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IMPROVED FULL-NEWTON STEP INTERIOR-POINT ALGORITHM FOR THE GENERAL FISHER MARKET EQUILIBRIUM PROBLEM

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Abstract. We present a feasible interior-point algorithm with full-Newton steps for solving the weighted linear complementarity problem of the general Fisher market equilibrium. By employing a kernel function, we derive an equivalent system of equations, which facilitates the full-Newton step feasible IPM for problem-solving. We offer a proof of global convergence and polynomial complexity, and validate the effectiveness of the algorithm through numerical examples.

1. Introduction

Recently there has been a renewed interest in the weighted linear complementarity problem (wLCP) [19, 20]. As an extension of the linear complementarity problem [1, 2, 5], wLCP has a wider range of applications [10] in economics, engineering, management science, etc. Additionally, interior-point algorithms [26] based on new search directions have become a hot research topic in wLCP [18, 20, 24].

The full-Newton step interior-point algorithm [7, 6, 15, 22] is an effective method for solving optimization problems [27]. It does not require line search during iteration, which saves a significant amount of work time, and has polynomial complexity [14, 25]. By perturbing wLCP [4, 20], the new algebraic equivalent form of the smooth central path [12, 17] is defined to obtain a new Newton search direction [8]. A feasible full-Newton step interior-point algorithm [4] is proposed to solve the LCP [16, 18, 30] model of Fisher market equilibrium problem [11, 28].

It is well known that Potra [21] introduced a new smooth central path, which is used in two IPM methods to solve monotone wLCP [12, 19, 20]. The central path [12] can be constructed by including a proposed IPM [4, 7, 15] starting point. The algorithm is shown to have both global and local convergence properties.

The full-Newton step feasible IPMs are extended to the wLCP [4, 3, 16, 19] model of the Fisher problem [11, 13], which does not require line searches [23]. There are n_p producers and n_c consumers, who buy goods from the producers to maximize the individual utility function with a given budget w. Price equilibrium is achieved

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when the supply and demand for a commodity are in balance, indicating that the budget and inventory are completely reset to zero.

In this paper, we consider a particular special case of the Fisher model. When each producer owns one unit of products, consumers i may have a linear utility function

$$u_i(x_i) = u(x_{i1}, \dots, x_{in}) = \sum_j u_{ij} x_{ij}.$$

The purchased quantity is assumed to be x_{ij}, while the utility coefficient of consumer i for producer j's goods is u_{ij}. Moreover, the following inequalities hold for all i and j

$$w_i > 0,\ u_{ij} \geq 0,\ \sum_{k=1}^{n_c} u_{kj} > 0,\ \sum_{k=1}^{n_p} u_{ik} > 0.$$

Thus, Eisenberg and Gale [9] gave the optimization formulation

$$\begin{aligned}
&\max_{u_i, x_{ij}} \sum_{i=1}^{n_c} w_i \log u_i, \\
&s.t. \sum_{i=1}^{n_c} x_{ij} = 1, \quad j = 1, 2, \dots, n_p, \\
&u_i - \sum_{j=1}^{n_p} u_{ij} x_{ij} = 0, \quad i = 1, 2, \dots, n_c, \\
&u_i \geq 0, x_{ij} \geq 0, \quad i = 1, 2, \dots, n_c, j = 1, 2, \dots, n_p.
\end{aligned} \tag{1.1}$$

Here the optimal Lagrange multipliers for the first n_p equation constraints are the market clearing prices.

Let $x = (u_1, \dots, u_{n_c}, x_{11}, \dots, x_{1n_p}, x_{21}, \dots, x_{2n_p}, \dots, x_{n_c 1}, \dots, x_{n_c n_p})^T$ be an $n-$dimensional vector, $b = (1, \dots, 1, 0, \dots, 0)^T$ be an $m-$dimensional vector, and $w = (w_1, \dots, w_{n_c}, 0, \dots, 0)^T \in \mathbb{R}^n_+$ be an $n-$dimensional vector. Set $m = n_c + n_p, n = n_c(n_p + 1)$.

$$A = \begin{pmatrix}
0 & 0 & \cdots & 0 & e_1^T & e_1^T & \cdots & e_1^T \\
0 & 0 & \cdots & 0 & e_2^T & e_2^T & \cdots & e_2^T \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0 & e_{n_p}^T & e_{n_p}^T & \cdots & e_{n_p}^T \\
1 & 0 & \cdots & 0 & a_1^T & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & a_2^T & \cdots & 0 \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & a_{n_c}^T
\end{pmatrix} \in \mathbb{R}^{m+n}$$

is a special full-rank matrix, with $a_i = -(u_{i1}, u_{i2}, \dots, u_{im_p})^T, i = 1, 2, \dots, n_c$. Then (1.1) can be formulated as

$$\begin{aligned} &\max_x \sum_{i=1}^{n} w_i \log x_i, \\ &s.t.\ Ax = b, \\ &\qquad x \geq 0. \end{aligned} \tag{1.2}$$

Consider the general optimization problem of (1.2). Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $w \in \mathbb{R}^n_+$ be arbitrary. Assuming A is full rank, the KKT conditions of (1.2) are given by the following wLCP

$$\begin{cases} Ax = b, x \geq 0, \\ s - A^T y = 0, s \geq 0, \\ xs = w, \end{cases} \tag{1.3}$$

where $A \in \mathbb{R}^{m\times n}$ is an arbitrary matrix, and $b \in \mathbb{R}^m$, $w \in \mathbb{R}^n_+$ are arbitrary vectors.

