Applied Analysis and Optimization
Volume 8, Number 2, 2024, 221–232

Yokohama Publishers
ISSN 2189-1664 Online Journal
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EXTENDED MONOTONIC RESULTS RELATED TO FRACTIONAL DERIVATIVES

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Abstract. In this paper, we discuss the monotonous results with fractional derivatives in interval $(1, 2)$. We have adopted some ideas of α order Riemann-Liouville fractional derivative and Caputo fractional derivative in our derivations. Some theoretical results are developed in this context. In addition to this, some numerical examples are included to justify these results. Further, we extend these results for generalized intervals.

1. Introduction

The concept of n^{th} order derivative (integration), where n is an integer was extended to fractional derivative (integration) by different researchers in different approaches. Recently fractional calculus has become a major area of research interest due to its applications in different fields like continuum mechanics, elasticity, signal analysis, quantum mechanics, bio-engineering, bio-medicine, financial systems, social systems, and machine learning, etc. (see [6, 10, 12–14, 23, 28]). Riemann-Liouville fractional derivative $({}_aD_x^{-\alpha})$ (see [27]) and Caputo fractional derivative $({}_a^cD_x^{\alpha})$ (see [27]) are widely used two different types of fractional derivatives. In most cases, the parameter α varies within the interval $(0, 1]$. Extension of mean value theorem, as well as Taylor-Reimann series in fractional derivatives, are discussed in [1]. Riemann-Liouville operators, Taylor-Riemann series when $\alpha = \frac{1}{2}$, and some relations to special functions are obtained by generating functions are discussed in [21]. Monotonicity and optimality using fractional derivatives are developed in [5, 8]. The necessary optimality condition for a discrete-time fractional calculus of variations is established by using new Euler-Lagrange and Legendre-type conditions in [4, 5]. In [8], monotonicity and non-positivity results are developed for discrete fractional difference operators in the specific interval $(1, 2)$. Monotonicity results depending on sign change in any interval for α order Caputo fractional derivative discussed in [9] for different values of α. The positivity of the fractional derivative of a function for a particular interval implies the convexity of the same. A particular class of fractional boundary value problems has no nontrivial positive solutions (see [15]). Cauchy problem for relaxation equation where $N(E(t))$ as renewal process, $N(t)$ is the Poisson process of intensity λ, and $E(t)$ is an inverse subordinator using fractional calculus is described in [16]. Abel's method for solving

2020 *Mathematics Subject Classification.* 26A33; 26A48, 26A42.

Key words and phrases. Fractional derivatives, fractional integral, monotonic functions, gamma value, and Riemann integral.

*The first author Barsha Shaw is grateful to the support provided by junior research fellowship (JRF) of the university grant commission (UGC), India.

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integral equations for a broader class of the integral equations of the convolution type of Taylor series is extended by Sonin in [18]. A different type fractional derivative is proposed using the generalized cardinal sine function as a non-singular analytic kernel in [17, 21]. Here a corresponding fractional integral operator is introduced in Legendre orthonormal polynomial basis with the science of Caputo and Riemann-Liouville.

Furthermore, gradient descent methods for nonlinear optimization using Caputo fractional derivatives are developed in [24]. The stability of fractional differential calculus as well as are discussed in [7]. Readers may see [2,3,11,20,22,25,26,29] for further development in fractional calculus.

In this paper, we have adopted some ideas from [27] and developed the monotonicity results for fractional derivatives in generalized intervals. Some preliminaries required for the current contribution are discussed in Section 2. Section 3 focuses on constructing the monotonicity results in interval $(1, 2)$. In Section 4 these results are generalized for any interval $(\beta, \beta + 1)$ where $\beta \in \mathbb{N}$. Some numerical examples are discussed in Section 3 and 4 to justify the theoretical results developed in these sections. Finally, some concluding remarks and the scope of future research is addressed in section 5.

2. Preliminaries

This section starts with some basic definitions, properties, and descriptions of monotonic function which is helpful for understanding the entire article. Throughout the discussion, the collection of n times differentiable and $n+1$ times continuous functions on $[a, b]$ is denoted by $C^n[a, b]$.

