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OPTIMALITY CONDITIONS AND DUALITY FOR ROBUST NONSMOOTH MULTIOBJECTIVE SEMI-INFINITE OPTIMIZATION PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. In this paper, we consider an uncertain non-smooth multiobjective semi-infinite programming problem (abbreviated as, UNMSIP) and its robust counterpart, namely, a robust non-smooth multiobjective semi-infinite programming problem (abbreviated as, RNMSIP) in the framework of Hadamard manifolds. To derive Karush-Kuhn-Tucker (abbreviated as, KKT)-type necessary optimality criteria for a local robust weak Pareto solution and a local robust Benson-proper solution to the problem RNMSIP, we introduce the notion of the Abadie constraint qualification (abbreviated as, ACQ) for the considered problem in terms of Clarke subdifferential on Hadamard manifolds. Moreover, under the appropriate assumptions of geodesic convexity on the Hadamard manifold, the necessary optimality criteria will be transformed into sufficient optimality criteria for the considered problem. To provide the applications of the derived results in this paper, we formulate Mond-Weir and Wolfe type dual models for the primal problem RNMSIP and establish several duality results by employing the concept of geodesic convexity. Furthermore, the paper offers numerous significant examples to demonstrate the validity of the established results. To the best of our knowledge, optimality criteria and duality results for RNMSIP in the context of Hadamard manifolds have not been studied yet.

1. Introduction

In the theory of optimization, multiobjective semi-infinite programming problems (abbreviated as, MSIPs) represent a class of mathematical optimization problems, characterized by a finite collection of decision variables, an infinite number of constraints, and a finite number of objective functions that are simultaneously minimized. Notably, when the number of objective functions reduces to one, then MSIPs are transformed into semi-infinite programming problems (abbreviated as, SIPs). The origin of semi-infinite programming can be attributed to Haar [17], who introduced its foundational concepts. Subsequently, in 1962, the term 'semi-infinite programming' was introduced by Charnes et al. [7]. Since then, this mathematical concept has found diverse applications in various real-world fields such as digital filter design [32], air pollution control [61], lapidary cutting problems [62], statistical design [18]. As a consequence of its significant applications and growing interest, SIPs have emerged as a prominent and actively researched area among scholars. Kanzi and Nobakhtian [24] discussed several constraint qualifications for MSIPs and derived necessary and sufficient optimality criteria for the considered problems.

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Kanzi and Nobakhtian [25] have derived Fritz–John and KKT-type necessary optimality conditions for non-smooth SIPs with mixed constraints. Necessary optimality conditions for non-smooth generalized SIPs have been established by Kanzi and Nobakhtian [26]. Furthermore, Kanzi and Nobakhtian [27] derived optimality conditions for non-smooth SIPs. The strong KKT necessary and sufficient optimality conditions for a nondifferentiable multiobjective semi-infinite optimization problem have been tremendously discussed by Kanzi [28] under invexity assumptions.

Over the last two decades, robust optimization (abbreviated as, RO) has emerged as a prominent and extensively studied approach in the fields of optimization and operations research, particularly for decision-making under uncertain conditions. The fundamental premise of robust optimization involves addressing situations where the exact values of certain parameters are uncertain. To handle this uncertainty, RO employs an "uncertainty set" that encompasses all potential realizations of these uncertain parameters. Rather than relying on specific parameter values, the focus is on finding a solution that remains feasible under any possible scenario within the uncertainty set. The ultimate objective of RO is to achieve the best possible outcome for a given objective function, considering the worst-case scenario within the uncertainty set. In other words, the goal is to devise a solution that performs optimally regardless of the specific realization of uncertain parameters. It has gained widespread popularity across various domains, including engineering, finance, economics, and healthcare, see, for example, [3, 6, 10, 13] and the references cited therein.

In the realm of mathematical optimization theory, the principle of duality asserts that optimization problems can be approached from one of two perspectives: primal or dual. By leveraging the dual problem associated with the primal problem, it is often feasible to analytically determine the solution of the primal problem. Moreover, duality theory serves as a source of interesting interpretations, which can form the basis of efficient and distributed solution methods. Furthermore, owing to the intrinsic uncertainty inherent in real-world data, significant attention has been devoted to investigating optimization problems involving uncertain data. Optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems for smooth and non-smooth cases have been widely discussed by Lee and Lee [31] and Pham [36], respectively. Tung [47] investigated constraint qualifications for semi-infinite multiobjective optimization with data uncertainty in both the objective and constraints functions, as well as derived optimality criteria for the considered problem under the assumption of generalized convexity. Constraint qualifications and optimality criteria for non-smooth multiobjective semi-infinite optimization problems with data uncertainty are extensively discussed by Tung and Duy [46]. Chuong [8] studied optimality and duality for robust multiobjective optimization problems. For further insights and comprehensive discussions regarding optimality conditions and duality results within the framework of robust optimization, we refer to [4, 5, 9, 21, 22, 23, 43], and the references cited therein.

In recent advancements in mathematical programming theory, the investigation of optimization within manifold contexts has emerged as a highly interesting and active field of research. Rapcsák [40] and Udriste [49] have contributed significantly to the field of optimization by introducing a concept known as geodesic convexity, which extends the concept of convexity to the context of manifolds. In this setting, the idea of a line segment is replaced by a geodesic, and the conventional linear space is exchanged for a Riemannian manifold. A Riemannian manifold is referred to as a Hadamard manifold if it is simply connected, geodesic complete, and has a nonpositive sectional curvature throughout. Extending and generalizing optimization techniques from Euclidean spaces to manifolds offer several important benefits. Notably, this approach facilitates the convenient transformation of numerous nonconvex optimization problems into convex optimization problems by introducing appropriate Riemannian metrics (see [37, 38]). This transformation simplifies the optimization process and makes use of well-established convex optimization methods, which often lead to efficient solutions. Furthermore, adopting the perspective of Riemannian geometry enables the conversion of several complex constrained mathematical optimization problems into unconstrained optimization problems. Certain mathematical programming problems exhibit substantial constraints that inherently possess a relative interior, which can be understood and modeled as Hadamard manifolds. For example, the set of symmetric positive definite matrices $\mathbb{S}^2$ equipped with the metric given by the Hessian of the barrier $-\log \det Z$, the positive orthant $\mathbb{R}^n_{++}$ with the Dikin metric $Z^{-2} = \text{diag}(1/z_1^2, 1/z_2^2, ..., 1/z_n^2)$, and the cone $C := \{(\sigma, w) \in \mathbb{R}^{1+n} : \sigma > \|w\|\}$ endowed with the Hessian of the barrier $-\ln(\sigma^2 - \|w\|^2)$, are Hadamard manifolds (see, for instance, [39]). Recently, numerous authors have expanded a wide range of exclusive concepts and intriguing ideas of optimization from the context of Euclidean spaces to Riemannian and Hadamard manifolds, see, for instance, [14, 15, 44, 45, 50, 51, 52, 53, 54, 56, 57, 58, 60] and the references cited therein.