2. Preliminaries

First, we define the strict feasible set for system (1.3) as

$$F^0 := \{(x, y, s) \in \mathbb{R}^n_{++} \times \mathbb{R}^m \times \mathbb{R}^n_{++} | Ax = b, A^T y + s = c\}.$$

Let $w \geq 0$, and propose the disturbed problem of (1.3)

$$\begin{cases} Ax = b, x \geq 0, \\ A^T + s = c, s \geq 0, \\ xs = w(t), \end{cases} \tag{2.1}$$

where

$$w(t) = tx^0 s^0 + (1 - t)w, \quad t \in [0, 1]. \tag{2.2}$$

Let $(x^0, y^0, s^0) \in F^0$ be an initial point. Assuming the interior-point condition holds for any parameter $0 < t < 1$, system (2.1) has a unique solution $((x(t), y(t), s(t))$. The central path of the system defined by system (1.3) is the set of optimal solution $\{(x(t), y(t), s(t))|t > 0\}$. When $t \to 0$, we get $w(t) \to w$, and hence the solution to the original problem can be obtained.

The system of equations for solving the Newton search direction $(\triangle x, \triangle y,\ \triangle s)$ is provided as follows

$$\begin{cases} A\triangle x = 0, \\ A^T \triangle y + \triangle s = 0, \\ s\triangle x + x\triangle s = w(t) - xs. \end{cases} \tag{2.3}$$

Let

$$v := \sqrt{\frac{xs}{w(t)}}, d_x := \frac{v\triangle x}{x}, d_s := \frac{v\triangle s}{s}. \tag{2.4}$$

Substituting (2.4) into (2.3) yields the scaled Newton system

$$\left\{\begin{array}{l}\bar{A}d_x = 0,\\ W^{-1}(t)\bar{A}\triangle y + d_s = 0,\\ d_x + d_s = v^{-1} - v,\end{array}\right. \tag{2.5}$$

where $\bar{A} := AV^{-1}X, V := diag(v), X := diag(x), W(t) := diag(w(t))$.

We consider the kernel function [22]

$$\psi(t) := \frac{1}{2}(t - \frac{1}{t})^2, \quad \Psi(v) := \sum_{i=1}^{n} \psi(v_i).$$

Since $\psi'(t) = t - \frac{1}{t^3}$, we replace d_x+d_s=$v^{-3} - v$ with the third equation in (2.5), and obtain

$$\left\{\begin{array}{l}\bar{A}d_x = 0,\\ W^{-1}(t)\bar{A}\triangle y + d_s = 0,\\ d_x + d_s = v^{-3} - v.\end{array}\right. \tag{2.6}$$

Define the proximity metric as

$$\delta(v) := \|d_x + d_s\| = \|v^{-3} - v\| \tag{2.7}$$

For any $t > 0$, we have

$$\delta(v) = 0 \Leftrightarrow \|v^{-3} - v\| = 0 \Leftrightarrow v = e \Leftrightarrow xs = w(t). \tag{2.8}$$

Since v is non-negative and equals zero when (x, s) is on the central path. The proximity function can be utilized to quantify the distance between the current iteration (x, y, s) and the corresponding t-central point $(x(t), y(t), s(t))$.

In this work, the full-Newton step feasible IPM for LO is extended to the wLCP model of the Fisher problem. We propose a new Newton search direction by incorporating a kernel function, and introduce an improved interior-point algorithm for solving the Fisher market equilibrium problem. The algorithm exclusively performs full-Newton steps, thereby avoiding the computation of step lengths. Then, we establish the global convergence of the algorithm and derive iteration bound with polynomial complexity.

This paper is organized as follows. In section 2, we define the center path of the problem, propose a new kernel function, and obtain a new search direction based on the kernel function. Then a full-Newton step feasible IPM is proposed for the wLCP model of the Fisher problem. In section 3, we complete a rigorous feasibility proof of the full-Newton step IPM, analyze the convergence, and derive polynomial complexity of our algorithm. Section 4 gives some numerical results. Section 5 concludes this paper.

Conventions Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ represent the sets of real numbers, nonnegative real numbers, and positive real numbers, respectively. The space of all $m \times n$ matrices is denoted as $\mathbb{R}^{m\times n}$. The symbol e denotes an n-dimensional vector consisting entirely of ones. The notation xs indicates the componentwise product of real vectors x and s. Furthermore, we also utilize $\frac{x}{s}$ to denote the real vector $\frac{x}{s} = [\frac{x_1}{s_1}, \ldots, \frac{x_i}{s_i}, \ldots, \frac{x_n}{s_n}]^T$, where $s_i \neq 0$ for $i = 1, 2, \ldots, n$. $X = diag(x)$ represents the diagonal matrix with entries from vector x placed along its diagonal.

Additionally, for any given vector x, $\|x\|$ and $\|x\|_\infty$ signify its 2-norm and infinity norm respectively. Moreover $\min x$ (or $\max x$) denotes the minimal (or maximal) component within vector x.

3. Analysis of algorithm

A new full-Newton step feasible IPM is proposed for solving the wLCP model of Fisher market problem. The following is the pseudocode of our algorithm.

