}$, i.e., for any given $x_0 \in C$, the sequence $\{x_i\}$ is generated by

$$\begin{cases} u_i = s_i x_i + (1 - s_i) y_i, \\ z_i = P_C(u_i - \mu_2 B_2 u_i), \\ y_i = P_C(z_i - \mu_1 B_1 z_i), \\ x_{i+1} = P_C[\alpha_i f(x_i) + (I - \alpha_i \rho F)T^i y_i] \quad \forall i \geq 0, \end{cases} \tag{1.8}$$

where $\{\alpha_i\}, \{s_i\} \subset (0,1]$ are such that (i) $\lim_{i\to\infty} \alpha_i = 0$, $\sum_{i=0}^{\infty} \alpha_i = \infty$, $\sum_{i=0}^{\infty} |\alpha_{i+1} - \alpha_i| < \infty$; (ii) $\lim_{i\to\infty} \theta_i/\alpha_i = 0$; (iii) $0 < \varepsilon \leq s_i \leq 1$, $\sum_{i=0}^{\infty} |s_{i+1} - s_i| < \infty$; and (iv) $\sum_{i=0}^{\infty} \|T^{i+1} y_i - T^i y_i\| < \infty$. They proved that the sequence constructed by (1.8) converges strongly to a point of $\mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T)$, which solves a certain HVI.

In a real Banach space E, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of countably many nonexpansive mappings. In this paper, we introduce a generalized extragradient implicit method for solving the GSVI (1.6) with the VI and CFPP constraints. We then prove the strong convergence of the suggested method to a solution of the GSVI (1.6) with the VI and CFPP constraints under some suitable assumptions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces. Our results improve and extend the corresponding results in Manaka and Takahashi [28], Sunthrayuth and Cholamjiak [36], and Cai et al. [6] to a certain extent.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Banach space E with the dual E^*. For simplicity, we shall use the following notations: $x_n \to x$ indicates the strong convergence of the sequence $\{x_n\}$ to x and $x_n \rightharpoonup x$ denotes the weak convergence of the sequence $\{x_n\}$ to x. Given a self-mapping T on C. We use the notations $\mathbf{R}$ and $\mathrm{Fix}(T)$ to stand for the set of all real numbers and the fixed point set of T, respectively. Recall that T is said to be nonexpansive if $\|Tu - Tv\| \leq \|u - v\|\ \forall u, v \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \delta \in [0, 1)$ s.t. $\|f(u) - f(v)\| \leq \delta\|u - v\|\ \forall u, v \in C$. Also, recall that the normalized duality mapping J defined by

$$J(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\} \quad \forall x \in E. \tag{2.1}$$

is the one from E into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of E^*, satisfying $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|u + v\|}{2} : u, v \in E,\ \|u\| = \|v\| = 1,\ \|u - v\| \geq \epsilon\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\{\frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : u, v \in E,\ \|u\| = \|v\| = 1\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0\ \forall \epsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau\to 0^+} \rho_E(\tau)/\tau = 0$. Also, it is said to be q-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \leq ct^q\ \forall t > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth and if E is uniformly convex, then E is also reflexive and strictly convex. It is known that Hilbert space H is 2-uniformly smooth. Further, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [44].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q,\ \|\phi\| = \|x\|^{q-1}\}, \tag{2.2}$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of E. It is known that

$J_q(x) = \|x\|^{q-2}J(x)$ $\forall x \neq 0$ and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued. Furthermore, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is q-uniformly smooth for $q > 2$; see [38] for more details.

Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then the following inequality is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$:

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle \quad \forall x, y \in E,\ j_q(x+y) \in J_q(x+y). \tag{2.3}$$

Lemma 2.1 (see [23]). *If $T : C \to C$ is a continuous and strong pseudocontraction mapping, then T has a unique fixed point in C.*

The following lemma can be obtained from the result in [44].

Lemma 2.2. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that*

(a) $\|\mu u + (1-\mu)v\|^q \leq \mu\|u\|^q + (1-\mu)\|v\|^q - \mu(1-\mu)g(\|u-v\|)$ with $\mu \in [0,1]$;
(b) $h(\|u-v\|) \leq \|u\|^q - q\langle u, j_q(v)\rangle + (q-1)\|v\|^q$
for all $u, v \in B_r$ and $j_q(v) \in J_q(v)$, where $B_r := \{y \in E : \|y\| \leq r\}$.

The following lemma is an analogue of Lemma 2.2 (a).

Lemma 2.3. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exists a strictly increasing, continuous and convex function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ such that*

$$\|\lambda u + \mu v + \nu w\|^q \leq \lambda\|u\|^q + \mu\|v\|^q + \nu\|w\|^q - \lambda\mu g(\|u-v\|)$$

for all $u, v, w \in B_r$ and $\lambda, \mu, \nu \in [0,1]$ with $\lambda + \mu + \nu = 1$.

Proposition 2.4 (see [3]). *Let $\{S_n\}_{n=0}^\infty$ be a sequence of self-mappings on C such that $\sum_{n=1}^\infty \sup_{x\in C}\|S_n x - S_{n-1}x\| < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty}\sup_{x\in C}\|S_n x - Sx\| = 0$.*

Proposition 2.5 (see [44]). *Let $q \in (1,2]$ a fixed real number and let E be q-uniformly smooth. Then $\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q$ $\forall x, y \in E$, where κ_q is the q-uniform smoothness coefficient of E.*

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. In terms of [32], we know that if E is smooth and Π is a retraction of C onto D, then the following statements are equivalent:

(i) Π is sunny and nonexpansive;

(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle \ \forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0 \ \forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \neq \emptyset \ \forall x \in C$. Let $q > 1$. An operator B is said to be accretive if for each $x, y \in C$, $\exists j_q(x - y) \in J_q(x - y)$ s.t. $\langle u - v, j_q(x - y)\rangle \geq 0 \ \forall u \in Bx, v \in By$. An accretive operator B is said to be α-inverse-strongly accretive of order q if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x - y)\rangle \geq \alpha\|u - v\|^q \ \forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J^B_\lambda : (I + \lambda B)C \to C$ by $J^B_\lambda = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J^B_λ is called the resolvent of B for $\lambda > 0$.

Lemma 2.6 (see [16, 26]). *Let $B : C \to 2^E$ be an m-accretive operator. Then the following statements hold:*

(i) *the resolvent identity: $J^B_\lambda x = J^B_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J^B_\lambda x) \ \forall \lambda, \mu > 0, \ x \in E$;*
(ii) *if J^B_λ is a resolvent of B for $\lambda > 0$, then J^B_λ is a firmly nonexpansive mapping with $\mathrm{Fix}(J^B_\lambda) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$;*
(iii) *if $E = H$ a Hilbert space, B is maximal monotone.*

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J^B_\lambda(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A) \ \forall \lambda > 0$.

Proposition 2.7 (see [26]). *The following statements hold:*

(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0 \ \forall \lambda > 0$;
(ii) $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ *for* $0 < \lambda \leq r$ *and* $y \in C$.