It is worthwhile to significant that in sharp contrast to Euclidean spaces, manifolds, in general, are not equipped with a linear structure, though globally diffeomorphic. Consequently, despite being globally homeomorphic to Euclidean spaces, the development of optimization methods within the framework of Hadamard manifolds is accompanied by various challenges. For instance, in sharp contrast to the Euclidean space setting, the concept of a unique line segment joining any two points is not possible in the manifold setting. Additionally, on Hadamard manifolds, both the exponential map and its inverse are nonlinear (see, for instance, [30]). Therefore, several researchers have developed new techniques over the past few decades to explore and solve optimization problems in manifold settings. For example, in manifold settings, the notion of geodesic convexity is introduced, and the concept of a unique minimal geodesic is employed to join any two points in the Hadamard manifold. Furthermore, in order to overcome the nonlinearity of manifolds, the notions of parallel transport and exponential on the tangent space of a Hadamard manifold (which has a vector space structure) are used.

The primary motivation and objective for studying RNMSIP in the Hadamard manifold setting is fourfold. Firstly, it is worth mentioning that several influential real-life problems arising in the fields of engineering, statistics, and design problems for instance, lapidary cutting problems (see, [62]), satistical design problems (see, [18]) can be modeled as a MSIP. The nonconvex constrained MSIP can be appropriately converted into unconstrained and convex MSIP by leveraging the structure of Hadamard manifolds (see, for instance, [37, 38]). Secondly, dealing with uncertainty poses a significant challenge in multiobjective semi-infinite programming problems. To the best of our knowledge, no techniques, methods, or theories have been developed to deal with MSIPs formulated in manifold setting that has data uncertainty associated with them. Therefore, robust multiobjective semi-infinite programming problems emerge as an evaluative and actively researched field, aiming to provide solutions in the face of such uncertainties within the context of the Hadamard manifold. Thirdly, it is widely acknowledged that non-smoothness is prevalent in various real-life optimization problems. Nonsmooth optimization problems have been extensively studied in Hadamard manifolds (see [1, 59]). Concerning robust MSIPs, it is crucial to recognize that the associated data is typically nonsmooth. To address this challenge, we explicitly account for the nonsmooth nature of the data. Fourthly, there are several research papers available in the literature that deal with the investigation of optimality criteria and duality results for multiobjective optimization problems, MSIPs, and SIPs, see, for instance, [8, 16, 27, 31, 46] (for Euclidean spaces), [2, 48, 55, 59] (for Hadamard manifolds). However, it is significant to observe that, optimality conditions and duality results for robust nonsmooth multiobjective semi-infinite programming problems on Hadamard manifolds have not been explored yet. Our aim is to fill this particular research gap by deriving necessary and sufficient optimality criteria for RNMSIP by employing the notions of ACQ and geodesic convexity, respectively, and deriving several duality results for the primal problem RNMSIP.

Motivated by the works of [8, 24, 27, 31, 46, 48, 59], in this paper, we consider an uncertain non-smooth MSIP and its corresponding robust counterpart, namely, a robust non-smooth MSIP on Hadamard manifolds. By employing ACQ, we derive KKT-type necessary optimality criteria for a local robust weak Pareto solution and a local robust Benson-proper solution of the problem RNMSIP in terms of Clarke subdifferentials. Moreover, we establish robust sufficient optimality conditions under the assumptions of geodesic convexity. Furthermore, we deduce Mond-Weir and Wolfe type duality results for the primal problem RNMSIP. Numerous non-trivial examples are furnished in the Hadamard manifold setting to demonstrate the validity of the established result.

The novelty and contributions of the present paper are fivefold. Firstly, the established results of this paper generalize the analogous results studied in [8, 24, 27, 31, 46] from the Euclidean space setting to the framework of Hadamard manifolds. Secondly, the findings of this paper extend the corresponding findings derived in [24, 27, 48, 59] in the domain of robust optimization. Thirdly, the outcomes in this paper extend the analogous outcomes in [27] from single objective nonsmooth SIPs to nonsmooth MSIPs within the framework of robust optimization. Fourthly, the results derived in this paper extend the analogous results derived in [31] (considering data uncertainty in the Euclidean space setting), [48] (concerning the lack of data uncertainty within the Hadamard manifold setting) from smooth MSIPs to non-smooth MSIPs. Fifthly, our paper extends the results derived in [8] from multiobjective optimization problems to MSIPs.

The paper is organized as follows: In Section 2, we revisit some fundamental concepts and definitions which will be employed throughout the subsequent parts of the paper. In Section 3, we consider UNMSIP and its robust counterpart, RNMSIP on the Hadamard manifold setting. By employing the notions of ACQ and geodesic convexity, we establish the necessary and sufficient optimality criteria for RNMSIP. In Section 4, we formulate Mond-Weir and Wolfe type dual models for the considered problem and derive several duality results. In Section 5, we give our conclusions as well as future research directions.

2. Notation and mathematical preliminaries

The conventional notation $\mathbb{R}^n$ represents the n-dimensional Euclidean space, while the symbol $\mathbb{N}$ is employed to signify the set of all natural numbers. The standard inner product on the Euclidean space $\mathbb{R}^n$ is represented by the symbol $\langle \cdot, \cdot \rangle$. The symbol $\emptyset$ is utilized to signify the empty set. Let γ and ν be two arbitrary elements of $\mathbb{R}^n$. We will employ the following notations in the sequel.

$$\gamma \prec \nu \iff \gamma_k < \nu_k, \ \forall k = 1, \ldots, n.$$

$$\gamma \preceq \nu \iff \begin{cases} \gamma_k \leq \nu_k, \ \forall k = 1, \ldots, n, \\ \gamma_s < \nu_s, \quad \text{for at least one } s \in \{1, 2, \ldots, n\}. \end{cases}$$

We use $\gamma \nprec \nu$ (respectively, $\gamma \npreceq \nu$) to indicate the negation of $\gamma \prec \nu$ (respectively, $\gamma \preceq \nu$).