Algorithm 1: Improved Full-Newton Step IPM for General Fisher Equilibrium

Require:
An update parameter $\theta \in (0,1)$;
a threshold parameter $0 < \tau < 1$;
An accuracy parameter $\varepsilon > 0$;
Give $(x^0, y^0, s^0) \in \mathcal{F}^0$ such that $x^0 s^0 \geq w$, and $\delta(x^0, y^0, s^0) \leq \tau t_0$;
Set $t_0 = 1$;
Let $x := x^0; y := y^0; s := s^0; t := t_0$;
while $\|xs - w\| \leq \varepsilon$ **do**
Combining (2.4) with (2.5), a new search direction $(\Delta x, \Delta y, \Delta s)$ can be obtained;
Following the new search direction, we take a full-Newton step and obtain the new iteration
$$x := x + \Delta x, \quad y := y + \Delta y, \quad s := s + \Delta s;$$
Set the update parameter $t := (1-\theta)t$;
end

Lemma 3.1 ([23])**.** *If u and v are both vectors in $\mathbb{R}^n$ and are orthogonal to each other, then*
$$\|uv\|_\infty \leq \frac{1}{4}\|u+v\|^2, \quad \|uv\| \leq \frac{\sqrt{2}}{4}\|u+v\|^2.$$

Since $d_x^T d_s = 0$, it follows from (2.7) and Lemma 3.1, that

$$\|d_x d_s\|_\infty \leq \frac{\delta^2}{4}, \quad \|d_x d_s\| \leq \frac{\sqrt{2}\delta^2}{4}. \tag{3.1}$$

Lemma 3.2. *For any vector $v \in R^n_+$, one has*
$$1 - \delta \leq v_i \leq 1 + \delta, \quad i = 1, 2, ..., n,$$
where $\delta := \delta(x, s; t)$.

Proof. We have
$$\delta(v) = \|v^{-3} - v\| = \left\| \frac{(e-v)(e+v+v^2+v^3)}{v^3} \right\| \geq \|e - v\|,$$
which implies
$$1 - \delta \leq v_i \leq 1 + \delta, \quad i = 1, 2, ..., n.$$
$\square$

3.1. **Feasibility Analysis of Full-Newton Steps.**

Lemma 3.3. *Let $\delta := \delta(x, s; t)$ and $v > 0$. If $\delta < 1$, then the full-Newton step is strictly feasible, that is, $x^+ > 0$, $s^+ > 0$.*

Proof. For any $\alpha \in [0, 1]$, define

$$x(\alpha) = x + \alpha \Delta x, \quad s(\alpha) = s + \alpha \Delta s.$$

For a given $v > 0$, we get

$$\begin{aligned} \frac{x(\alpha)s(\alpha)}{w(t)} &= \frac{xs}{w(t)} + \frac{\alpha(s\Delta x + x\Delta s)}{w(t)} + \frac{\alpha^2 \Delta x \Delta s}{w(t)} \\ &= v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s \\ &= v^2 + \alpha v(v^{-3} - v) + \alpha^2 d_x d_s \\ &= (1-\alpha)v^2 + \alpha\left(v^{-2} + \alpha d_x d_s\right). \end{aligned} \tag{3.2}$$

In order to ensure that $x(\alpha)s(\alpha) > 0$, we only need to prove that $v^{-2} + \alpha d_x d_s > 0$. If $\delta < 1$, we get from (3.1)

$$v^{-2} + \alpha d_x d_s \geq v^{-2} - \left\|d_x d_s\right\|_\infty e \geq \left(\frac{1}{(1+\delta)^2} - \frac{\delta^2}{4}\right) e.$$

Thus, $x(\alpha)s(\alpha) > 0$ for $\alpha \in [0, 1]$. Also, since $x(\alpha)$ and $s(\alpha)$ are linear functions of α and $x(0) = x^0 > 0$, $s(0) = s^0 > 0$, it follows that $x(\alpha) > 0$, $s(\alpha) > 0$. Hence $x^+ = x(1) > 0$, $s^+ = s(1) > 0$, which complete the proof. $\square$

3.2. **Convergence Analysis.**
Define

$$v_+ = \sqrt{\frac{x^+ s^+}{w(t)}}. \tag{3.3}$$

In this subsection, we demonstrate the convergence of Algorithm 1. Lemma 3.4 and Lemma 3.5 offer lower bounds on v_+ and $\left\|e - v_+^2\right\|$ respectively.

Lemma 3.4. *If $\delta < 1$ and $v > 0$, then* $\min v_+ \geq \sqrt{\frac{1}{(1+\delta)^2} - \frac{\delta^2}{4}}$.

Proof. When $\alpha = 1$ in (3.2), it follows from (3.1) and (3.3) that

$$\begin{aligned} v_+^2 &= \frac{x^+ s^+}{w(t)} = v^{-2} + d_x d_s \\ &\geq v^{-2} - \|d_x d_s\|_\infty \\ &\geq \left(\frac{1}{(1+\delta)^2} - \frac{\delta^2}{4}\right) e. \end{aligned} \tag{3.4}$$

Then, we get by (3.4)

$$\min v_+ \geq \sqrt{\frac{1}{(1+\delta)^2} - \frac{\delta^2}{4}}.$$

$\square$

Lemma 3.5. *If $\delta < 1$ and $v > 0$, one has* $\left\|e - v_+^2\right\| \leq \frac{\delta}{2} + \frac{\sqrt{2}}{4}\delta^2$.