Proposition 2.8 (see [45]). *Let E be uniformly smooth, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $f : C \to C$ be a fixed contraction. For each $t \in (0, 1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1 - t)Tz$ on C, i.e., $z_t = tf(z_t) + (1 - t)Tz_t$. Then $\{z_t\}$ converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which solves the hierarchical variational inequality (HVI): $\langle (I - f)x^*, J(x^* - x)\rangle \leq 0 \ \forall x \in \mathrm{Fix}(T)$.*

Proposition 2.9 (see [26]). *Let E be q-uniformly smooth with $q \in (1, 2]$. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \geq 0$,*

$$\|(I - \lambda A)u - (I - \lambda A)v\|^q \leq \|u - v\|^q - \lambda(\alpha q - \kappa_q\lambda^{q-1})\|Au - Av\|^q \quad \forall u, v \in C,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 \leq \lambda \leq (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Proposition 2.10 (see [35]). *Let E be q-uniformly smooth with $q \in (1, 2]$. Let Π_C be a sunny nonexpansive retraction from E onto C. Suppose that $B_1, B_2 : C \to E$ are α-inverse-strongly accretive mapping of order q and β-inverse-strongly accretive mapping of order q, respectively. Let $G : C \to C$ be a mapping defined by $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$, and GSVI$(C, B_1, B_2)$ denote the fixed point set of G. If $0 \leq \mu_1 \leq (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$ and $0 \leq \mu_2 \leq (\frac{q\beta}{\kappa_q})^{\frac{1}{q-1}}$, then G is nonexpansive*

Lemma 2.11 (see [35]). *Let E be q-uniformly smooth with $q \in (1, 2]$. Let Π_C be a sunny nonexpansive retraction from E onto C. Suppose that $B_1, B_2 : C \to E$ are two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.6) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$, and $y^* = \Pi_C(I - \mu_2 B_2)x^*$.*

Lemma 2.12 (see [2]). *Let E be smooth, $A : C \to E$ be accretive and Π_C be a sunny nonexpansive retraction from E onto C. Then $\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I-\lambda A))\ \forall \lambda > 0$, where $\mathrm{VI}(C, A)$ is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z - y)\rangle \leq 0\ \forall y \in C$.*

Recall that if $E = H$ a Hilbert space, then the sunny nonexpansive retraction Π_C from E onto C coincides with the metric projection P_C from H onto C. Moreover, if E is uniformly smooth and T is a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \neq \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from E onto C [33]. By Lemma 2.12 we know that, $x^* \in \mathrm{Fix}(T)$ solves the HVI in Proposition 2.8 if and only if x^* solves the fixed point equation $x^* = \Pi_{\mathrm{Fix}(T)} f(x^*)$.

Lemma 2.13 (see [27]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each integer $i \geq 1$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \geq 1$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}\ \forall n \geq n_0$.

Lemma 2.14 (see [4]). *Let E be strictly convex, and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x\ \forall x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 2.15 (see [45]). *Let $\{a_n\}$ be a sequence in $[0, \infty)$ such that $a_{n+1} \leq (1 - s_n)a_n + s_n\nu_n\ \forall n \geq 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions: (i) $\{s_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \leq 0$ or $\sum_{n=0}^{\infty} |s_n\nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main results

Throughout this paper, we assume that E is a q-uniformly smooth and uniformly convex Banach space with $q \in (1, 2]$. Let C be a nonempty closed convex subset of E and Π_C be a sunny nonexpansive retraction from E onto C. Let $f : C \to C$ be a δ-contraction with constant $\delta \in [0, 1)$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Suppose that $B_1, B_2 : C \to E$ are α-inverse-strongly accretive mapping of order q and β-inverse-strongly accretive mapping of order q, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0 \neq \emptyset$.

Algorithmn 3.1. Combined extragradient implicit rule for the GSVI (1.6) with the VI and CFPP constraints.

Initial Step: Given $\zeta \in (0,1)$ and $x_0 \in C$ arbitrarily.
Step 1: Calculate $w_n = s_n x_n + (1-s_n)(\zeta S_n w_n + (1-\zeta)Gw_n)$;
Step 2: Calculate $y_n = J^B_{\lambda_n}(u_n - \lambda_n A u_n)$ with $u_n = Gw_n$;
Step 3: Calculate $z_n = J^B_{\lambda_n}(u_n - \lambda_n A y_n + r_n(y_n - u_n))$;
Step 4: Calculate $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n G z_n$, where $\{r_n\}$, $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0,\infty)$.
Set $n := n+1$ and go to Step 1.

Lemma 3.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A+B)^{-1}0$. Then we observe that

$$p = Gp = S_n p = J^B_{\lambda_n}(p - \lambda_n Ap) = J^B_{\lambda_n}\left((1-r_n)p + r_n\Big(p - \frac{\lambda_n}{r_n}Ap\Big)\right).$$

By Propositions 2.9 and 2.10, we know that $I - \mu_1 B_1,\ I - \mu_2 B_2$ and $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ are nonexpansive mappings. Moreover, it can be readily seen that for each $n \geq 0$, there is only an element $w_n \in C$ s.t.

$$w_n = s_n x_n + (1-s_n)(\zeta S_n w_n + (1-\zeta)Gw_n). \tag{3.1}$$

In fact, consider the mapping $F_n u = s_n x_n + (1-s_n)(\zeta S_n u + (1-\zeta)Gu)\ \forall u \in C$. Note that $S_n : C \to C$ is a continuous pseudocontraction. Hence we obtain that for all $u, v \in C$,

$$\begin{aligned}&\langle F_n u - F_n v, J(u-v)\rangle \\ &\quad= (1-s_n)\langle(\zeta S_n u + (1-\zeta)Gu) - (\zeta S_n v + (1-\zeta)Gv), J(u-v)\rangle \\ &\quad= (1-s_n)[\zeta\langle S_n u - S_n v, J(u-v)\rangle + (1-\zeta)\langle Gu - Gv, J(u-v)\rangle] \\ &\quad\leq (1-s_n)\|u-v\|^2.\end{aligned}$$

Also, from $\{s_n\} \subset (0,1]$, we get $0 \leq 1 - s_n < 1\ \forall n \geq 0$. Thus, F_n is a continuous and strong pseudocontractive self-mapping on C. Using Lemma 2.1, we deduce that for each $n \geq 0$, there is only an element $w_n \in C$, satisfying (3.1). Since each $S_n : C \to C$ is a pseudocontraction mapping, we get

$$\begin{aligned}\|w_n - p\|^2 &= s_n\langle x_n - p, J(w_n - p)\rangle + (1-s_n)\langle \zeta S_n w_n \\ &\quad + (1-\zeta)Gw_n - p, J(w_n - p)\rangle \\ &\leq s_n\|x_n - p\|\|w_n - p\| \\ &\quad + (1-s_n)[\zeta\|w_n - p\|^2 + (1-\zeta)\|w_n - p\|^2] \\ &= s_n\|x_n - p\|\|w_n - p\| + (1-s_n)\|w_n - p\|^2,\end{aligned}$$

and hence

$$\|w_n - p\| \leq \|x_n - p\| \quad \forall n \geq 0.$$

Using $u_n = Gw_n$, we deduce from the nonexpansivity of G that

$$\|u_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| \quad \forall n \geq 0. \tag{3.2}$$