Definition 2.1. [27] Let $f \in C[a, b]$, the α order left Riemann-Liouville fractional derivative for $x \in [a, b]$ $\alpha > 0$ are defined as,

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\gamma)^{\alpha-1} f(\gamma) d\gamma. $$

Definition 2.2. Let $f \in C[a, b]$. the α order left Riemann-Liouville fractional derivative for $x \in [a, b]$ is formed as,

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} (\frac{d}{dx})^\beta \int_a^x (x-\gamma)^{\beta-\alpha-1} f(\gamma) d\gamma, $$

where $\alpha \in (\beta, \beta + 1), \beta \in \mathbb{N}$.

Definition 2.3. Let $f \in C[a, b]$, the α order Caputo fractional derivative for $x \in [a, b]$ is formed as,

$$ {}_a^cD_x^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-\gamma)^{\beta-\alpha-1} f^{(\beta)}(\gamma) d\gamma, $$

where $\alpha \in (\beta, \beta + 1), \beta \in \mathbb{N}$.

The following results hold for the Riemann-Liouville and Caputo fractional derivative:

(1) The relationship between left Riemann-Liouville and Caputo fractional derivative is

$$ {}_aD_x^\alpha f(x) = {}_a^cD_x^\alpha f(x) + \sum_{i=0}^{\beta} \frac{(x-a)^{i-\alpha}}{\Gamma(i-\alpha+1)} f^{(k)}(a). \tag{2.1}$$

(2) The following rule helds for the left Riemann-Liouville and Caputo fractional derivative

$$ {}_aI_x^\alpha \, {}_aD_x^\alpha f(x) = f(x) - \sum_{i=0}^{\beta} \frac{(x-a)^{\alpha-i}}{\Gamma(\alpha-i+1)} {}_aD_x^{\alpha-i} f(x), \tag{2.2}$$

$$ {}_aI_x^\alpha \, {}_a^cD_x^\alpha f(x) = f(x) - \sum_{i=0}^{\beta} \frac{(x-a)^{i}}{\Gamma(i+1)} f^{(i)}(a), \tag{2.3}$$

where $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$.

A real-valued function f is said to be monotonic increasing or decreasing in interval $[a, b]$ if $f^{'}(x) > 0$ or $f^{'}(x) < 0$. If $f^{(n+1)}(x) \geq 0$ then concluded that $f^{(n)}$ is monotonic increasing interval $[a, b]$.

3. Construction of relations between fractional derivatives with monotonic nature of a function in a specific interval

In this section we develope the following theorems, corollary with the help of previous basic definitions, properties in a specific interval $(1, 2)$. The results provide a new idea to develop theorems in generalizing interval which is discussed briefly in the article.

Theorem 3.1. *Let $f \in C^2[a, b]$ and $\alpha \in (1, 2)$ then f' is monotonic increasing on $[a, x]$ iff Caputo fractional derivative that is ${}_a^cD_x^\alpha f(x) \geq 0$.*

Proof. Let f' is monotonic increasing on $[a, x]$, then $f^{''}(x) \geq 0$ on $[a, x]$. Let $t \in [a, x]$, and $\alpha \in (1, 2)$, therefore $(x-t)^{1-\alpha} > 0$ and $\Gamma(2-\alpha) > 0$.

Now by the definition of the Caputo fractional derivative of order α is,

$$ {}_a^cD_x^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)} \int_a^x (x-\gamma)^{1-\alpha} f^{''}(\gamma) d\gamma \geq 0.$$

For the converse part, let $h(x) = {}_a^cD_x^\alpha f(x) \geq 0$. Then,

$$ {}_aI_x^\alpha h(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\gamma)^{\alpha-1} h(\gamma) d\gamma \geq 0.$$

Therefore, for given $x \geq a$

$$\begin{aligned} 0 &\leq {}_aI_x^\alpha \, {}_a^cD_x^\alpha f(x) \\ &= f(x) - \sum_{j=0}^{1} \frac{(x-a)^j}{\Gamma(j+1)} f^{(j)}(a), \text{(from (2.3))} \end{aligned}$$

$$= f(x) - f(a) - \frac{(x-a)}{\Gamma(1)} f'(a)$$
$$= f'(x) - f'(a).$$

Hence, $f^{'}$ is increasing function on $[a, x]$.