Proof. From (2.7) and Lemma 3.2, we have

$$v^{-3} - v = \frac{(e-v)(e+v)}{v^2} \cdot \frac{\left(e + v + v^2 + v^3\right)}{v(e+v)} = \left(e - \frac{e}{v^2}\right) \frac{\left(e + v + v^2 + v^3\right)}{v(e+v)}.$$

It follows from the aforementioned relation that

$$\begin{aligned}
\left\|e - \frac{e}{v^2}\right\| &= \left\|(v^{-3} - v)\frac{v(e+v)}{(e + v + v^2 + v^3)}\right\| \\
&\leq \left\|v^{-3} - v\right\| \left\|\frac{v(e+v)}{(e+v)(e+v^2)}\right\|_\infty \\
&= \left\|v^{-3} - v\right\| \left\|\frac{e}{v^{-1} + v}\right\|_\infty \\
&\leq \frac{1}{2}\left\|v^{-3} - v\right\| \\
&= \frac{\delta}{2}.
\end{aligned}$$

Therefore, we get by (3.1) and (3.4)

$$\left\|e - v_+^2\right\| = \left\|e - \frac{e}{v^2} - d_x d_s\right\| \leq \left\|e - \frac{e}{v^2}\right\| + \|d_x d_s\| \leq \frac{\delta}{2} + \frac{\sqrt{2}}{4}\delta^2.$$

□

Subsequently, prior to updating t, we provide an upper bound for the neighborhood function $\delta(v_+) := \delta(x^+, s^+; t)$ of Algorithm 1 after a full-Newton step.

Lemma 3.6. *If $\delta < 1$ and $v > 0$, then* $\delta\left(v_+\right) \leq \frac{\phi\left(\delta\right)}{2\sqrt{2}}\delta^2$, *where*

$$\phi(\delta) = \left(\frac{1}{\sqrt{\frac{1}{(1+\delta)^2} - \frac{\delta^2}{4}}} + \frac{1}{\left(\sqrt{\frac{1}{(1+\delta)^2} - \frac{\delta^2}{4}}\right)^3}\right)\left(1 + \frac{\sqrt{2}}{\delta}\right). \tag{3.5}$$

Proof. From (2.7), we have

$$\begin{aligned}
\delta(v_+) &= \left\|v_+^{-3} - v_+\right\| \\
&= \left\|\frac{(e - v_+)(e + v_+ + v_+^2 + v_+^3)}{v_+^3}\right\| \\
&\leq \left\|\frac{e + v_+ + v_+^2 + v_+^3}{v_+^3}\right\|_\infty \left\|e - v_+\right\|.
\end{aligned} \tag{3.6}$$

For any $u > 0$, consider the function $f\left(u\right) = \frac{1+u+u^2+u^3}{u^3}$ and thus $f'\left(u\right) = -\frac{1}{u^2} - \frac{2}{u^3} - \frac{3}{u^4} < 0$. Since f is strictly decreasing on the interval $(0, +\infty)$, it follows from (3.6), Lemma 3.4 and Lemma 3.5 that

$$\delta\left(v_+\right) \leq \frac{1 + \min v_+ + \min v_+^2 + \min v_+^3}{\min v_+^3}\left\|e - v_+\right\|$$

$$
\begin{aligned}
&= \frac{(1+\min v_+)\left(1+\min v_+^2\right)}{\min v_+^3}\left\|\frac{e-v_+^2}{e+v_+}\right\| \\
&\leq \frac{(1+\min v_+)\left(1+\min v_+^2\right)}{\min v_+^3\,(1+\min v_+)}\left\|e-v_+^2\right\| \\
&= \left(\frac{1}{\min v_+}+\frac{1}{\min {v_+}^3}\right)\left\|e-v_+^2\right\| \\
&\leq \left(\frac{1}{\sqrt{\frac{1}{(1+\delta)^2}-\frac{\delta^2}{4}}}+\frac{1}{\left(\sqrt{\frac{1}{(1+\delta)^2}-\frac{\delta^2}{4}}\right)^3}\right)\left(\frac{\sqrt{2}}{4}\delta^2+\frac{\delta}{2}\right) \\
&= \frac{\phi(\delta)}{2\sqrt{2}}\delta^2.
\end{aligned}
$$

□

Define

$$
v^+ = \sqrt{\frac{x^+s^+}{\omega(t_+)}}. \tag{3.7}
$$

Now we proceed to investigate the impact of updating t on the proximity function $\delta(v^+) := \delta(x^+, s^+; t_+)$, where $t_+ = (1-\theta)t$.

Lemma 3.7. *If $\delta < 1$, $v > 0$ and $x^0s^0 \geq w$, then*

$$
\delta(v^+) \leq \frac{\phi(\delta)}{2\sqrt{2}}\delta^2 + \frac{2\theta\beta t}{\gamma}\sqrt{\frac{1}{(1-\delta)^2}+\frac{\delta^2}{4}}, \tag{3.8}
$$

where $\beta = \left\|x^0s^0-\omega\right\|$, $\gamma = \min\omega$.