Using Lemma 2.6 (ii) and Proposition 2.9, we have

$$\begin{aligned} \|y_n - p\|^q &= \|J^B_{\lambda_n}(u_n - \lambda_n A u_n) - J^B_{\lambda_n}(p - \lambda_n A p)\|^q \\ &\le \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^q \\ &\le \|u_n - p\|^q - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q, \end{aligned} \tag{3.3}$$

which hence leads to

$$\|y_n - p\| \le \|u_n - p\|.$$

By the convexity of $\|\cdot\|^q$ for all $q \in (1, 2]$ and (3.3), we deduce that

$$\begin{aligned} \|z_n - p\|^q &= \Big\| J^B_{\lambda_n}\Big((1 - r_n)u_n + r_n\Big(y_n - \frac{\lambda_n}{r_n} A y_n\Big)\Big) \\ &\quad - J^B_{\lambda_n}\Big((1 - r_n)p + r_n\Big(p - \frac{\lambda_n}{r_n} A p\Big)\Big)\Big\|^q \\ &\le (1 - r_n)\|u_n - p\|^q + r_n \Big\|\Big(I - \frac{\lambda_n}{r_n} A\Big) y_n - \Big(I - \frac{\lambda_n}{r_n} A\Big) p\Big\|^q \\ &\le (1 - r_n)\|u_n - p\|^q + r_n \Big[\|y_n - p\|^q - \frac{\lambda_n}{r_n}\Big(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ap\|^q\Big] \qquad (3.4) \\ &\le (1 - r_n)\|u_n - p\|^q + r_n \Big[\|u_n - p\|^q - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q \\ &\quad - \frac{\lambda_n}{r_n}\Big(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ap\|^q\Big] \\ &= \|u_n - p\|^q - r_n \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q \\ &\quad - \lambda_n\Big(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ap\|^q. \end{aligned}$$

This ensures that

$$\|z_n - p\| \le \|u_n - p\|.$$

So it follows from (3.2) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(Gz_n - p)\| \\ &\le \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|Gz_n - p\| \\ &\le \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|Gz_n - p\| \\ &\le \alpha_n(\delta\|x_n - p\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= (1 - \alpha_n(1 - \delta))\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\le \max\Big\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \delta}\Big\}. \end{aligned}$$

By induction, we get $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\delta}\}$ $\forall n \ge 0$. Consequently, $\{x_n\}$ is bounded, and so are $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Gz_n\}$, $\{Au_n\}$, $\{Ay_n\}$. This completes the proof. □

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$, $0 < c \le s_n \le d < 1$;
(C3) $0 < r \le r_n < 1$ *and* $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < (\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$;
(C4) $0 < \mu_1 < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $0 < \mu_2 < (\frac{\beta q}{\kappa_q})^{\frac{1}{q-1}}$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset* D *of* C. *Let* $S : C \to C$ *be a mapping defined by* $Sx = \lim_{n\to\infty} S_n x\ \forall x \in C$, *and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. *Then* $x_n \to x^* \in \Omega$, *which is the unique solution to the HVI:* $\langle (I-f)x^*, J(x^*-p)\rangle \le 0\ \forall p \in \Omega$, *i.e., the fixed point equation* $x^* = \Pi_\Omega f(x^*)$.

Proof. First of all, let $x^* \in \Omega$ and $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$. Moreover, we put $v_n := \Pi_C(I - \mu_2 B_2)w_n$ for all $n \ge 0$. Then $u_n = \Pi_C(I - \mu_1 B_1)v_n\ \forall n \ge 0$. Using Proposition 2.9 we get

$$\begin{aligned}\|v_n - y^*\|^q &= \|\Pi_C(w_n - \mu_2 B_2 w_n) - \Pi_C(x^* - \mu_2 B_2 x^*)\|^q \\ &\le \|w_n - x^*\|^q - \mu_2(\beta q - \kappa_q \mu_2^{q-1})\|B_2 w_n - B_2 x^*\|^q,\end{aligned}$$

and

$$\begin{aligned}\|u_n - x^*\|^q &= \|\Pi_C(v_n - \mu_1 B_1 v_n) - \Pi_C(y^* - \mu_1 B_1 y^*)\|^q \\ &\le \|v_n - y^*\|^q - \mu_1(\alpha q - \kappa_q \mu_1^{q-1})\|B_1 v_n - B_1 y^*\|^q.\end{aligned}$$

Combining the last two inequalities, we have

$$\begin{aligned}\|u_n - x^*\|^q \le{}& \|w_n - x^*\|^q - \mu_2(\beta q - \kappa_q \mu_2^{q-1})\|B_2 w_n - B_2 x^*\|^q \\ &- \mu_1(\alpha q - \kappa_q \mu_1^{q-1})\|B_1 v_n - B_1 y^*\|^q.\end{aligned}$$

Using Lemma 2.3, from (2.3), (3.2) and (3.4) we have that

$$\begin{aligned}\|x_{n+1} - x^*\|^q \le{}& \|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(Gz_n - x^*)\|^q \\ &+ q\alpha_n\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \\ \le{}& \alpha_n\|f(x_n) - f(x^*)\|^q + \beta_n\|x_n - x^*\|^q \\ &+ \gamma_n\|Gz_n - x^*\|^q - \beta_n\gamma_n g(\|x_n - Gz_n\|) \\ &+ q\alpha_n\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \\ \le{}& \alpha_n\delta\|x_n - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[\|u_n - x^*\|^q \\ &- r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q \\ &- \lambda_n\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ax^*\|^q] - \beta_n\gamma_n g(\|x_n - Gz_n\|) \\ &+ q\alpha_n\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \\ \le{}& \alpha_n\delta\|x_n - x^*\|^q + \beta_n\|x_n - x^*\|^q \\ &+ \gamma_n[\|x_n - x^*\|^q - \mu_2(\beta q - \kappa_q\mu_2^{q-1})\|B_2 w_n - B_2 x^*\|^q \\ &- \mu_1(\alpha q - \kappa_q\mu_1^{q-1})\|B_1 v_n - B_1 y^*\|^q \\ &- r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q \\ &- \lambda_n\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ax^*\|^q] - \beta_n\gamma_n g(\|x_n - Gz_n\|)\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+ q\alpha_n\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \\
&= (1 - \alpha_n(1-\delta))\|x_n - x^*\|^q \\
&- \gamma_n[\mu_2(\beta q - \kappa_q\mu_2^{q-1})\|B_2w_n - B_2x^*\|^q \\
&+ \mu_1(\alpha q - \kappa_q\mu_1^{q-1})\|B_1v_n - B_1y^*\|^q \\
&+ r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q \\
&+ \lambda_n\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ax^*\|^q] - \beta_n\gamma_n g(\|x_n - Gz_n\|) \\
&+ q\alpha_n\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle.
\end{aligned}
$$