□

Curiosity about knowing more about the condition for the monotonic decreasing character of a function in terms of Caputo fractional alpha order derivative is as follows.

Theorem 3.2. *Let $f \in C^2[a, b]$ and $\alpha \in (1, 2)$ then f' is monotonic decreasing on $[a, x]$ if and only if Caputo fractional derivative that is ${}_a^cD_x^\alpha f(x) \le 0$.*

Proof. The result can be proved by replacing f by $-f$ in Theorem 3.1. □

Corollary 3.3. *Let $f \in C^2[a, b]$ and $\alpha \in (1, 2)$, f' is monotonic increasing on $[a, x]$, $f(a) < 0$ and $f^{'}(a) \ge 0$ then α order left Reimann-Liouville that is ${}_aD_x^\alpha f(x) \ge 0$.*

Proof. From the (2.1) for $\alpha \in (1, 2)$ is,

$${}_aD_x^\alpha f(x) = {}_a^cD_x^\alpha f(x) + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) + \frac{(x-a)^{1-\alpha}}{\Gamma(-\alpha)} f^{'}(a).$$

Since $f^{'}(x)$ is increasing on $[a, x]$ which implies ${}_a^cD_x^\alpha f(x) \ge 0$. Next, $\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) \ge 0$ and $\frac{(x-a)^{1-\alpha}}{\Gamma(-\alpha)} f^{'}(a) \ge 0$ by the given hypotheses $f(a) < 0$, $\frac{1}{\Gamma(1-\alpha)} < 0$, $f^{'}(a) \ge 0$. Hence, we get ${}_aD_x^\alpha f(x) \ge 0$. □

In order to find, the basic behavior of function with some conditions and positivity of alpha order left Riemann-Liouville where alpha is in a specific interval.

Corollary 3.4. *Let $f \in C^2[a, b]$ and $\alpha \in (1, 2)$, if α order left Reimann-Liouville ${}_aD_x^\alpha f(x) \ge 0$, $f(a) > 0$ and $f^{'}(a) < 0$ then f' is monotonic increasing on $[a, x]$.*

Proof. Calculating from equation (2.3),

$${}_a^cD_x^\alpha f(x) = {}_aD_x^\alpha f(x) - \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) - \frac{(x-a)^{1-\alpha}}{\Gamma(-\alpha)} f^{'}(a).$$

Theorem [3.1] states that $f'(x)$ is monotonic increasing on $[a, x]$ if and only if Caputo fractional derivative that is ${}_a^cD_x^\alpha f(x) \ge 0$ and given ${}_aD_x^\alpha f(x) \ge 0$, $f(a) > 0$ which completes the proof by simple calculation. □

The following interesting theorem can be proved by combining the above two corollaries.

Theorem 3.5. *Let $f \in C^2[a, b]$ and $\alpha \in (1, 2)$, if f' is monotonic increasing on $[a, x]$, $f(a) = 0$ if and only if α order left Riemann-Liouville fractional derivative that is ${}_aD_x^\alpha f(x) \ge 0$.*

Furthermore, the $\alpha+1$ order Caputo fractional derivative is nothing but one more extra derivative of the α order Caputo fractional derivative. Functions are dealt different conditions with Caputo fractional derivative which is discussed here very briefly.