Proof. Due to the fact that $x^0s^0 \geq w$, and in light of (2.2), we get

$$
\omega(t_+) = \omega(t) + \theta t\left(\omega - x^0s^0\right) \leq \omega(t). \tag{3.9}
$$

By (3.3) and (3.7), it can be demonstrated that

$$
v^+ = \sqrt{\frac{x^+s^+}{\omega(t_+)}} = \sqrt{\frac{x^+s^+}{\omega(t)}}\sqrt{\frac{\omega(t)}{\omega(t_+)}} = \sqrt{\frac{\omega(t)}{\omega(t_+)}}v_+. \tag{3.10}
$$

From (2.2) and (3.9), we have

$$
\frac{\omega(t_+)}{\omega(t)} \leq e.
$$

By (3.9), (3.10) and Lemma 3.6, we obtain

$$\begin{aligned}\delta(v^+) &= \left\|\left(v^+\right)^{-3} - v^+\right\| = \left\|\left(\sqrt{\frac{\omega(t)}{\omega(t_+)}}v_+\right)^{-3} - \sqrt{\frac{\omega(t)}{\omega(t_+)}}v_+\right\| \\ &\le \left\|\left(\sqrt{\frac{\omega(t_+)}{\omega(t)}}\right)^3 \left(v_+^{-3} - v_+\right)\right\| + \left\|v_+\left(\left(\sqrt{\frac{\omega(t_+)}{\omega(t)}}\right)^3 - \sqrt{\frac{\omega(t)}{\omega(t_+)}}\right)\right\| \\ &\le \delta(v_+) + \|v_+\|_\infty \left\|\frac{\omega(t_+)^2 - \omega(t)^2}{\omega(t)^{\frac{3}{2}}\omega(t_+)^{\frac{1}{2}}}\right\|.\end{aligned} \tag{3.11}$$

Now we consider the upper bound of the last term in the aforementioned inequation. From (3.1) and (3.4), it follows that

$$\begin{aligned}\left\|v_+\right\|_\infty &= \max v_+ = \max \sqrt{v^{-2} + d_x d_s} \\ &\le \sqrt{\frac{1}{(1-\delta)^2} + \frac{\delta^2}{4}}.\end{aligned} \tag{3.12}$$

Furthermore, by (3.9), we get

$$\begin{aligned}\left\|\frac{\omega(t_+)^2 - \omega(t)^2}{\omega(t)^{\frac{3}{2}}\omega(t_+)^{\frac{1}{2}}}\right\| &= \left\|\frac{(\omega(t)+\omega(t_+))(\omega(t_+)-\omega(t))}{\omega(t)^{\frac{3}{2}}\omega(t_+)^{\frac{1}{2}}}\right\| \\ &= \left\|\left(\frac{e}{\omega(t)^{\frac{1}{2}}\omega(t_+)^{\frac{1}{2}}} + \frac{\omega(t_+)^{\frac{1}{2}}}{\omega(t)^{\frac{3}{2}}}\right)\theta t(x^0 s^0 - \omega)\right\| \\ &\le \frac{2\theta\|x^0 s^0 - \omega\| t}{\min \omega(t_+)} \\ &\le \frac{2\theta\beta t}{\gamma}.\end{aligned} \tag{3.13}$$

Here the last inequality is due to $x^0 s^0 \ge \omega, \beta = \left\|x^0 s^0 - \omega\right\|$ and $\gamma = \min \omega$. As a consequence of (3.11)-(3.13) and Lemma 3.6, we immediately have

$$\delta\left(v^+\right) \le \frac{\phi(\delta)}{2\sqrt{2}}\delta^2 + \frac{2\theta\beta t}{\gamma}\sqrt{\frac{1}{(1-\delta)^2} + \frac{\delta^2}{4}}.$$

□

In the following result, we select the optimal update parameter θ and threshold parameter τ such that the new iterate (x^+, s^+, y^+) lies in the τt_+-neighborhood of the central path. Let $\delta(v) = \frac{1}{3}\|v^{-3} - v\|$.

Theorem 3.8. *Let* $\theta \leq \frac{\left(27\sqrt{2} - t\phi\left(\frac{2t}{3}\right)\right)\gamma}{27\sqrt{2}\left(\varphi(t)\beta + \gamma\right)}$, *where*

$$\phi\left(\frac{2t}{3}\right) = \left(\frac{1}{\sqrt{\frac{1}{(1+\frac{2t}{3})^2} - \frac{\left(\frac{2t}{3}\right)^2}{4}}} + \frac{1}{\left(\sqrt{\frac{1}{(1+\frac{2t}{3})^2} - \frac{\left(\frac{2t}{3}\right)^2}{4}}\right)^3}\right)\left(1 + \frac{\sqrt{2}}{\frac{2t}{3}}\right),$$

$$\varphi(t) = \sqrt{\frac{1}{\left(1 - \frac{2t}{3}\right)^2} + \frac{1}{4}\left(\frac{2t}{3}\right)^2}$$

with $\beta = \left\|x^0 s^0 - \omega\right\|$ *and* $\gamma = \min \omega$. *If* $\delta \leq \frac{2}{3}t$, *then* $\delta(v^+) \leq \frac{2}{3}t_+$.