For any infinite set I, the symbol $\mathbb{R}_+^{|I|}$ represents the set of functions $\beta : I \to \mathbb{R}_+$, where these functions take positive values β_i only at a finite number of points within I and are equal to zero at all other points of I. For any arbitrary non-empty set $C \subset \mathbb{R}^n$, $\operatorname{int} C$, $\operatorname{cl}C$, and $\operatorname{co}C$ will denote the topological interior, closure, and convex hull of C, respectively. The cone and the convex cone (containing the origin) generated by $C \subset \mathbb{R}^n$ are denoted by the symbols $\operatorname{cone}C$ and $\operatorname{pos}C$, respectively.

The negative and strictly negative polar cones of $C(\subset \mathbb{R}^n)$ are denoted by the symbols $C^{\leqq}$ and $C^{<}$, respectively, and are defined in the following way:

$$C^{\leqq} := \{\gamma \in \mathbb{R}^n : \langle \gamma, \nu \rangle \leq 0, \ \forall \nu \in C\},$$

$$C^{<} := \{\gamma \in \mathbb{R}^n : \langle \gamma, \nu \rangle < 0, \ \forall \nu \in C\}.$$

From the bipolar theorem, we have $(C^{\leqq})^{\leqq} = \operatorname{cl}(\operatorname{pos}C)$.

For $n \in \mathbb{N}$, we use the notation $\mathscr{H}$ to represent n-dimensional smooth manifold. For any $z \in \mathscr{H}$, the tangent space at z is represented by the symbol $T_z\mathscr{H}$, which is a linear space of dimension n. In the case of real manifolds, $T_z\mathscr{H}$ is isomorphic to $\mathbb{R}^n$. On a smooth manifold $\mathscr{H}$, the notation $\mathscr{G}$, denotes the Riemannian metric which is a 2-tensor field, symmetric, and positive-definite. For any two elements $p_1, \ p_2 \in T_z\mathscr{H}$, the inner product of p_1 and p_2 is given by: $\langle p_1, p_2 \rangle_z = \mathscr{G}_z(p_1, p_2)$, where $\mathscr{G}_z$ represents the Riemannian metric at the point $z \in \mathscr{H}$. The norm associated with the inner product $\langle p_1, p_2 \rangle_z$ is represented by the symbol $\| \cdot \|_z$. A smooth manifold endowed with a Riemannian metric is termed a Riemannian manifold. For any $u \in \mathscr{H}$ and $\nu \in T_u\mathscr{H}$, $\Gamma_{u,\nu}$ be the geodesic, starting at the point u with velocity ν, and the exponential map $\exp_u : T_u\mathscr{H} \to \mathscr{H}$ is defined as $\exp_u(\nu) = \Gamma_{u,\nu}(1)$.

A Riemannian manifold $\mathscr{H}$ is called geodesically complete, if the exponential map $\exp_u(\nu)$ is defined for all $\nu \in T_u\mathscr{H}$ and for any $u \in \mathscr{H}$. A Riemannian manifold is known as a Cartan-Hadamard manifold, or simply a Hadamard manifold, if it satisfies the conditions of being complete, simply connected, and having nonpositive sectional curvature everywhere. It is notable that, if $\mathscr{H}$ is a Hadamard manifold, then for any $u \in \mathscr{H}$, the exponential map $\exp_u(\nu) : T_u\mathscr{H} \to \mathscr{H}$ is a diffeomorphism, and the inverse exponential map $\exp_u^{-1} : \mathscr{H} \to T_u\mathscr{H}$ satisfying $\exp_u^{-1}(u) = 0_u$. Furthermore, for any element $s \in \mathscr{H}$, there exists a unique minimal geodesic denoted as $\Gamma_{u,s} : [0,1] \to \mathscr{H}$, that satisfies the property $\Gamma_{u,s}(\mu) = \exp_u(\mu \exp_u^{-1}(s))$. The gradient of a differentiable function $\mathcal{R} : \mathscr{H} \to \mathbb{R}$, which is symbolized by the notation grad $\mathcal{R}$, is a vector field on $\mathscr{H}$, and it is defined in such a way that $d\mathcal{R}(Z) = \langle \text{grad } \mathcal{R}, Z\rangle = Z(\mathcal{R})$, where Z is also a vector field on the manifold $\mathscr{H}$. From now onwards, unless explicitly stated otherwise, we will use the notation $\mathscr{H}$ to represent a Hadamard manifold of dimension n (where n is a natural number).

For any $u \in \mathscr{H}$, and $C \subset T_u\mathscr{H}$, the negative and strictly negative polar cone of the set C are denoted by $C^{\leq}$ and $C^{<}$, respectively, and are defined in the following manner:

$$C^{\leq} := \{q \in T_u\mathscr{H} : \langle q, t\rangle_u \leq 0, \ \forall t \in C\},$$

$$C^{<} := \{q \in T_u\mathscr{H} : \langle q, t\rangle_u < 0, \ \forall t \in C\}.$$

Now, we recall some fundamental definitions and results related to non-smooth analysis from [1, 48, 49, 59].

Definition 2.1. The real-valued function $\Theta : \mathscr{H} \to \mathbb{R}$ is said to be locally Lipschitz near $z \in \mathscr{H}$ with rank $L > 0$, if the following inequality is satisfied:

$$|\Theta(\gamma) - \Theta(\nu)| \leq L \ \text{dis}(\gamma, \nu),$$

for any $\gamma, \ \nu$, which are lying in the neighbourhood of z. If Θ is locally Lipschitz near any point $z \in \mathscr{H}$, then Θ is locally Lipschitz on $\mathscr{H}$.

If $\Psi : \mathscr{H} \rightrightarrows T_\wp\mathscr{H}$, $\wp \in \mathscr{H}$, is a set-valued map, then graph of Ψ is defined as follows:

$$\text{gph}\Psi := \{(\wp, z) \in \mathscr{H} \times T_\wp\mathscr{H} \,|\, z \in \Psi(\wp)\}.$$

Definition 2.2. A set $\mathcal{B} \subset \mathscr{H}$ is referred to as geodesic convex set in $\mathscr{H}$, if for every $z, \wp \in \mathcal{B}$, with $z \neq \wp$ and for any geodesic $\Gamma_{z,\wp} : [0,1] \to \mathscr{H}$ connecting the points z and $\wp$, such that $\Gamma_{z,\wp}(\mu) \in \mathcal{B}$, $\forall \mu \in [0,1]$, where $\Gamma_{z,\wp}(\mu) = \exp_z(\mu \exp_z^{-1} \wp)$.

The following definitions provide the notions of geodesic and strictly geodesic convex functions in the framework of the Hadamard manifolds.