Theorem 3.6. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$, if f' is monotonic increasing on $[a,x]$ then Caputo fractional derivative of order $\alpha+1$, ${}^c_aD^{\alpha+1}_x f(x) \leq 0$.*

Proof. Caputo fractional derivative for $\alpha \in (1,2)$ is

$$ {}^c_aD^{\alpha}_x f(x) = \frac{1}{\Gamma(2-\alpha)} \int_a^x (x-\gamma)^{1-\alpha} f^{''}(\gamma) d\gamma $$

it follows that,

$$ {}^c_aD^{\alpha+1}_x f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\gamma)^{-\alpha} f^{''}(\gamma) d\gamma. $$

For $\frac{1}{\Gamma(1-\alpha)} < 0$, $(x-t)^{-\alpha} > 0$, $t \in [a,x]$ and $f^{''}(x) > 0$, then
${}^c_aD^{\alpha+1}_x f(x) \leq 0$. □

Theorem 3.7. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$, if Caputo fractional derivative ${}^c_aD^{\alpha+1}_x f(x) \geq 0$ then f' is monotonic increasing on $[a,x]$.*

Proof. From previous calculation,

$$ {}^c_aD^{\alpha+1}_x f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\gamma)^{-\alpha} f^{''}(\gamma) d\gamma \geq 0. $$

Let ${}^c_aD^{\alpha+1}_x f(x) = {}^c_aD^{\alpha}_x h(x) \geq 0$ where $h(x) = f'(x)$. We know, for given $x \geq a$,

$$ \begin{aligned} 0 &\leq {}_aI^{\alpha}_x \, {}^c_aD^{\alpha}_x h(x) \\ &= g(x) - \sum_{i=0}^{1} \frac{(x-a)^i}{\Gamma(j+1)} g^i(a), \text{(from (2.3))} \\ &= g(x) - \frac{g(a)}{\Gamma(1)} - \frac{(x-a)}{\Gamma(1)} g^{'}(a) \\ &= g(x) - g(a) \\ &= f^{'}(x) - f^{'}(a). \end{aligned} $$

Thus $f^{'}$ is monotonic increasing functions. □

Theorem 3.8. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$, if $f'(x)$ is monotonic decreasing on $[a,x]$ then Caputo fractional derivative of order $\alpha+1$, ${}^c_aD^{\alpha+1}_x f(x) \geq 0$.*

Proof. From the definition of Caputo fractional derivative,

$$ {}^c_aD^{\alpha+1}_x f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\gamma)^{-\alpha} f^{''}(\gamma) d\gamma. $$

Note that, $\frac{1}{\Gamma(1-\alpha)} < 0$, $f'(x)$ is monotonic decreasing implies $f^{''}(x) \leq 0$ that is ${}^c_aD^{\alpha+1}_x f(x) \geq 0$. This proves the theorem. □

Theorem 3.9. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$, if Caputo fractional derivative that is ${}^c_aD^{\alpha+1}_x f(x) \leq 0$ then f' is monotonic decreasing on $[a,x]$.*

Proof. Let, ${}_a^cD_x^{\alpha+1}f(x) = {}_a^cD_x^{\alpha}\ h(x) \leq 0$ where $h(x) = f'(x)$. For any $x \leq a$,

$$\begin{aligned} 0 &\geq {}_aI_x^{\alpha}\ {}_a^cD_x^{\alpha}h(x) \\ &= g(x) - \sum_{i=0}^{1} \frac{(x-a)^i}{\Gamma(i+1)} g^i(a), (\text{from } (2.3)) \\ &= g(x) - g(a) - (x-a)g'(a) \\ &= f'(x) - f'(a) \end{aligned}$$

Thus, $f^{'}$ is monotonic decreasing. □

Next theorems state monotonic results for α order Caputo fractional derivative with some basic assumptions. More required corollary data also helps to provide monotonic increasing or decreasing behavior of Caputo fractional derivatives.

Theorem 3.10. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$, if $f'(x)$ is monotonic decreasing on $[a,x]$, $f^{''}(a) \geq 0$ then α order Caputo fractional derivative, ${}_a^cD_x^{\alpha}f(x)$ is monotonic increasing on $[a,x]$.*

Proof. It is sufficient to show that the derivative of Caputo fractional is greater than equal to zero,

$$\begin{aligned} \frac{d}{dx}\ [{}_a^cD_x^{\alpha}f(x)] &= \frac{d}{dx}\frac{1}{\Gamma(2-\alpha)} \int_a^x (x-\gamma)^{1-\alpha} f^{''}(\gamma) d\gamma \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\gamma)^{1-\alpha} f^{''}(\gamma) d\gamma \\ &\quad + \frac{1}{\Gamma(2-\alpha)} (x-a) f^{''}(a). \end{aligned}$$

Since $f^{''} \leq 0$ and $\frac{1}{\Gamma(1-\alpha)} \leq 0$, then $\frac{d}{dx}[{}_a^cD_x^{\alpha}f(x)] \geq 0$ for $x \geq a$. Therefore, α order Caputo's fractional derivative of f is monotonic increasing on $[a,x]$.