Proof. If $\delta \leq \frac{2}{3}t$, we have from (3.8)

$$\begin{aligned}\delta(v^+) &\leq \frac{2t^2\phi\left(\frac{2t}{3}\right)}{81\sqrt{2}} + \frac{2\theta\beta t}{3\gamma}\sqrt{\frac{1}{\left(1 - \frac{2t}{3}\right)^2} + \frac{1}{4}\left(\frac{2t}{3}\right)^2} \\ &= \frac{2t^2\phi\left(\frac{2t}{3}\right)}{81\sqrt{2}} + \frac{2\theta\beta t}{3\gamma}\varphi(t).\end{aligned} \tag{3.14}$$

To guarantee that $\delta\left(v^+\right) \leq \frac{2}{3}t_+$ holds, we assume $\frac{1}{3}\left(\frac{2t^2\phi\left(\frac{2t}{3}\right)}{27\sqrt{2}} + \varphi(t)\frac{2\theta\beta t}{\gamma}\right) \leq \frac{2t_+}{3}$. Substituting $t_+ = (1-\theta)\,t$ into the last relation and simplifying, we get

$$\theta \leq \frac{\left(27\sqrt{2} - t\phi\left(\frac{2t}{3}\right)\right)\gamma}{27\sqrt{2}\left(\varphi(t)\beta + \gamma\right)}.$$

□

3.3. **Complexity Analysis.**

For $t \in [0,1)$, $\phi(\frac{2t}{3})$ and $\varphi(t)$ are defined by Theorem 3.8. Let $\tau = \frac{2}{3}$ and the initial point $(x^0, s^0, y^0) \in \mathcal{F}^0$ such that $\delta(x^0, s^0, y^0) \leq \tau t_0$ and $x^0 s^0 \geq \omega$. We speculate that $\delta(v^+) \leq \tau t_0$, which means that the new iterate (x^+, s^+, y^+) generated by Algorithm 1 after a full-Newton step remains within the τt_+-neighborhood of the central path. The following result demonstrates that Algorithm 1 for solving wLCP (1.3) has polynomial complexity.

Theorem 3.9. *Let*

$$\theta = \frac{\left(27\sqrt{2} - t\phi\left(\frac{2t}{3}\right)\right)\gamma}{27\sqrt{2}\left(\varphi(t)\beta + \gamma\right)}, \quad t \in (0, t_0]. \tag{3.15}$$

Select $(x^0, s^0, y^0) \in \mathcal{F}^0$ such that $x^0 s^0 \geq \omega$. Then Algorithm 1 requires at most

$$k \geq \frac{116\beta + 39\gamma}{6\gamma} \log\left(\frac{\frac{\max\left(x^0 s^0\right)}{2} + \beta}{\varepsilon}\right) + 1$$

iterations to obtain an ε-approximate solution to wLCP (1.3) satisfying $\|xs - \omega\| \leq \varepsilon$.

Proof. Since $x^0 s^0 \geq \omega$, it follows from (2.2), (3.3), Lemma 3.5 and Theorem 3.8 that

$$\begin{aligned}
\left\|x^+ s^+ - \omega\right\| &\leq \left\|x^+ s^+ - \omega\left(t\right)\right\| + \left\|\omega\left(t\right) - \omega\right\| \\
&= \left\|\left[{v_+}^2 - e\right]\omega\left(t\right)\right\| + \left\|\omega\left(t\right) - \omega\right\| \\
&\leq \left\|\omega\left(t\right)\right\|_\infty \left\|e - {v_+}^2\right\| + \left\|x^0 s^0 - \omega\right\| t \\
&\leq \left(\frac{\delta}{2} + \frac{\sqrt{2}\delta^2}{4}\right)\max\left(x^0 s^0\right) + \beta t \\
&\leq \frac{4t^2}{9}\left(\frac{\sqrt{2}}{4} + \frac{3}{4t}\right)\max\left(x^0 s^0\right) + \beta t \\
&\leq \left(\frac{\max\left(x^0 s^0\right)}{2} t + \beta\right) t.
\end{aligned}$$

After k iterations, we get for $t_{k-1} \in (0, 1]$

$$\begin{aligned}
\left\|x^k s^k - \omega\right\| &< \left[\frac{\max\left(x^0 s^0\right)}{2} t + \beta\right] t_{k-1} \\
&< \left[\frac{\max\left(x^0 s^0\right)}{2} + \beta\right] (1 - \theta_{min})^{k-1}.
\end{aligned} \tag{3.16}$$

The value of $\theta_{\min}$ in (3.16) is discussed below. From the definitions of $\varphi\left(t\right)$ and $\phi\left(\frac{2t}{3}\right)$, $\varphi\left(t\right)$ and $\phi\left(\frac{2t}{3}\right) t$ are increasing functions with respect to t. When $t = 1$, we have $\varphi\left(1\right) = \sqrt{\frac{1}{9} + 9}$, and

$$\phi\left(\frac{2}{3}\right) t \leq \left(\frac{15}{\sqrt{56}} + \left(\frac{15}{\sqrt{56}}\right)^3\right)\left(1 + \frac{3\sqrt{2}}{2}\right).$$

Moreover, since the function θ in (3.15) is a monotonically decreasing function with respect to t, we get

$$\begin{aligned}
\theta &= \frac{\left(27\sqrt{2} - t\phi\left(\frac{2t}{3}\right)\right)\gamma}{27\sqrt{2}\left(\varphi(t)\beta + \gamma\right)} \\
&\geq \frac{\left(27\sqrt{2} - t\phi\left(\frac{2}{3}\right)\right)\gamma}{27\sqrt{2}\left(\varphi(1)\beta + \gamma\right)}
\end{aligned}$$

$$\geq \frac{\left(27\sqrt{2}-\left(\frac{15}{\sqrt{56}}+\left(\frac{15}{\sqrt{56}}\right)^3\right)\left(1+\frac{3\sqrt{2}}{2}\right)\right)\gamma}{27\sqrt{2}\left(\sqrt{\frac{1}{9}+9}\beta+\gamma\right)}$$
$$> \frac{6\gamma}{116\beta+39\gamma}.$$