Definition 2.3. Let $\Theta : \mathcal{B} \to \mathbb{R}$ be any real-valued locally Lipschitz function, defined on the geodesic convex subset $\mathcal{B}$ of $\mathscr{H}$. Then,

(1) Θ is said to be geodesic convex, provided the following inequality satisfied

$$\Theta(\exp_\wp(\mu \exp_\wp^{-1} z)) \leq \mu\Theta(z) + (1-\mu)\Theta(\wp),$$

for any $z, \wp \in \mathcal{B}$, $\mu \in [0,1]$.

(2) Θ is said to be strictly geodesic convex provided the above inequality holds strictly, for any $z, \wp \in \mathcal{B}$ ($z \neq \wp$) and $\mu \in [0,1]$.

The following definition gives the notion of a generalized directional derivative of a real-valued locally Lipschitz function in the setting of Hadamard manifolds.

Definition 2.4. Let $z, \wp \in \mathscr{H}$ and $\Theta : \mathscr{H} \to \mathbb{R}$ be real-valued locally Lipschitz function. Then, the symbol $\Theta^\circ(\wp; \nu)$ represents the generalized directional derivative of Θ at $\wp$ in the direction $\nu \in T_\wp\mathscr{H}$, is defined as follows:

$$\Theta^\circ(\wp; \nu) = \limsup_{z \to \wp, \mu \downarrow 0} \frac{\Theta\left(\exp_z \mu (d \exp_\wp)_{\exp_\wp^{-1} z} \nu\right) - \Theta(z)}{\mu},$$

where the differential of the exponential function at the point $\exp_\wp^{-1} z$ is given by, $(d \exp_\wp)_{\exp_\wp^{-1} z} : T_{\exp_\wp^{-1} z}(T_\wp\mathscr{H}) \simeq T_\wp\mathscr{H} \to T_z\mathscr{H}$.

Remark 2.5. (1) By considering, $0_\wp \in T_\wp\mathscr{H}$, we have

$$\Theta^\circ(\wp; \nu) = (\Theta \circ \exp_\wp)^\circ(0_\wp, \nu), \text{ (see, for instance, [20])}.$$

(2) For a locally Lipschitz function $\Theta : \mathscr{H} \to \mathbb{R}$ of rank L, and for an open neighborhood $\mathcal{U}_z$ of z, we have the following assertion. (see, for instance, [20])

For every $\wp \in \mathcal{U}_z$, the function $\nu \mapsto \Theta^\circ(\wp; \nu)$ is finite, positive homogeneous, and subadditive on $T_\wp\mathscr{H}$ and satisfies the following property:

$$|\Theta^\circ(\wp; \nu)| \le L\|\nu\|_\wp, \quad \forall \nu \in T_\wp\mathscr{H}.$$

(3) For a given geodesic convex function $\Theta : \mathscr{H} \to \mathbb{R}$, the symbol $\Theta'(\wp; \nu)$, represents the directional derivative of Θ at $\wp$ in the direction $\nu \in T_\wp\mathscr{H}$ and is given by (see, for instance, [1])

$$\Theta'(\wp; \nu) := \lim_{\mu \to 0} \frac{\Theta(\exp_\wp \mu\nu) - \Theta(\wp)}{\mu}.$$

Notably, if Θ is a geodesic convex, then $\Theta'(\wp; \nu) = \Theta^\circ(\wp; \nu)$, $\forall \nu \in T_\wp\mathscr{H}$.

In the Hadamard manifold setting, the notion of the generalized gradient of a real-valued locally Lipschitz function is represented by the following definition.

Definition 2.6. For a real-valued locally Lipschitz function $\Theta : \mathscr{H} \to \mathbb{R}$, the generalized gradient (also known as the Clarke subdifferential) of Θ at $\wp \in \mathscr{H}$ is a subset of the tangent space $T_\wp\mathscr{H}$, denoted by $\partial_C\Theta(\wp)$, is defined as follows:

$$\partial_C\Theta(\wp) := \{\xi \in T_\wp\mathscr{H} \mid \Theta^\circ(\wp; \nu) \ge \langle \xi, \nu \rangle_\wp, \ \forall \nu \in T_\wp\mathscr{H}\}.$$

Remark 2.7. For a geodesic convex function $\Theta : \mathscr{H} \to \mathbb{R}$, the subdifferential of Θ at $\wp \in \mathscr{H}$ is defined by (see, for instance, [1])

$$\begin{aligned}\partial\Theta(\wp) : &= \{\xi \in T_\wp\mathscr{H} : \langle \xi, \exp_\wp^{-1} \acute{x} \rangle_\wp \le \Theta(\acute{x}) - \Theta(\wp) \ \forall \acute{x} \in \mathscr{H}\} \\ &= \{\xi \in T_\wp\mathscr{H} : \langle \xi, \nu \rangle_\wp \le \Theta'(\wp; \nu), \ \forall \nu \in T_\wp\mathscr{H}\}.\end{aligned}$$

Moreover, in the case, when Θ is geodesic convex, then $\partial\Theta(\wp) = \partial_C\Theta(\wp)$.

For a locally Lipschitz function, the following proposition from Barani [1] and Hosseini and Pouryayevali [20], illustrates the relationship between the generalized directional derivative and the generalized gradient, as well as presents the subdifferential rules in the framework of Hadamard manifolds.

Proposition 2.8. *Let $\Theta_1, \ \Theta_2 : \mathscr{H} \to \mathbb{R}$ be real-valued functions which are locally Lipschitz near the point $\wp \in \mathscr{H}$ and $\nu \in T_\wp\mathscr{H}$. Then,*

(1) $\Theta^{\circ}(\wp;\nu) = \max_{\xi\in\partial\Theta(\wp)}\langle\xi,\nu\rangle_{\wp}$.
(2) $\partial_C\Theta(\wp)$ *is non-empty geodesic convex, compact subset of* $T_{\wp}\mathscr{H}$ *and is upper semi-continuous on* $\mathscr{H}$.
(3) $\partial_C(\lambda\Theta)(\wp) = \lambda\partial_C\Theta(\wp)$ *for* $\lambda \in \mathbb{R}$.
(4) $\partial_C(\Theta_1+\Theta_2)(\wp) \subset \partial_C\Theta_1(\wp) + \partial_C\Theta_2(\wp)$.
(5) *If* Θ *has a local minima at the point* $\wp \in \mathscr{H}$, *then* $0 \in \partial_C\Theta(\wp)$.

The following definition of geodesic convexity in the notion of Clarke subdifferential is from [49, 59].