□

Theorem 3.11. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$, if f' is monotonic increasing on $[a,x]$, $f^{''}(a) \leq 0$ then α order Caputo fractional derivative that is ${}_a^cD_x^{\alpha}f(x)$ is monotonic decreasing on $[a,x]$.*

Proof. By a similar process, we can easily show that $\frac{d}{dx}[{}_a^cD_x^{\alpha}f(x)] \leq 0$
then it follows that Caputo fractional derivatives are monotonic decreasing on $[a,x]$.
□

Corollary 3.12. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$ if f' is monotonic decreasing on $[a,x]$, $f(a) \geq 0$, $f'(a) \leq 0$, $f''(a) \geq 0$ then left Riemann-Liouville fractional derivative of order $\alpha + 1$ that is ${}_aD_x^{\alpha+1}f(x) \geq 0$.*

Proof. The relationship between left Riemann-Liouville and Caputo fractional derivative (2.1) for $2 < \alpha < 3$ is,

$$ {}_aD_x^{\alpha+1}f(x) = {}_a^cD_x^{\alpha+1}f(x) + \sum_{i=0}^{2} \frac{(x-a)^{i-(\alpha+1)}}{\Gamma(i-(\alpha+1)+1)} f^{(i)}(a) $$

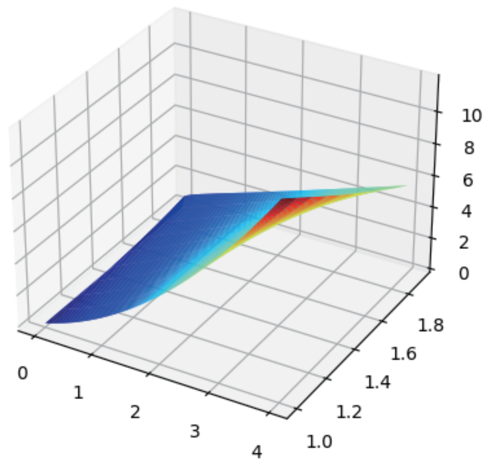

FIGURE 1. Plot of α order Caputo fractional derivative of $f(x) = \frac{x^3}{6}$.

$$= {}_a^cD_x^{\alpha+1}f(x) + \frac{(x-a)^{-(\alpha+1)}}{\Gamma(-\alpha)}f(a) + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}f^{'}(a) + \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}f^{''}(a).$$

Since, $\Gamma(1-\alpha) < 0$ and $\Gamma(-\alpha) > 0$, then by simple calculation left Riemann-Liouville fractional derivative ${}_aD_x^{\alpha+1}f(x) \geq 0$. □

Corollary 3.13. *Let $f \in C^2[a,b]$ and $\alpha \in (1,2)$ if left Riemann-Liouville fractional derivative of $\alpha+1$ order ${}_aD_x^{\alpha+1}f(x) \geq 0$, $f(a) \leq 0$, $f'(a) \geq 0$, $f''(a) \leq 0$ then f' is monotonic increasing on $[a,x]$.*

Proof. The proof is similar to the above theorems, hence not explained in details. □

All the above theorem deals with the monotonic character of a function to fractional derivative, which one may understand clearly by the example below.