There we have

$$\theta_{min} \geq \frac{6\gamma}{116\beta+39\gamma}. \tag{3.17}$$

In order that $\left\|xs-\omega\right\| \leq \varepsilon$ holds, it is sufficient to have by (3.16)

$$\left[\frac{\max\left(x^0s^0\right)}{2}t+\beta\right](1-\theta_{min})^{k-1} \leq \varepsilon.$$

Taking the logarithm of both sides yields

$$(k-1)\log\left(1-\theta_{\min}\right) \leq \log\left(\frac{\varepsilon}{\frac{\max\left(x^0s^0\right)}{2}+\beta}\right).$$

For any $0<\theta_{\min}<1$, we have $\log\left(1-\theta_{\min}\right) \leq -\theta_{\min}$, hence

$$k-1 \geq \frac{1}{\theta_{\min}}\log\left(\frac{\frac{\max\left(x^0s^0\right)}{2}+\beta}{\varepsilon}\right). \tag{3.18}$$

Combining (3.17) and (3.18) yields

$$k \geq \frac{116\beta+39\gamma}{6\gamma}\log\left(\frac{\frac{\max\left(x^0s^0\right)}{2}+\beta}{\varepsilon}\right)+1.$$

□

4. Numerical results

In this section, Algorithm 1 was tested on several Fisher market equilibrium problems. All experiments were performed on an ASUS PC equipped with an 11th Gen Intel(R) Core(TM) i5-11400H @ 2.70GHz 2.69 GHz with 16GB RAM, and the operating system was Windows 11. The implementations were done in MATLAB (R2023b).

Set Gap $:= \|xs-w\|$ and $\delta(v) := \left\|v^{-3}-v\right\|$. We let the algorithm terminate, when $\|xs-\omega\| \leq \varepsilon$ with $\varepsilon = 10^{-4}$.

Problem 1: We consider a general Fisher market equilibrium problem, where

$$A = \begin{pmatrix} 2 & 5 & 1 & 8 & 2 & 1 & 8 \\ 9 & 9 & 9 & 8 & -8 & 2 & 5 \\ 1 & 0 & 0 & 0 & -4 & 1 & 5 \\ 8 & -4 & 7 & 3 & 9 & -1 & 1 \end{pmatrix}$$

is a 4×7-dimensional full row-rank matrix, and $b = \begin{pmatrix} 25 & 25 & 34 & 17 \end{pmatrix}^T \in \mathbb{R}^4$ with a weight vector $w = \boldsymbol{e} \in \mathbb{R}^7$.

We can find a strictly feasible initial point

$$\begin{aligned} x^0 &= \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{pmatrix}^T, \\ y^0 &= \begin{pmatrix} 1 & 5 & -3 & 3 \end{pmatrix}^T, \\ s^0 &= A^T y^0. \end{aligned}$$

Set the update parameter $\theta = 0.1$. Our algorithm finds an ε-approximate solution of the Problem 1

$$\begin{aligned} x^* &= \begin{pmatrix} 1.54 & 0.85 & 1.46 & 0.68 & 3.04 & 10.60 & 2.00 \end{pmatrix}^T, \\ s^* &= \begin{pmatrix} 1.06 & 1.45 & 0.43 & 1.40 & 0.38 & 0.35 & 0.28 \end{pmatrix}^T, \\ y^* &= \begin{pmatrix} 0.12 & 0.06 & -0.15 & 0 \end{pmatrix}^T, \end{aligned}$$

It took 0.0004596 seconds and 14 iterations.

Problem 2: We consider another general Fisher market equilibrium problem, where

$$A = \begin{pmatrix} 8 & 1 & 13 & 0 & -8 & 5 & 7 & 5 & 3 & 4 \\ 6 & 2 & 9 & -1 & 0 & 2 & 0 & -5 & 9 & 2 \\ 3 & 4 & 3 & 9 & 0 & 7 & 9 & 8 & 8 & 8 \\ 5 & 0 & 1 & -9 & 1 & 1 & 7 & 7 & 5 & 1 \\ 4 & 9 & 7 & 5 & 6 & 6 & 4 & 6 & 6 & 2 \\ 0 & 9 & -3 & 0 & 7 & 1 & 4 & 3 & 5 & 1 \\ 2 & 4 & 2 & 2 & 6 & 3 & -4 & 8 & 2 & 2 \\ 1 & 4 & -4 & 3 & 4 & 6 & 3 & 5 & 3 & 4 \end{pmatrix},$$

and $b = \begin{pmatrix} 15 & 2 & 26 & 18 & 2 & 26 & 15 & 4 \end{pmatrix}^T$.