For each $n \geq 0$, we set

$$
\begin{aligned}
\Gamma_n &= \|x_n - x^*\|^q, \\
\varepsilon_n &= \alpha_n(1-\delta), \\
\eta_n &= \gamma_n[\mu_2(\beta q - \kappa_q\mu_2^{q-1})\|B_2w_n - B_2x^*\|^q + \mu_1(\alpha q - \kappa_q\mu_1^{q-1})\|B_1v_n - B_1y^*\|^q \\
&\quad + r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q + \lambda_n\Big(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\Big)\|Ay_n - Ax^*\|^q] \\
&\quad + \beta_n\gamma_n g(\|x_n - Gz_n\|) \\
\delta_n &= q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle.
\end{aligned}
$$

Then (3.5) can be rewritten as the following formula:

$$\Gamma_{n+1} \leq (1 - \varepsilon_n)\Gamma_n - \eta_n + \delta_n \quad \forall n \geq 0, \tag{3.6}$$

and hence

$$\Gamma_{n+1} \leq (1 - \varepsilon_n)\Gamma_n + \delta_n \quad \forall n \geq 0. \tag{3.7}$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

Case 1. Suppose that there exists an integer $n_0 \geq 1$ such that $\{\Gamma_n\}$ is non-increasing. Then

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

From (3.6), we get

$$0 \leq \eta_n \leq \Gamma_n - \Gamma_{n+1} + \delta_n - \varepsilon_n\Gamma_n.$$

Since $\varepsilon_n \to 0$ and $\delta_n \to 0$, we have $\eta_n \to 0$. This ensures that

$$\lim_{n\to\infty} g(\|x_n - Gz_n\|) = 0,$$

$$\lim_{n\to\infty} \|B_2w_n - B_2x^*\| = \lim_{n\to\infty} \|B_1v_n - B_1y^*\| = 0 \tag{3.8}$$

and

$$\lim_{n\to\infty} \|Au_n - Ax^*\| = \lim_{n\to\infty} \|Ay_n - Ax^*\| = 0. \tag{3.9}$$

Note that g is a strictly increasing, continuous and convex function with $g(0) = 0$. So it follows that

$$\lim_{n\to\infty} \|x_n - Gz_n\| = 0. \tag{3.10}$$

On the other hand, using Lemma 2.2 (b) and the firm nonexpansivity of Π_C, we have

$$\begin{aligned}\|v_n - y^*\|^q &= \|\Pi_C(w_n - \mu_2 B_2 w_n) - \Pi_C(x^* - \mu_2 B_2 x^*)\|^q \\ &\le \langle w_n - \mu_2 B_2 w_n - (x^* - \mu_2 B_2 x^*), J_q(v_n - y^*)\rangle \\ &= \langle w_n - x^*, J_q(v_n - y^*)\rangle + \mu_2\langle B_2 x^* - B_2 w_n, J_q(v_n - y^*)\rangle \\ &\le \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q \\ &\quad - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] \\ &\quad + \mu_2\langle B_2 x^* - B_2 w_n, J_q(v_n - y^*)\rangle,\end{aligned}$$

which hence attains

$$\begin{aligned}\|v_n - y^*\|^q \le \|w_n - x^*\|^q &- \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &+ q\mu_2\|B_2 x^* - B_2 w_n\|\|v_n - y^*\|^{q-1}.\end{aligned}$$

In a similar way, we get

$$\begin{aligned}\|u_n - x^*\|^q &= \|\Pi_C(v_n - \mu_1 B_1 v_n) - \Pi_C(y^* - \mu_1 B_1 y^*)\|^q \\ &\le \langle v_n - \mu_1 B_1 v_n - (y^* - \mu_1 B_1 y^*), J_q(u_n - x^*)\rangle \\ &= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \mu_1\langle B_1 y^* - B_1 v_n, J_q(u_n - x^*)\rangle \\ &\le \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q \\ &\quad - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\ &\quad + \mu_1\langle B_1 y^* - B_1 v_n, J_q(u_n - x^*)\rangle,\end{aligned}$$

which hence attains

$$\begin{aligned}\|u_n - x^*\|^q &\le \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) \\ &\quad + q\mu_1\|B_1 y^* - B_1 v_n\|\|u_n - x^*\|^{q-1} \\ &\le \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ &\quad + q\mu_2\|B_2 x^* - B_2 w_n\|\|v_n - y^*\|^{q-1} \\ &\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + q\mu_1\|B_1 y^* - B_1 v_n\|\|u_n - x^*\|^{q-1}.\end{aligned} \tag{3.11}$$

Since $J^B_{\lambda_n}$ is firmly nonexpansive (due to Lemma 2.6 (ii)), by Lemma 2.2 (b) we get

$$\begin{aligned}\|y_n - x^*\|^q &= \|J^B_{\lambda_n}(u_n - \lambda_n A u_n) - J^B_{\lambda_n}(x^* - \lambda_n A x^*)\|^q \\ &\le \langle (u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*), J_q(y_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^q + (q-1)\|y_n - x^*\|^q \\ &\quad - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)],\end{aligned}$$

which together with (3.3), implies that

$$\|y_n - x^*\|^q \le \|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^q$$

$$
\begin{aligned}
&- h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\\
&\le \|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|).
\end{aligned}
$$

This together with (3.4) and (3.11), implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^q &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|Gz_n - x^*\|^q\\
&\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[(1 - r_n)\|u_n - x^*\|^q\\
&\quad + r_n\|y_n - x^*\|^q]\\
&\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\{(1 - r_n)\|u_n - x^*\|^q\\
&\quad + r_n[\|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)]\}\\
&= \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\{\|u_n - x^*\|^q\\
&\quad - r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\}\\
&\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q\\
&\quad + \gamma_n\{\|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|)\\
&\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1}\\
&\quad + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1}\\
&\quad - r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\}\\
&\le \alpha_n\|f(x_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|)\\
&\quad + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\}\\
&\quad + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1}\\
&\quad + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1},
\end{aligned}
$$

which immediately yields

$$
\begin{aligned}
&\gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|)\\
&\quad + r_n h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\}\\
&\le \alpha_n\|f(x_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1} + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1}\\
&\quad + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1}.
\end{aligned}
$$

Since $\tilde{h}_1, \tilde{h}_2$ and h_1 are strictly increasing, continuous and convex functions with $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$, from (3.8) and (3.9) we conclude that $\|w_n - v_n - x^* + y^*\| \to 0$, $\|v_n - u_n + x^* - y^*\| \to 0$ and $\|u_n - y_n\| \to 0$ as $n \to \infty$. This immediately implies that

$$
\lim_{n\to\infty} \|w_n - u_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.12}
$$