Example 3.14. Let $f(x) = \frac{x^3}{6}$, $x \geq 0$. Clearly $f^{'}$ is monotonically increasing for $x \geq 0$ as $f^{''} = x \geq 0$ for $\alpha \in (1,2)$and $n = 2$.
First, consider Caputo fractional derivative of f is,

$$\begin{aligned} {}_0^cD_x^{\alpha}f(x) &= \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-\gamma)^{1-\alpha}\,\gamma\,d\gamma, \\ &= \frac{1}{\Gamma(3-\alpha)}\,x^{2-\alpha} + \frac{1}{\Gamma(4-\alpha)}\,x^{3-\alpha}\ \forall\, x \geq 0, \end{aligned}$$

since $\Gamma(3-\alpha)$and $\Gamma(4-\alpha) \geq 0$ for given α which clearly indicate
${}_0^cD_x^{\alpha}f(x) \geq 0$, (see figure 1).

If we consider the left Riemann-Liouville fractional derivative,

$$\begin{aligned} {}_0D_x^{\alpha}f(x) &= \frac{1}{\Gamma(2-\alpha)}\big(\frac{d}{dx}\big)^2\int_0^x (x-\gamma)^{1-\alpha}\frac{\tau^3}{6}\ d\gamma \\ &= \frac{1}{\Gamma(2-\alpha)}\big[\frac{d}{dx}\big(\int_0^x (x-\gamma)^{1-\alpha}\frac{\tau^3}{6}d\gamma\big)^{\frac{1}{2}}\ \big]^2 \\ &= \frac{1}{\Gamma(2-\alpha)}\big[\ \frac{1}{2}\big(\int_0^x (x-\gamma)^{1-\alpha}\ \frac{\gamma^3}{6}d\gamma\big)^{-\frac{1}{2}}\big) \\ &\quad \times\int_0^x \frac{(x-\gamma)^{2-\alpha}}{2-\alpha}\ \frac{\tau^3}{6}\ d\tau\big]\ \geq 0. \end{aligned}$$

4. Construction of relations between fractional derivatives, integrals with monotonic nature of a function on a generalized interval

Till now, all the above information has been calculated for a particular interval, then new thoughts arise what about generalizing the interval for any non-negative number? We get some new, interesting, and very different results by following the new idea. The monotonicity character of functions deals in very different ways of conditions to Caputo and Left Riemann-Liouville fractional derivative in this generalized interval. These ideas bring an interesting concept in fractional calculus to mathematics.

Theorem 4.1. *Let $f \in C^{\beta}[a,b]$ and $\alpha \in (\beta,\beta+1), \beta \in \mathbb{N}$ then $f^{(\beta)}$ is monotonic increasing on $[a,x]$ if and only if Caputo fractional derivative that is ${}_a^cD_x^{\alpha}f(x) \geq 0$.*

Proof. From the definition of Caputo fractional derivative,

$${}_a^cD_x^{\alpha}f(x) = \frac{1}{\Gamma(\beta+1-\alpha)}\int_a^x (x-\gamma)^{1+\beta}f^{(\beta+1)}(\gamma)d\gamma.$$

For any positive integer β, $\frac{1}{\Gamma(\beta+1-\alpha)} > 0$, $(x-t)^{\beta-\alpha} > 0$ and $f^{(\beta+1)}(t) > 0$, for any $t > 0$, therefore ${}_a^cD_x^{\alpha}f(x) \geq 0$.

For the converse part, let $h(x) = {}_a^cD_x^{\alpha}f^{(\beta)}(x) \geq 0$.
Then, ${}_aI_x^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}h(t)dt \geq 0$.

Therefore, for any $x \geq a$,

$$\begin{aligned} 0 &\leq {}_aI_x^{\alpha}\ {}_a^cD_x^{\alpha}f(x) \\ &= f(x) - \sum_{i=0}^{\beta}\frac{(x-a)^i}{\Gamma(i+1)}f^{(i)}(a), (\text{from } (2.3)) \\ &= f(x) - f(a) - \frac{(x-a)}{\Gamma(2)}f'(a) - \frac{(x-a)^2}{\Gamma(3)}f''(a) \\ &\quad - \cdots - \frac{(x-a)^{\beta}}{\Gamma(\beta+1)}f^{(\beta)}(a). \end{aligned}$$

By differentiated β times, we get $f^{(\beta)}$ is monotonic increasing on $[a, x]$. □

Theorem 4.2. *Let $f \in C^{\beta}[a, b]$ and $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$ then $f^{(\beta)}$ is monotonic decreasing on $[a, x]$ if and only if Caputo fractional derivative that is ${}_{a}^{c}D_{x}^{\alpha} f(x) \leq 0$.*

Proof. The following result can be proved by replacing f by $-f$. □

The next example enables us to understand the criteria of α order fractional scene monotonicity of a function in a generalized interval.