Definition 2.9. Let $\Theta : \mathcal{B} \to \mathbb{R}$ be a real-valued locally Lipschitz function defined on the geodesic convex subset $\mathcal{B}$ of $\mathscr{H}$. Then, Θ is considered as a geodesic convex function at the point $\wp$, if for each $\acute{x} \in \mathcal{B}$ and $\xi \in \partial_C\Theta(\wp)$, the following inequality holds:

$$\Theta(\acute{x}) - \Theta(\wp) \geq \langle\xi, \exp_{\wp}^{-1}\acute{x}\rangle_{\wp}.$$

Likewise, Θ is said to possess strictly geodesic convexity at the point $\wp$, if for every $\acute{x} \in \mathcal{B}$, with $\acute{x} \neq \wp$, such that $\Theta(\acute{x}) - \Theta(\wp) > \langle\xi, \exp_{\wp}^{-1}\acute{x}\rangle_{\wp}$, holds true for every $\xi \in \partial_C\Theta(\wp)$.

The following Lebourg's mean value theorem in the setting of Hadamard manifolds is from [20, 59].

Theorem 2.10 (Lebourg's Mean Value Theorem)**.** *Let* $\Theta : \mathscr{H} \to \mathbb{R}$ *be a real-valued locally Lipschitz function. Then, for any pair of points* $\acute{x}, \wp \in \mathscr{H}$, *there are always points* $t_0 \in (0,1)$ *and* $z_0 = \Gamma(t_0)$, *such that*

$$\Theta(\wp) - \Theta(\acute{x}) \in \langle\partial_C\Theta(z_0), \Gamma'(t_0)\rangle_{z_0},$$

where $\Gamma(\mu) := exp_{\wp}(\mu exp_{\wp}^{-1}\acute{x})$ *and* $\mu \in [0,1]$.

Remark 2.11. Let us consider the real-valued locally Lipschitz function $\Theta : T_u\mathscr{H} \to \mathbb{R}$ defined on the tangent space $T_u\mathscr{H}$ at the point $u \in \mathscr{H}$. Then, for any pair of points $\acute{x}, \wp \in T_u\mathscr{H}$, there is a point z_0 lies within the open line segment $(\acute{x}, \wp) := \{t_0\acute{x} + (1-t_0)\wp \mid 0 < t_0 < 1\}$, such that $\Theta(\wp) - \Theta(\acute{x}) \in \langle\partial_C\Theta(z_0), \wp - \acute{x}\rangle_{z_0}$.

The existence of an isometry that establishes a correspondence between $\mathbb{R}^n$ and an n-dimensional Euclidean space $\mathbb{E}$ is widely recognized. Unless explicitly stated otherwise, the symbol $\mathbb{E}$ represents any n-dimensional Euclidean space.

The proof of the following theorems can be verified in a manner analogous to that presented in the Euclidean space setting, see, for instance, [19, 41, 48, 59, 63].

Lemma 2.12. *Consider an arbitrary collection of non-empty convex sets* $\{P_i | i \in I\}$ *in* $\mathbb{E}$. *Let* C *be the convex cone generated by the union of this collection. Then, every nonzero vector in* C *can be represented as a non-negative linear combination of* n *or fewer linearly independent vectors, with each vector belonging to a distinct* P_i *in the collection.*

Lemma 2.13. *Let us consider two arbitrary index sets* $\mathcal{I}$ *and* $\mathcal{J}$, *which can be finite or infinite. Further, let* $b_i : \mathcal{I} \to \mathbb{E}$ *and* $b_j : \mathcal{J} \to \mathbb{E}$ *be two maps that can be defined in the following way:*

$$b_i = b(i) = (b_1(i), \ldots, b_n(i)),$$

$$b_j = b(j) = (b_1(j), \ldots, b_n(j)).$$

Moreover, if the set $co\{b_i, i \in I\} + pos\{b_j : j \in J\}$ *is closed, then we have the following equivalent statements:*
Statement I: The following inclusion is satisfied

$$-co\{b_i, i \in I\} \in pos\{b_j : j \in J\}.$$

Statement II : The following system

$$\langle b_i, \acute{x} \rangle < 0, \ i \in \mathcal{I}, \ \mathcal{I} \neq \emptyset,$$
$$\langle b_j, \acute{x} \rangle \leq 0 \ j \in \mathcal{J},$$

has no solution $\acute{x} \in \mathbb{E}$.

Lemma 2.14. *For any non-empty set* $\mathcal{B} \subset \mathbb{E}$*, which is closed as well as convex, then for any* $x \notin \mathcal{B}$*, there exists* $p \in \mathbb{E}$ *such that*

$$\langle p, x \rangle > \sup_{b \in \mathcal{B}} \langle p, b \rangle.$$

3. Optimality conditions for robust nonsmooth multiobjective semi-infinite programming problems

In this section, an uncertain non-smooth multiobjective semi-infinite programming problem and its corresponding robust counterpart, namely, a robust nonsmooth multiobjective semi-infinite programming problem, are considered. Moreover, we derive Karush-Kuhn-Tucker type robust necessary optimality conditions for RNMSIP by employing the Abadie constraint qualification. Furthermore, we establish robust sufficient optimality conditions for RNMSIP under the assumptions of geodesic convexity.

Now, we consider the following uncertain non-smooth multiobjective semi-infinite programming problem:

$$\text{UNMSIP: } \min \ \Theta(z)$$
$$\text{subject to } h_i(z, w_i) \leq 0, \ \forall i \in \mathcal{I},$$

where $\Theta(z) : \mathscr{H}_1 \to \mathbb{R}^r$, is locally Lipschitz function and the constraint functions $h_i : \mathscr{H}_1 \times \mathscr{W}_i \to \mathbb{R}$, $i \in \mathcal{I}$ are given and $w_i \in \mathscr{W}_i \subset \mathscr{H}_2$ are uncertain parameters. Here, $\mathscr{H}_1$, $\mathscr{H}_2$ are Hadamard manifolds of dimensions n_1 and n_2, respectively ($n_1, n_2 \in \mathbb{N}$). The index set $\mathcal{I}$ is assumed to be non-empty and may be infinite. The uncertainty mapping $\mathscr{W} : \mathcal{I} \rightrightarrows \mathscr{H}_2$ can be defined as $\mathscr{W}(i) := \mathscr{W}_i$.