Furthermore, we put $p_n := \zeta S_n w_n + (1 - \zeta)Gw_n$ for all $n \ge 0$. Then we obtain that

$$
\begin{aligned}
\|w_n - x^*\|^q &= \langle s_n x_n + (1 - s_n)(\zeta S_n w_n + (1 - \zeta)Gw_n) - x^*, J_q(w_n - x^*)\rangle\\
&\le s_n\langle x_n - x^*, J_q(w_n - x^*)\rangle\\
&\quad + (1 - s_n)\langle(\zeta S_n w_n + (1 - \zeta)Gw_n) - x^*, J_q(w_n - x^*)\rangle\\
&\le s_n\langle x_n - x^*, J_q(w_n - x^*)\rangle + (1 - s_n)\|w_n - x^*\|^q.
\end{aligned}
$$

Using Lemma 2.2 (b), we get

$$\begin{aligned}\|w_n - x^*\|^q &\le \langle x_n - x^*, J_q(w_n - x^*)\rangle \\ &\le \frac{1}{q}[\|x_n - x^*\|^q + (q-1)\|w_n - x^*\|^q - h_3(\|x_n - w_n\|)].\end{aligned}$$

This together with (3.2) implies that

$$\|u_n - x^*\|^q \le \|w_n - x^*\|^q \le \|x_n - x^*\|^q - h_3(\|x_n - w_n\|). \tag{3.13}$$

In a similar way, we have

$$\begin{aligned}\|z_n - x^*\|^q &= \|J^B_{\lambda_n}(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\ &\le \langle (u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*), J_q(z_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*)\|^q \\ &\quad + (q-1)\|z_n - x^*\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)],\end{aligned}$$

which together with (3.4), implies that

$$\begin{aligned}\|z_n - x^*\|^q &\le \|(u_n - \lambda_n Ay_n + r_n(y_n - u_n)) - (x^* - \lambda_n Ax^*)\|^q \\ &\qquad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|) \\ &\le \|u_n - x^*\|^q \\ &\qquad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|).\end{aligned}$$

This together with (3.13), ensures that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|Gz_n - x^*\|^q \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[\|u_n - x^*\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)] \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[\|x_n - x^*\|^q \\ &\quad - h_3(\|x_n - w_n\|) \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)] \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n[h_3(\|x_n - w_n\|) \\ &\quad + h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)],\end{aligned}$$

which immediately leads to

$$\begin{aligned}&\gamma_n[h_3(\|x_n - w_n\|) + h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)] \\ &\quad \le \alpha_n\|f(x_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1}.\end{aligned}$$

Note that h_2 and h_3 are strictly increasing, continuous and convex functions with $h_2(0) = h_3(0) = 0$. Using (3.9) and (3.12), we obtain

$$\lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.14}$$

So, it follows from (3.10), (3.12) and (3.14) that

$$\|x_n - z_n\| \le \|x_n - w_n\| + \|w_n - u_n\| + \|u_n - z_n\| \to 0 \quad (n \to \infty),$$

and hence

$$\begin{aligned}\|x_n - Gx_n\| &\le \|x_n - Gz_n\| + \|Gz_n - Gx_n\| \\ &\le \|x_n - Gz_n\| + \|z_n - x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.15}$$

Since $w_n = s_n x_n + (1-s_n)p_n$ and $p_n = \zeta S_n w_n + (1-\zeta)u_n$, from (3.12) and (3.14) we get

$$\|p_n - w_n\| = \frac{s_n}{1-s_n}\|x_n - w_n\| \le \frac{d}{1-d}\|x_n - w_n\| \to 0 \quad (n \to \infty),$$

and hence

$$\begin{aligned}\zeta\|S_n w_n - w_n\| &= \|p_n - w_n - (1-\zeta)(u_n - w_n)\| \\ &\le \|p_n - w_n\| + \|u_n - w_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Since $\{S_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we deduce from (3.14) that

$$\begin{aligned}\|S_n x_n - x_n\| &\le \|S_n x_n - S_n w_n\| + \|S_n w_n - w_n\| + \|w_n - x_n\| \\ &\le (\ell+1)\|x_n - w_n\| + \|S_n w_n - w_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.16}$$

We next claim that $\|x_n - \widetilde{S}x_n\| \to 0$ as $n \to \infty$ where $\widetilde{S} := (2I - S)^{-1}$. In fact, it is first clear that $S : C \to C$ is pseudocontractive and ℓ-Lipschitzian where $Sx = \lim_{n\to\infty} S_n x \ \forall x \in C$. We claim that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Using the boundedness of $\{x_n\}$ and setting $D = \overline{\text{conv}}\{x_n : n \ge 0\}$ (the closed convex hull of the set $\{x_n : n \ge 0\}$), by the assumption we have $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$. Hence, by Proposition 2.4 we get $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$, which immediately arrives at

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0.$$

Thus, from (3.16) we have

$$\|x_n - Sx_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad (n \to \infty). \tag{3.17}$$

Now, let us show that if we define $\widetilde{S} := (2I - S)^{-1}$, then $\widetilde{S} : C \to C$ is nonexpansive, $\text{Fix}(\widetilde{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ and $\lim_{n\to\infty} \|x_n - \widetilde{S}x_n\| = 0$. As a matter of fact, put $\widetilde{S} := (2I - S)^{-1}$, where I is the identity operator of E. Then it is known that $\widetilde{S}$ is nonexpansive and $\text{Fix}(\widetilde{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ as a consequence of [29, Theorem 6]. From (3.17) it follows that

$$\begin{aligned}\|x_n - \widetilde{S}x_n\| &= \|\widetilde{S}\widetilde{S}^{-1}x_n - \widetilde{S}x_n\| \le \|\widetilde{S}^{-1}x_n - x_n\| \\ &= \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.18}$$

For each $n \ge 0$, we put $T_{\lambda_n} := J_{\lambda_n}^B(I - \lambda_n A)$. Then from (3.12) and (3.14) we get

$$\begin{aligned}\|x_n - T_{\lambda_n}x_n\| &\le \|x_n - u_n\| + \|u_n - T_{\lambda_n}u_n\| + \|T_{\lambda_n}u_n - T_{\lambda_n}x_n\| \\ &\le 2\|x_n - u_n\| + \|u_n - y_n\| \\ &\le 2(\|x_n - w_n\| + \|w_n - u_n\|) + \|u_n - y_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Noticing $0 < \lambda \le \lambda_n$ for all $n \ge 0$ and using Proposition 2.7 (ii), we obtain

$$\|T_\lambda x_n - x_n\| \le 2\|T_{\lambda_n}x_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.19}$$

We define the mapping $\mathit{\Psi} : C \to C$ by $\mathit{\Psi} x := \theta_1 \widetilde{S}x + \theta_2 Gx + (1-\theta_1-\theta_2)T_\lambda x \ \forall x \in C$ with $\theta_1 + \theta_2 < 1$ for constants $\theta_1, \theta_2 \in (0,1)$. Then by Lemma 2.14 and Proposition 2.7 (i), we know that $\mathit{\Psi}$ is nonexpansive and

$$\begin{aligned}\mathrm{Fix}(\mathit{\Psi}) &= \mathrm{Fix}(\widetilde{S}) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) \\ &= \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A+B)^{-1}0 \ (=: \mathit{\Omega}).\end{aligned}$$