The feasible initial point and weight vector are selected as follows:

$$\begin{aligned} x^0 &= \begin{pmatrix} 1 & 2 & 1 & 1 & 4 & 5 & 2 & 3 & 2 & 4 \end{pmatrix}^T, \\ y^0 &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T, \\ s^0 &= \begin{pmatrix} 4 & 6 & 10 & 18 & 2 & 7 & 6 & 8 & 5 & 1 \end{pmatrix}^T, \\ w &= \tfrac{x^0 s^0}{2}. \end{aligned}$$

Problem 2 was solved by Algorithm 1 and the solution was obtained as

$$\begin{aligned} x^* &= \begin{pmatrix} 0.76 & 1.85 & 1.38 & 0.77 & 4.59 & 4.86 & 2.25 & 2.74 & 1.47 & 4.90 \end{pmatrix}^T, \\ s^* &= \begin{pmatrix} 2.61 & 3.24 & 3.63 & 11.74 & 0.87 & 3.60 & 2.660 & 4.38 & 3.40 & 0.41 \end{pmatrix}^T, \\ y^* &= \begin{pmatrix} 1.18 & 1.25 & 0.73 & 0.98 & -0.07 & 1.53 & 1.24 & 1.38 \end{pmatrix}^T. \end{aligned}$$

The algorithm was iterated 11 times, and the computation time is 0.0003172 seconds.

TABLE 1. Numerical results of Problem 3.

DIM	$\theta = 0.05$		$\theta = 0.075$		$\theta = 0.1$		$\theta = 0.125$		$\theta = 0.15$	
	CPU/s	Iter	CPU/s	Iter	CPU/s	Iter	CPU/s	Iter	CPU/s	Iter
6×12	0.00086	13	0.00040	10	0.00047	8	0.00032	8	0.00036	8
16×72	0.00913	13	0.0021	10	0.00188	9	0.00194	9	0.00243	11
26×182	0.02676	14	0.02054	10	0.01891	10	0.02053	10	0.02614	13
36×342	0.09929	14	0.08980	10	0.07056	10	0.08730	11	0.11583	16
46×522	0.32168	14	0.26993	11	0.24201	11	0.28945	12	0.41728	19

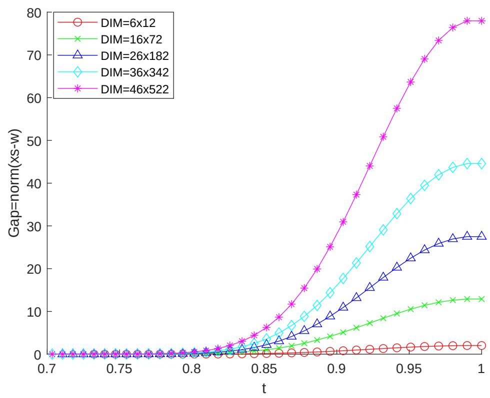

FIGURE 1. The value of Gap

Problem 3: We solve five random general Fisher market equilibrium problems, where the problem dimension is $m \times n$ with $m \in \{6, 16, 26, 36, 46\}$, $n = \frac{m(m/2+1)}{2}$, and $n \in \{12, 72, 182, 342, 522\}$.

We randomly generate a coefficient matrix $A = rand(m,n)$ and a vector $x^0 > 0$. Let $b = Ax^0$ and $s^0 = A^T y^0 > 0$. Thus we obtain a strictly feasible initial point $\left(x^0, y^0, s^0\right)$. To satisfy the condition of Lemma 3.7, we take the weight vector as $w = \frac{x^0 s^0}{2}$.

The dimension of Problem 3 is denoted by "DIM", and the running time (in seconds) and the iterations of our algorithm for Problem 3 are denoted by "CPU" and "Iter", respectively. Let $\theta \in \{0.05, 0.075, 0.1, 0.125, 0.15\}$. The corresponding numerical results of solving Problem 3 by Algorithm 1 are shown in Table 1. Fig. 1 and Fig. 2 show the variation of Gap and $\delta(v)$ with $\theta = 0.01$.

According to Table 1, when the value of θ is constant, the number of iterations and the computation time required by Algorithm 1 increase with the dimension of the problem, but the change is not significant. When the dimension of the general Fisher market equilibrium problem is given, the number of iterations and computation time required by Algorithm 1 do not increase as θ becomes larger.

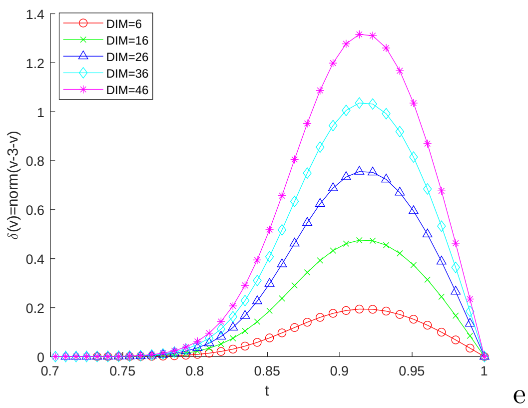

FIGURE 2. The value of $\delta(v)$

Instead, the time needed for the computation of Algorithm 1 and the number of iterations decrease as θ approaches 0.1.

According to Fig. 1 and Fig. 2, $\delta(v)$ first increases to a great value, and then decreases to 0. Gap gradually decreases to 0. In the middle of the change, Gap decreases at a larger rate, while in the early and late stages of the change, Gap decreases at a slower rate, when t tends to 0.

In a word, Algorithm 1 could efficiently solve the general Fisher market equilibrium problem.

5. CONCLUSION

In this paper, we propose a full-Newton step feasible IPM to solve the wLCP model of the general Fisher market equilibrium problem. Based on the kernel function

$$\psi(t) = \frac{1}{2}\left(t - \frac{1}{t}\right)^2,$$

we obtain a new search direction. We prove the strict feasibility and the polynomial complexity of Algorithm 1 for solving the general Fisher market equilibrium problem. The effectiveness of the algorithm is illustrated by numerical examples.

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