Example 4.3. Let $f(x) = \frac{x^{\beta+2}}{(\beta+2)!}$, $x \geq 0$. Clearly $f^{(\beta)}$ is monotonically increasing for $x \geq 0$ as $f^{(\beta+1)}(x) = x \geq 0$. For $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$, consider α order Caputo fractional derivative of f is,

$$ {}_{0}^{c}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(\beta+1-\alpha)(\beta+2)!} \int_{0}^{x} (x-\gamma)^{\beta-\alpha} \gamma^{\beta+2} d\gamma, $$

since $\Gamma(\beta+1-\alpha) \geq 0$ for given α it is clear ${}_{0}^{c}D_{x}^{\alpha} f(x) \geq 0$.

Corollary 4.4. *Let $f \in C^{\beta}[a, b]$ and $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$, $f^{(\beta-1)}$ is monotonic increasing on $[a, x]$,*

$$ f^{(\beta)}(x) = \begin{cases} \leq 0 & \text{if } \beta > \alpha \\ \geq 0 & \text{otherwise,} \end{cases} $$

then α order left Riemann-Liouville ${}_{a}D_{x}^{\alpha} f(x) \geq 0$.

Proof. From the previous description, the relationship properties of Left Riemann-Liouville and Caputo fractional derivative is (from(2.1)),

$$ \begin{aligned} {}_{a}D_{x}^{\alpha} f(x) &= {}_{a}^{c}D_{x}^{\alpha} f(x) + \sum_{i=0}^{\beta} \frac{(x-a)^{i-\alpha}}{\Gamma(i-\alpha+1)} f^{(i)}(a) \\ &= {}_{a}^{c}D_{x}^{\alpha} f(x) + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) + \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} f'(a) \\ &\quad + \cdots + \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} f^{(\beta)}(a) \end{aligned} $$

Note that, for all $x \geq a$, $\frac{1}{\Gamma(\beta-\alpha)} > 0$, then $f^{(\beta)}(x) > 0$, the result follows. □

Corollary 4.5. *Let $f \in C^{\beta}[a, b]$ and $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$,*

$$ f^{(\beta-1)}(x) = \begin{cases} \geq 0 & \text{if } \beta < \alpha \\ \leq 0 & \text{if } \beta+1 \geq \alpha, \end{cases} $$

and α order left Riemann-Liouville ${}_{a}D_{x}^{\alpha} f(x) \geq 0$ then $f^{(\beta)}$ is monotonic increasing on $[a, x]$.

Proof. One can observe from (2.1) that for all $x \geq a$,

$$ {}_{a}^{c}D_{x}^{\alpha} f(x) = {}_{a}D_{x}^{\alpha} f(x) - \sum_{i=0}^{\beta} \frac{(x-a)^{i-\alpha}}{\Gamma(i-\alpha+1)} f^{(i)}(a) $$

$$= \; {}_aD_x^\alpha f(x) - \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) - \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} f^{'}(a) - \cdots - \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)} f^{(\beta)}(a).$$

Since $\Gamma(1-\alpha), \Gamma(2-\alpha), \ldots \Gamma(\beta-\alpha) < 0$ and $\Gamma(\beta+1-\alpha) > 0$, therefore ${}_a^cD_x^\alpha f(x) \geq 0$. Thus $f^{(\beta)}$ is monotonic increasing on $[a, x]$.

□

The statement and proof of the Corollaries (4.4) and (4.5) as stated above arrive to the following theorems.