Taking into account that

$$\|\mathit{\Psi} x_n - x_n\| \le \theta_1 \|\widetilde{S}x_n - x_n\| + \theta_2 \|Gx_n - x_n\| + (1-\theta_1-\theta_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.15), (3.18) and (3.19) that

$$\lim_{n\to\infty} \|\mathit{\Psi} x_n - x_n\| = 0. \tag{3.20}$$

Let $z_s = sf(z_s) + (1-s)\mathit{\Psi} z_s \ \forall s \in (0,1)$. Then it follows from Proposition 2.8 that $\{z_s\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\mathit{\Psi}) = \mathit{\Omega}$, which solves the HVI:

$$\langle (I-f)x^*, J(x^*-p)\rangle \le 0 \quad \forall p \in \mathit{\Omega}.$$

Also, from (2.3) we get

$$\begin{aligned}\|z_s - x_n\|^q &= \|s(f(z_s) - x_n) + (1-s)(\mathit{\Psi} z_s - x_n)\|^q \\ &\le (1-s)^q \|\mathit{\Psi} z_s - x_n\|^q + qs\langle f(z_s) - x_n, J_q(z_s - x_n)\rangle \\ &= (1-s)^q \|\mathit{\Psi} z_s - x_n\|^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle \\ &\quad + qs\langle z_s - x_n, J_q(z_s - x_n)\rangle \\ &\le (1-s)^q (\|\mathit{\Psi} z_s - \mathit{\Psi} x_n\| + \|\mathit{\Psi} x_n - x_n\|)^q \\ &\quad + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\|z_s - x_n\|^q \\ &\le (1-s)^q (\|z_s - x_n\| + \|\mathit{\Psi} x_n - x_n\|)^q \\ &\quad + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\|z_s - x_n\|^q,\end{aligned}$$

which immediately attains

$$\begin{aligned}&\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle \\ &\quad \le \frac{(1-s)^q}{qs}(\|z_s - x_n\| + \|\mathit{\Psi} x_n - x_n\|)^q + \frac{qs-1}{qs}\|z_s - x_n\|^q.\end{aligned}$$

From (3.20), we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(z_s) - z_s, J_q(x_n - z_s)\rangle &\le \frac{(1-s)^q}{qs} M + \frac{qs-1}{qs} M \\ &= (\frac{(1-s)^q + qs - 1}{qs})M,\end{aligned} \tag{3.21}$$

where M is a constant such that $\|z_s - x_n\|^q \le M$ for all $n \ge 0$ and $s \in (0,1)$. It is easy to see that $((1-s)^q + qs - 1)/qs \to 0$ as $s \to 0$. Since J_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_s \to x^*$, we get

$$\|J_q(x_n - z_s) - J_q(x_n - x^*)\| \to 0 \quad (s \to 0).$$

So we obtain

$$\begin{aligned}
&|\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\
&\quad = |\langle f(z_s) - f(x^*), J_q(x_n - z_s)\rangle + \langle f(x^*) - x^*, J_q(x_n - z_s)\rangle \\
&\qquad + \langle x^* - z_s, J_q(x_n - z_s)\rangle \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\
&\quad \le |\langle f(x^*) - x^*, J_q(x_n - z_s) - J_q(x_n - x^*)\rangle| \\
&\qquad + |\langle f(z_s) - f(x^*), J_q(x_n - z_s)\rangle| + |\langle x^* - z_s, J_q(x_n - z_s)\rangle| \\
&\quad \le \|f(x^*) - x^*\| \|J_q(x_n - z_s) - J_q(x_n - x^*)\| \\
&\qquad + (1+\delta)\|z_s - x^*\| \|x_n - z_s\|^{q-1}.
\end{aligned}$$

Hence, for each $n \ge 0$, we get

$$\lim_{s\to 0} \langle f(z_s) - z_s, J_q(x_n - z_s)\rangle = \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle.$$

From (3.21), as $s \to 0$, it follows that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle \le 0. \tag{3.22}$$

By (C1) and (3.10), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n G z_n - x_n\| \\
&= \|\alpha_n (f(x_n) - x_n) + \gamma_n (G z_n - x_n)\| \\
&\le \alpha_n \|f(x_n) - x_n\| + \gamma_n \|G z_n - x_n\| \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.23}$$

Using (3.22) and (3.23), we have

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \le 0. \tag{3.24}$$

Using Lemma 2.15 and (3.24), we can infer that $\Gamma_n \to 0$ as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that $\exists \{\Gamma_{i_k}\} \subset \{\Gamma_i\}$ s.t. $\Gamma_{i_k} < \Gamma_{i_k+1}$ $\forall k \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau : \mathbf{N} \to \mathbf{N}$ by

$$\tau(i) := \max\{j \le i : \Gamma_j < \Gamma_{j+1}\}.$$

Using Lemma 2.13, we get

$$\Gamma_{\tau(i)} \le \Gamma_{\tau(i)+1} \quad \text{and} \quad \Gamma_i \le \Gamma_{\tau(i)+1}.$$

Putting $\Gamma_i = \|x_i - x^*\|^q$ $\forall i \in \mathbf{N}$ and using the same inference as in Case 1, we can obtain

$$\lim_{i\to\infty} \|x_{\tau(i)+1} - x_{\tau(i)}\| = 0 \tag{3.25}$$

and

$$\limsup_{i\to\infty} \langle f(x^*) - x^*, J_q(x_{\tau(i)+1} - x^*)\rangle \le 0. \tag{3.26}$$

Owing to $\Gamma_{\tau(i)} \le \Gamma_{\tau(i)+1}$ and $\alpha_{\tau(i)} > 0$, we conclude from (3.7) that

$$\|x_{\tau(i)} - x^*\|^q \le \frac{q}{1-\delta} \langle f(x^*) - x^*, J_q(x_{\tau(i)+1} - x^*)\rangle$$

and hence

$$\limsup_{i\to\infty} \|x_{\tau(i)} - x^*\|^q \le 0.$$

Thus, we get

$$\lim_{i\to\infty} \|x_{\tau(i)} - x^*\|^q = 0.$$

Using Proposition 2.5 and (3.25), we obtain

$$\begin{aligned}
&\|x_{\tau(i)+1} - x^*\|^q - \|x_{\tau(i)} - x^*\|^q \\
&\quad \leqq \langle x_{\tau(i)+1} - x_{\tau(i)}, J_q(x_{\tau(i)} - x^*)\rangle + \kappa_q \|x_{\tau(i)+1} - x_{\tau(i)}\|^q \\
&\quad \le q\|x_{\tau(i)+1} - x_{\tau(i)}\|\|x_{\tau(i)} - x^*\|^{q-1} + \kappa_q \|x_{\tau(i)+1} - x_{\tau(i)}\|^q \to 0 \ (i \to \infty).
\end{aligned}$$