The associated robust counterpart for UNMSIP can be formulated in the following manner:

$$\text{RNMSIP: } \min \ \Theta(z)$$
$$\text{subject to } h_i(z, w_i) \leq 0, \ \forall w_i \in \mathscr{W}_i, \ \forall i \in \mathcal{I}.$$

Let, the set $\mathcal{F}$ is containing all feasible elements of the problem RNMSIP, equivalently,

$$\mathcal{F} := \{z \in \mathscr{H}_1 | \ h_i(z, w_i) \leq 0, \ \forall w_i \in \mathscr{W}_i, \ \forall i \in \mathcal{I}\},$$

and in the rest of this article, the set $\mathcal{F}$ is assumed to be non-empty.

The following definition introduces the notions of local robust weak Pareto, local robust Pareto, and local robust Benson-proper solutions for RNMSIP on Hadamard manifolds.

Definition 3.1. Let $\overline{z} \in \mathcal{F}$. Then,

(1) $\overline{z}$ is said to be a local robust weak Pareto solution, provided there exists a neighborhood $\mathcal{U}_{\overline{z}}$ of $\overline{z}$ such that
$$\Theta(z) \nprec \Theta(\overline{z}),$$
for all $z \in \mathcal{U}_{\overline{z}} \cap \mathcal{F}$.

(2) $\overline{z}$ is said to be a local robust Pareto solution, if there exists a neighborhood $\mathcal{U}_{\overline{z}}$ of $\overline{z}$ such that
$$\Theta(z) \npreceq \Theta(\overline{z}),$$
for all $z \in \mathcal{U}_{\overline{z}} \cap \mathcal{F}$.

(3) $\overline{z}$ is said to be a local robust Benson-proper solution, if there exists a neighborhood $\mathcal{U}_{\overline{z}}$ of $\overline{z}$ such that
$$\operatorname{clcone}(\Theta(\mathcal{U}_{\overline{z}} \cap \mathcal{F}) + \mathbb{R}^r_+ - \Theta(\overline{z})) \cap (-\mathbb{R}^r_+ \setminus \{0\}) = \emptyset.$$

Henceforth, we employ the notations LRWP(RNMSIP), LRP(RNMSIP), and LRBP(RNMSIP) to represent the sets containing all local robust weak Pareto solutions, local robust Pareto solutions, and local robust Benson-proper solutions, respectively.

Remark 3.2. (1) If $\mathcal{U}_{\overline{z}} = \mathscr{H}_1$, then the term local in Definition 3.1 can be omitted.

(2) From Definition 3.1, one has the following containments
$$\text{LRBP(RNMSIP)} \subset \text{LRP(RNMSIP)} \subset \text{LRWP(RNMSIP)}.$$

As a result, the necessary conditions for the term on the right are also applicable to the other terms and the sufficient conditions for the term on the left are also valid for the other terms.

Before moving further, we will study the following sets which will be used in our subsequent analysis.
For any $z \in \mathscr{H}_1$ and $\operatorname{gph}\mathscr{W} = \{(i, w) \in \mathcal{I} \times \mathscr{H}_2 : w \in \mathscr{W}(i)\}$, we define
$$\mathcal{K}(z) := \{(i, w) \in \operatorname{gph}\mathscr{W} \mid h_i(z, w) = 0\}, \ \mathcal{H}(z) := \sup_{(i,w)\in \operatorname{gph}\mathscr{W}} h_i(z, w),$$
$$\mathcal{R}(z) := \textstyle\bigcup_{(i,w)\in\mathcal{K}(z)} \partial^z_C h_i(z, w),$$
where $\partial^z_C h_i(z, w)$ denotes the Clarke's subdifferential of $h_i(z, w)$ with respect to z.

Remark 3.3. (1) In the rest of this paper, we assume that the set $\mathcal{K}(z)$ is always non-empty. If not, we can add an element (i', w') into $\operatorname{gph}\mathscr{W}$ for which $h_{i'}(z, w')$ is equivalent to 0 and one has $\mathcal{R}(z) = \{0\}$. Consequently, the condition $\mathcal{R}(z) \neq \emptyset$ is excluded from our consideration.

The following definition from [29] represents the notion of the contingent cone for a subset Υ in the framework of the Hadamard manifold.

Definition 3.4. Let $\Upsilon \subset \mathscr{H}_1$ and $z \in \operatorname{cl}\Upsilon$. Then, the contingent cone of Υ at the point z is defined as follows:

$$\mathscr{T}(\Upsilon, z) := \{u \in T_z\mathscr{H}_1 : \exists \nu_j \downarrow 0,\ \exists u_j \in T_z\mathscr{H}_1,\ u_j \to u,\ \forall j \in \mathbb{N},\ \exp_z(\nu_j u_j) \in \Upsilon\}.$$

In the following definition, we introduce the notion of Abadie constraint qualification at the feasible point of the problem RNMSIP in terms of Clarke subdifferential within the framework of Hadamard manifolds.

Definition 3.5. Let, $\overline{z} \in \mathcal{F}$. Then, the Abadie constraint qualification is fulfilled at $\overline{z}$, provided the following inclusion satisfies:

$$(\mathcal{R}(\overline{z}))^{\leq} \subset \mathscr{T}(\mathcal{F}, \overline{z}).$$

In the following theorem, we establish KKT-type robust necessary conditions for a local robust weak Pareto solution of the problem RNMSIP.

Theorem 3.6. *Let $\overline{z}$ be a local robust weak Pareto solution for RNMSIP and the set $pos\mathcal{R}(\overline{z})$ is closed. Furthermore, if the ACQ is satisfied at $\overline{z}$, then there exist multipliers $(\alpha_k)_k \in \mathbb{R}^r_+ \setminus \{0\}$, $(\mu_l)_l \in \mathbb{R}^n_+$, and $(i_l, w_l)_l \in (gph\mathscr{W})^n$, such that*

$$0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(\overline{z}) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(\overline{z}, w_l), \tag{3.1}$$

$$\mu_l h_{i_l}(\overline{z}, w_l) = 0,\ \forall l = 1, 2, \ldots, n. \tag{3.2}$$

Proof. To begin with, we define the set $\mathcal{C}(\subset T_{\overline{z}}\mathscr{H}_1)$ as follows:

$$\mathcal{C} := \{v \in \mathscr{T}(\mathcal{F}, \overline{z}) : \Theta_k^\circ(\overline{z}; v) < 0,\ \forall k \in \{1, 2, \ldots, r\}\}.$$

Now, our claim is to show that the set $\mathcal{C}$ is empty. Equivalently, the following system does not possess any solution

$$\begin{cases} \Theta_1^\circ(\overline{z}; v) < 0,\ \Theta_2^\circ(\overline{z}; v) < 0, \ldots, \Theta_r^\circ(\overline{z}; v) < 0, \\ v \in \mathscr{T}(\mathcal{F}, \overline{z}). \end{cases}$$