Theorem 4.6. *Let $f \in C^\beta[a, b]$ and $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$, $f(a) = 0$ then $f^{(\beta)}$ is monotonic increasing on $[a, x]$ if and only if Riemann-Liouville fractional derivative ${}_aD_x^\alpha f(x) \geq 0$.*

Proof. From the given data $f(a) = 0$ which implies that the higher order of f is also zero at point $a \in [a, x]$ then Caputo fractional derivative ${}_a^cD_x^\alpha f(x) \geq 0$ concluded that ${}_aD_x^\alpha f(x) \geq 0$. □

Theorem 4.7. *Let $f \in C^\beta[a, b]$ and $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$, if $f^{(\beta)}$ is monotonic increasing on $[a, x]$ then ${}_a^cD_x^{\alpha+1} f(x) \leq 0$.*

Proof. α order fractional derivative is

$${}_a^cD_x^\alpha f(x) = \frac{1}{\Gamma(\beta+1-\alpha)} \int_a^x (x-\gamma)^{\beta-\alpha} f^{(\beta)}(\gamma) d\gamma$$
$${}_a^cD_x^{\alpha+1} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-\gamma)^{\beta-\alpha-1} f^{(\beta)}(\gamma) d\gamma.$$

Here $f^{(\beta+1)}(x) \geq 0$ and $\frac{1}{\Gamma(\beta-\alpha)} < 0$.
Thus ${}_a^cD_x^{\alpha+1} f(x) \leq 0$.

□

Theorem 4.8. *Let $f \in C^\beta[a, b]$ and $\alpha \in (\beta, \beta+1), \beta \in \mathbb{N}$, if ${}_a^cD_x^{\alpha+1} f(x) \leq 0$ then $f^{(\beta)}$ is monotonic decreasing on $[x, a]$.*

Proof. Let ${}_a^cD_x^{\alpha+1} f(x) = {}_a^cD_x^\alpha h(x) \leq 0$ where $h(x) = f'(x)$.

Thus for given $x \geq a$,

$$\begin{aligned}
0 \;&\geq\; {}_aI_x^\alpha \; {}_a^cD_x^\alpha h(x) \\
&= g(x) - \sum_{i=0}^{\beta} \frac{(x-a)^i}{\Gamma(i+1)} g^i(a), \text{(from (2.3))} \\
&= g(x) - g(a) - (x-a)g'(a) - \frac{(x-a)^2}{\Gamma(2)} g^{''}(a) \\
&\quad - \cdots - \frac{(x-a)^{(\beta)}}{\Gamma(\beta+1)} g^{(\beta)}(a) \\
&= f^{(\beta)}(x) - f^{(\beta)}(a)
\end{aligned}$$

Thus, $f^{(\beta)}$ is monotonic decreasing on $[a,x]$. □

Theorem 4.9. *Let $f \in C^\beta[a,b]$ and $\alpha \in (\beta,\beta+1), \beta \in \mathbb{N}$, if Caputo fractional derivative of $\alpha+1$ order, ${}_a^cD_x^{\alpha+1}f(x) \geq 0$ then $f^{(\beta)}$ is monotonic increasing on $[a,x]$.*

Proof. The proof similarly follows from theorem 4.8. □

Theorem 4.10. *Let $f \in C^\beta[a,b]$ and $\alpha \in (\beta,\beta+1), \beta \in \mathbb{N}$, Caputo fractional derivative of $\alpha+1$ order ${}_a^cD_x^{\alpha+1}f(x) \geq 0$ if $f^{(\beta)}(x)$ is monotonic decreasing on $[x,a]$.*

Proof. The proof similarly follows from theorem 4.7. □

5. Conclusion

In this article, we have discussed the monotonous results using the concept of Reimann-Liouville and Caputo fraction derivatives. Monotonicity results are established for any generalised interval $(\beta,\beta+1)$, where $\beta \in \mathbb{N}$. However many important topics like convexity and optimality in generalized interval calculus remain unfolded. In addition to this, the extension of these results in the function of several variables is not studied here. These are left as the scope of future research.

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Manuscript received July 2 2024
revised July 3 2023

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