Noticing $\Gamma_i \le \Gamma_{\tau(i)+1}$, we get

$$\begin{aligned}
\|x_i - x^*\|^q &\le \|x_{\tau(i)+1} - x^*\|^q \\
&\le \|x_{\tau(i)} - x^*\|^q + q\|x_{\tau(i)+1} - x_{\tau(i)}\|\|x_{\tau(i)} - x^*\|^{q-1} \\
&\quad + \kappa_q \|x_{\tau(i)+1} - x_{\tau(i)}\|^q.
\end{aligned}$$

It is easy to see from (3.25) that $x_i \to x^*$ as $i \to \infty$. This completes the proof. □

We also obtain the strong convergence result for the combined extragradient implicit rule in a real Hilbert space H. It is well known that $\kappa_2 = 1$ [44]. Thus, by Theorem 3.3 we derive the following conclusion.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a δ-contraction with constant $\delta \in [0, 1)$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Let $A : C \to H$ and $B : C \to 2^H$ be a σ-inverse-strongly monotone mapping and a maximal monotone operator, respectively. Suppose that $B_1, B_2 : C \to H$ are α-inverse-strongly monotone mapping and β-inverse-strongly monotone mapping, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0 \ne \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0, 1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases}
w_n = s_n x_n + (1 - s_n)(\zeta S_n w_n + (1 - \zeta)G w_n), \\
u_n = G w_n, \\
y_n = J_{\lambda_n}^B (u_n - \lambda_n A u_n), \\
z_n = J_{\lambda_n}^B (u_n - \lambda_n A y_n + r_n(y_n - u_n)), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n G z_n \quad \forall n \ge 0,
\end{cases} \tag{3.27}$$

where $J_{\lambda_n}^B = (I + \lambda_n B)^{-1}$, $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$. Suppose that the following conditions hold:

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (C2) $0 < a \le \beta_n \le b < 1$ *and* $0 < c \le s_n \le d < 1$;
- (C3) $0 < r \le r_n < 1$ *and* $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < 2\sigma$;
- (C4) $0 < \mu_1 < 2\alpha$ *and* $0 < \mu_2 < 2\beta$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset* D *of* C*. Let* $S : C \to C$ *be a mapping defined by* $Sx = \lim_{n\to\infty} S_n x \ \forall x \in C$*, and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$*. Then* $x_n \to x^* \in \Omega$*, which is the unique solution to the HVI:* $\langle (I - f)x^*, p - x^*\rangle \geq 0 \ \forall p \in \Omega$*, i.e., the fixed point equation* $x^* = P_\Omega f(x^*)$*.*

Remark 3.5. Compared with the corresponding results in Manaka and Takahashi [28], Sunthrayuth and Cholamjiak [36], and Cai et al. [6], our results improve and extend them in the following aspects.

(i) The problem of solving the VI for two monotone operators A, B with the FPP constraint of a nonexpansive mapping S in [28, Theorem 3.1] is extended to develop our problem of solving the GSVI (1.6) with the constraints of the VI for two accretive operators A, B and the CFPP of countably many pseudocontractive mappings $\{S_n\}_{n=0}^{\infty}$. The Mann-type iterative scheme with weak convergence in [28, Theorem 3.1] is extended to develop our combined extragradient implicit rule with strong convergence.

(ii) The problem of solving the GSVI (1.7) with the FPP constraint of an asymptotically nonexpansive mapping T in [6, Theorem 3.1], is extended to develop our problem of solving the GSVI (1.6) with the constraints of the VI for two accretive operators A, B and the CFPP of countably many pseudocontractive mappings $\{S_n\}_{n=0}^{\infty}$. The modified viscosity implicit rule in [6, Theorem 3.1] is extended to develop our combined extragradient implicit rule.

(iii) The problem of solving the VI for two accretive operators A, B with the FPP constraint of a nonexpansive mapping S in [36, Theorem 3.3] is extended to develop our problem of solving the GSVI (1.6) with the constraints of the VI for two accretive operators A, B and the CFPP of countably many pseudocontractive mappings $\{S_n\}_{n=0}^{\infty}$. The modified viscosity-type extragradient method in [36, Theorem 3.3] is extended to develop our combined extragradient implicit rule.

4. Some applications

In this section, we give some applications of Corollary 3.4 to important mathematical problems in the setting of Hilbert spaces.

4.1. **Application to variational inequality problem.** Given a nonempty closed convex subset $C \subset H$ and a nonlinear monotone operator $A : C \to H$. Consider the classical VIP of finding $u^* \in C$ s.t.

$$\langle Au^*, v - u^*\rangle \geq 0 \quad \forall v \in C. \tag{4.1}$$

The solution set of problem (4.1) is denoted by $\mathrm{VI}(C, A)$. It is clear that $u^* \in C$ solves VIP (4.1) if and only if it solves the fixed point equation $u^* = P_C(u^* - \lambda Au^*)$ with $\lambda > 0$. Let i_C be the indicator function of C defined by

$$i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{if } u \notin C. \end{cases}$$

We use $N_C(u)$ to indicate the normal cone of C at $u \in H$, i.e., $N_C(u) = \{w \in H : \langle w, v - u\rangle \leq 0 \ \forall v \in C\}$. It is known that i_C is a proper, convex and lower semicontinuous function and its subdifferential ∂i_C is a maximal monotone mapping [10]. We define the resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x) \quad \forall x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{w \in H : i_C(u) + \langle w, v-u \rangle \le i_C(v) \ \forall v \in H\} \\ &= \{w \in H : \langle w, v-u \rangle \le 0 \ \forall v \in C\} = N_C(u) \quad \forall u \in C. \end{aligned}$$

Hence, we get

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) &\Leftrightarrow \quad x - u \in \lambda N_C(u) \\ &\Leftrightarrow \quad \langle x-u, v-u \rangle \le 0 \quad \forall v \in C \\ &\Leftrightarrow \quad u = P_C(x), \end{aligned}$$

where P_C is the metric projection of H onto C. Moreover, we also have $(A + \partial i_C)^{-1}0 = \mathrm{VI}(C, A)$ [39].

Thus, putting $B = \partial i_C$ in Corollary 3.4, we obtain the following result:

Theorem 4.1. *Let f, A, B_1, B_2 and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.4. Suppose that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{VI}(C, A) \neq \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0, 1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$(4.2) \qquad \begin{cases} w_n = s_n x_n + (1 - s_n)(\zeta S_n w_n + (1-\zeta) G w_n), \\ u_n = G w_n, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = P_C(u_n - \lambda_n A y_n + r_n(y_n - u_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n G z_n \quad \forall n \ge 0, \end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C4) in Corollary 3.4 hold. Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (I-f)x^*, p-x^* \rangle \ge 0 \ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

4.2. **Application to split feasibility problem.** Let H_1 and H_2 be two real Hilbert spaces. Consider the following split feasibility problem (SFP) of finding

$$(4.3) \qquad u \in C \text{ s.t } Tu \in Q,$$

where C and Q are closed convex subsets of H_1 and H_2, respectively, and $T : H_1 \to H_2$ is a bounded linear operator with its adjoint T^*. The solution set of SFP is denoted by $\mho := C \cap T^{-1}Q = \{u \in C : Tu \in Q\}$. In 1994, Censor and Elfving [7] first introduced the SFP for modeling inverse problems of radiation therapy treatment planning in a finite dimensional Hilbert space, which arise from phase retrieval and in medical image reconstruction.