On the contrary, assume that $\mathcal{C} \neq \emptyset$. Then, there exists $v \in \mathscr{T}(\mathcal{F}, \overline{z})$, such that $\Theta_1^\circ(\overline{z}; v) < 0,\ \Theta_2^\circ(\overline{z}; v) < 0, \ldots, \Theta_r^\circ(\overline{z}; v) < 0$. Since $v \in \mathscr{T}(\mathcal{F}, \overline{z})$, there are sequences $\tau_j \downarrow 0$, and $v_j \to v$, such that $\exp_{\overline{z}}(\tau_j v_j) \in \mathcal{F}$. Now, for every $k = 1, 2, \ldots r$, and in light of Remark 2.11 there exists $z_j' \in (0, \tau_j v_j)$ and $a_{kj}' \in \partial_C(\Theta_k \circ \exp_{\overline{z}})(z_j')$ such that

$$\Theta_k \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta_k \circ \exp_{\overline{z}}(0) = \tau_j \langle a_{kj}', v_j \rangle_{\overline{z}}. \tag{3.3}$$

Let us define the set-valued mapping $\Phi : T_\nu\mathscr{H}_1 \rightrightarrows T_\nu\mathscr{H}_1$ as follows:

$$\Phi(\nu) := \partial_C(\Theta_k \circ \exp_\nu)(z)$$

The upper semi-continuity of the set-valued map Φ and taking into account the fact that $z_j' \to 0$, one has $a_{kj}' \to a_k' \in \Phi(0) = \partial_C(\Theta_k \circ \exp_{\overline{z}})(0)$. Moreover, from the continuity property of the inner product, one has $\langle a_{kj}', v_j \rangle_{\overline{z}} \to \langle a_k', v \rangle_{\overline{z}}$. Furthermore, we have

$$\partial_C(\Theta_k \circ \exp_{\overline{z}})(0) = \partial_C \Theta_k(\overline{z}),$$

(see, for instance, [20]). Since for each $k = 1, 2, \ldots, r$, $\Theta_k^\circ(\overline{z}; v) < 0$, therefore from Proposition 2.8, it follows that $\langle a_k', v \rangle_{\overline{z}} < 0$. Employing (3.3),

$$\Theta_k \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta_k \circ \exp_{\overline{z}}(0) < 0,$$

for sufficiently large j and for every $k = 1, 2, \ldots, r$. Therefore, for sufficiently large values of j, we have $\Theta(\exp_{\bar{z}}(\tau_j v_j)) \prec \Theta(\bar{z})$. Further, as, $\exp_{\bar{z}}(\tau_j v_j)$ lies in the neighborhood of $\bar{z}$, this implies that $\bar{z}$ is not local robust weak Pareto solution for RNMSIP. This concludes the fact that the set $\mathcal{C}$ is empty. Moreover, from the satisfaction of ACQ at $\bar{z}$, it follows that the following system

$$(I)\ \begin{cases} \Theta_1^\circ(\bar{z}; v) < 0,\ \Theta_2^\circ(\bar{z}; v) < 0, \ldots, \Theta_r^\circ(\bar{z}; v) < 0, \\ v \in (\mathcal{R}(\bar{z}))^{\leq}, \end{cases}$$

has no solution.

Now, our claim is to show that there exist $\alpha_1, \alpha_2, \ldots, \alpha_r \geq 0$, not all zero such that

$$\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; v) \geq 0,\ \forall v \in (\mathcal{R}(\bar{z}))^{\leq}.$$

On contrary, suppose that for each $k = 1, 2, \ldots, r$, there exist $v_k \in (\mathcal{R}(\bar{z}))^{\leq}$, such that $\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; v_k) < 0$, holds true for any $\alpha_1, \alpha_2, \ldots, \alpha_r \geq 0$. From the convexity of the function $\Theta_k^\circ(\bar{z}; v)$ with respect to v and for $\sum_{k=1}^{r} \alpha_k = 1$, $\alpha_k \geq 0,\ k = 1, 2, \ldots, r$, we have

$$\Theta_k^\circ\Big(\bar{z}; \sum_{k=1}^{r} \alpha_k v_k\Big) \leq \sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; v_k) < 0.$$

This implies together with the fact that the set $(\mathcal{R}(\bar{z}))^{\leq}$ is convex, one has $\sum_{k=1}^{r} \alpha_k v_k$ is a solution of system (I).

Moreover, since $0 \in (\mathcal{R}(\bar{z}))^{\leq}$ and $\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; 0) = 0$, it follows that, for any $v \in T_{\bar{z}}\mathscr{H}_1$, 0 is a minimizer of

$$\Big\{ \sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; v) + \delta_{(\mathcal{R}(\bar{z}))^{\leq}}(v) \Big\},$$

where $\delta_{(\mathcal{R}(\bar{z}))^{\leq}}$ is the indicator function of $(\mathcal{R}(\bar{z}))^{\leq}$. Now, by employing Fermat's rule and sum rule, we have

$$\begin{aligned} 0 \in \partial_C &\left(\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; \cdot) + \delta_{(\mathcal{R}(\bar{z}))^{\leq}}(\cdot) \right)(0) \\ &\subset \partial_C \left(\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; \cdot) \right)(0) + \partial_C \delta_{(\mathcal{R}(\bar{z}))^{\leq}}(\cdot)(0), \end{aligned}$$

Furthermore, it can be readily deduced that

$$\partial_C \left(\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\bar{z}; \cdot) \right)(0) = \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(\bar{z}),$$

$$\partial_C \delta_{(\mathcal{R}(\bar{z}))^{\leq}}(0) = ((\mathcal{R}(\bar{z}))^{\leq})^{\leq}.$$

By invoking the bipolar theorem and the fact that $\mathrm{pos}\mathcal{R}(\overline{z})$ is closed, we have $\partial_C \delta_{(\mathcal{R}(\overline{z}))^{\leq}}(0) = \mathrm{pos}\mathcal{R}(\overline{z})$. Therefore we have

$$0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(\overline{z}) + \mathrm{pos}\mathcal{R}(\overline{z}).$$

In view of Lemma 2.12, there exists an integer $q \leq n$, $(\mu_l)_l \in \mathbb{R}^q_+$, and $(i_l, w_l)_l \in (\mathcal{R}(\overline{z}))^q$, such that

$$0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(\overline{z}) + \sum_{l=1}^{q} \mu_l \partial_C^z h_{i_l}(\overline{z}, w_l). \tag{3.4}$$

If $q = n$, then (3.1) and (3.2) hold, as $(i_l, w_l)_l \in (\mathcal{R}(\overline{z}))^q$. Whenever, $q < n$, then by adding some multipliers $\mu_{q+1} = \cdots = \mu_n = 0$, we get our desired result. □

Remark 3.7.