It is known that $z \in C$ solves the SFP (4.3) if and only if z is a solution of the minimization problem: $\min_{y \in C} g(y) := \frac{1}{2}\|Ty - P_Q Ty\|^2$. Note that the function g is differentiable convex and has the Lipschitzian gradient defined by $\nabla g = T^*(I - P_Q)T$. Moreover, ∇g is $\frac{1}{\|T\|^2}$-inverse-strongly monotone, where $\|T\|^2$ is the spectral radius of T^*T [5]. So, $z \in C$ solves the SFP if and only if it solves the variational inclusion problem of finding $z \in H_1$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial i_C(z) \quad &\Leftrightarrow \quad 0 \in z + \lambda \partial i_C(z) - (z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z - \lambda \nabla g(z) \in z + \lambda \partial i_C(z) \\ &\Leftrightarrow \quad z = (I + \lambda \partial i_C)^{-1}(z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z = P_C(z - \lambda \nabla g(z)). \end{aligned}$$

Now, setting $A = \nabla g,\ B = \partial i_C$ and $\sigma = \frac{1}{\|T\|^2}$ in Corollary 3.4, we obtain the following result:

Theorem 4.2. *Let f, B_1, B_2 and $\{S_n\}_{n=0}^{\infty}$ be the same as in Corollary 3.4. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mho \neq \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0,1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$(4.4) \qquad \begin{cases} w_n = s_n x_n + (1 - s_n)(\zeta S_n w_n + (1 - \zeta) G w_n), \\ u_n = G w_n, \\ y_n = P_C(u_n - \lambda_n T^*(I - P_Q) T u_n), \\ z_n = P_C(u_n - \lambda_n T^*(I - P_Q) T y_n + r_n(y_n - u_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n G z_n \quad \forall n \geq 0, \end{cases}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C4) in Corollary 3.4 hold where $\sigma = \frac{1}{\|T\|^2}$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (I - f)x^*, p - x^* \rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

4.3. **Application to LASSO problem.** In this subsection, we first recall the least absolute shrinkage and selection operator (LASSO) [41], which can be formulated as a convex constrained optimization problem:

$$(4.5) \qquad \min_{y \in H} \frac{1}{2} \|Ty - b\|_2^2 \quad \text{subject to } \|y\|_1 \leq s,$$

where $T : H \to H$ is a bounded operator on H, b is a fixed vector in H and $s > 0$. Let $\mho$ be the solution set of LASSO (4.5). The LASSO has received much attention because of the involvement of the ℓ_1 norm which promotes sparsity, phenomenon of many practical problems arising in statics model, image compression, compressed sensing and signal processing theory.

In terms of the optimization theory, ones know that the solution to the LASSO problem (4.5) is a minimizer of the following convex unconstrained minimization problem so-called Basis Pursuit denoising problem:

$$\min_{y \in H} g(y) + h(y), \tag{4.6}$$

where $g(y) := \frac{1}{2}\|Ty - b\|_2^2,\ h(y) := \lambda \|y\|_1$ and $\lambda \geq 0$ is a regularization parameter. It is known that $\nabla g(y) = T^*(Ty - b)$ is $\frac{1}{\|T^*T\|}$-inverse-strongly monotone. Hence, we have that z solves the LASSO if and only if z solves the variational inclusion problem of finding $z \in H$ s.t.

$$0 \in \nabla g(z) + \partial h(z) \quad \Leftrightarrow \quad 0 \in z + \lambda \partial h(z) - (z - \lambda \nabla g(z))$$

$$\begin{aligned}
&\Leftrightarrow \quad z - \lambda\nabla g(z) \in z + \lambda\partial h(z)\\
&\Leftrightarrow \quad z = (I + \lambda\partial h)^{-1}(z - \lambda\nabla g(z))\\
&\Leftrightarrow \quad z = \mathrm{prox}_h(z - \lambda\nabla g(z)),
\end{aligned}$$

where $\mathrm{prox}_h(y)$ is the proximal of $h(y) := \lambda\|y\|_1$ given by

$$\mathrm{prox}_h(y) = \mathrm{argmin}_{u\in H}\{\lambda\|u\|_1 + \frac{1}{2}\|u-y\|_2^2\} \quad \forall y \in H,$$

which is separable in indices. Then, for $y \in H$,

$$\begin{aligned}
\mathrm{prox}_h(y) &= \mathrm{prox}_{\lambda\|\cdot\|_1}(y)\\
&= (\mathrm{prox}_{\lambda|\cdot|}(y_1), \mathrm{prox}_{\lambda|\cdot|}(y_2), ..., \mathrm{prox}_{\lambda|\cdot|}(y_n)),
\end{aligned}$$

where $\mathrm{prox}_{\lambda|\cdot|}(y_i) = \mathrm{sgn}(y_i)\max\{|y_i| - \lambda, 0\}$ for $i = 1, 2, ..., n$.

In 2014, Xu [46] suggested the following proximal-gradient algorithm (PGA):

$$x_{k+1} = \mathrm{prox}_h(x_k - \lambda_k\Gamma^*(\Gamma x_k - b)).$$

He proved the weak convergence of the PGA to a solution of the LASSO problem (4.5).

Next, putting $C = H$, $A = \nabla g$, $B = \partial h$ and $\sigma = \frac{1}{\|T^*T\|}$ in Corollary 3.4, we obtain the following result:

Theorem 4.3. *Let f, B_1, B_2 and $\{S_n\}_{n=0}^{\infty}$ be the same as in Corollary 3.4 with $C = H$. Assume that $\Omega := \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(H, B_1, B_2) \cap \mho \neq \emptyset$. For any given $x_0 \in H$ and $\zeta \in (0, 1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases}
w_n = s_nx_n + (1 - s_n)(\zeta S_nw_n + (1 - \zeta)Gw_n),\\
u_n = Gw_n,\\
y_n = \mathrm{prox}_h(u_n - \lambda_nT^*(Tu_n - b)),\\
z_n = \mathrm{prox}_h(u_n - \lambda_nT^*(Ty_n - b) + r_n(y_n - u_n)),\\
x_{n+1} = \alpha_nf(x_n) + \beta_nx_n + \gamma_nGz_n \quad \forall n \geq 0,
\end{cases} \tag{4.6}$$

where the sequences $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C4) in Corollary 3.4 hold where $\sigma = \frac{1}{\|T^*T\|}$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (I - f)x^*, p - x^*\rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

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