(1) Theorem 3.6 extends Corollary 3.8 derived by Kanzi and Nobakhtian [27] from single objective nonsmooth SIPs to the more general optimization problems, namely, RNMSIP. In addition, Theorem 3.6 generalizes Corollary 3.8 from the context of Euclidean spaces to the framework of Hadamard manifolds.

(2) If $\mathscr{H}_1 = \mathbb{R}^{n_1}$, then Theorem 3.6 extends Theorem 3.4 derived by Kanzi and Nobakhtian [24] in the domain of robust optimization.

(3) Theorem 3.6 generalizes Theorem 3.1 derived in [46] from the framework of Hadamard manifolds to the setting of Euclidean space.

(4) If $\mathscr{H}_1 = \mathbb{R}^{n_1}$, then Theorem 3.6 generalizes Theorem 1 derived in [31] from smooth MSIPs to nonsmooth MSIPs.

(5) Theorem 3.6 extends Theorem 3.3 derived in [8] from multiobjective optimization problems to MSIPs and generalizes it from Euclidean spaces to Hadamard manifolds.

(6) In the Hadamard manifold setting, Theorem 3.6 extends Proposition 3.3 in [48] from smooth MSIPs to nonsmooth MSIPs. Moreover, Theorem 3.6 extends Proposition 3.3 derived in [48] in the domain of robust optimization.

(7) In the Hadamard manifold setting, Theorem 3.6 extends Theorem 3.5 deduced in [59] from nonsmooth MSIPs to more general optimization problems, namely, RNMSIP.

To highlight the importance of the results derived in Theorem 3.6, we provide the following example on the cone of symmetric positive definite matrices of order 2×2.

Example 3.8. Let $\mathbb{S}^2$ and $\mathscr{P}^2_+ (\subset \mathbb{S}^2)$ denote the set of all symmetric matrices and the set of all symmetric positive definite matrices, respectively. For any matrix $\mathcal{Z}$, the notation trace$(\mathcal{Z})$ denotes the trace of the matrix $\mathcal{Z}$. From [12], it follows that $\mathscr{P}^2_+$ is a Riemannian manifold endowed with the Riemannian metric

$$\langle \mathcal{A}_1, \mathcal{A}_2 \rangle_{\mathcal{Z}} := \text{ trace } (\mathcal{A}_2 \mathcal{Z}^{-1} \mathcal{A}_1 \mathcal{Z}^{-1}),\ \mathcal{Z} \in \mathscr{P}^2_+,\ \mathcal{A}_1,\ \mathcal{A}_2 \in T_{\mathcal{Z}} \mathscr{P}^2_+.$$

Moreover, from [12], it follows that $\mathscr{P}^2_+$ is a Hadamard manifold with $T_{\mathcal{Z}}\mathscr{P}^2_+ = \mathbb{S}^2$. Let $\mathcal{Z}_1$, $\mathcal{Z}_2$ be two arbitrary elements of $\mathscr{P}^2_+$ and $\mathcal{A} \in T_{\mathcal{Z}}\mathscr{P}^2_+$. Then, the exponential map $\exp_{\mathcal{Z}_1}(\mathcal{A}) : T_{\mathcal{Z}_1}\mathscr{P}^2_+ \to \mathscr{P}^2_+$ is given by:

$$\exp_{\mathcal{Z}_1}(\mathcal{A}) = \mathcal{Z}_1^{1/2} \mathrm{Exp}\big(\mathcal{Z}_1^{-1/2} \mathcal{A} \mathcal{Z}_1^{-1/2}\big) \mathcal{Z}^{1/2}, \tag{3.5}$$

where $\operatorname{Exp} : \mathscr{P}^2_+ \to \mathscr{P}^2_+$ is the usual exponential on $\mathscr{P}^2_+$ and the corresponding inverse exponential map $\exp^{-1}_{\mathcal{Z}_1} : \mathscr{P}^2_+ \to T_{\mathcal{Z}_1}\mathscr{P}^2_+$ is given by:

$$\exp^{-1}_{\mathcal{Z}_1}(\mathcal{Z}_2) = \mathcal{Z}_1^{1/2}\operatorname{Log}\big(\mathcal{Z}_1^{-1/2}\mathcal{Z}_2\mathcal{Z}_1^{-1/2}\big)\mathcal{Z}_1^{1/2}, \tag{3.6}$$

where $\operatorname{Log} : \mathscr{P}^2_+ \to \mathbb{S}^2$ is the usual logarithm on $\mathscr{P}^2_+$ (see, for instance, [33]).
For any real valued function $\mathscr{H} : \mathscr{P}^2_+ \to \mathbb{R}$, the Riemannian gradient is given by:

$$\operatorname{grad}(\mathscr{H}(\mathcal{Z})) = \mathcal{Z}\mathscr{H}'(\mathcal{Z})\mathcal{Z},$$

for any $\mathcal{Z} \in \mathscr{P}^2_+$, where $\mathscr{H}'(\mathcal{Z})$ denotes the Euclidean gradient of the function $\mathscr{H}$ at $\mathcal{Z}$ (see, for instance, [12]).
Let $\Theta : \mathscr{P}^2_+ \to \mathbb{R}^2$ be defined by

$$\Theta(\mathcal{Z}) = (\Theta_1(\mathcal{Z}),\ \Theta_2(\mathcal{Z})) = (-\ln\det \mathcal{Z},\ -\ln z_1 - \ln z_4).$$

Let $\mathcal{I} = [0,1]$, $\mathscr{W}_\circ = [0,\frac{\pi}{2}]$, $\mathscr{W}_{\frac{1}{3}} = [-1,0]$, $\mathscr{W}_{\frac{1}{2}} = [0,1]$, $\mathscr{W}_1 = [0,\frac{\pi}{2}]$, $\mathscr{W}_i = [\frac{1}{2}, i]$, $i \in \mathcal{I}\setminus\{0,\frac{1}{3},\frac{1}{2},1\}$, and $h_i : \mathscr{P}^2_+ \times \mathscr{W}_i \to \mathbb{R}$ be defined by

$$h_i(\mathcal{Z},w) = \begin{cases} |z_4| + \sin w - 2 & \text{if } (i,w) \in \operatorname{gph}\mathscr{W} \text{ and } i = 0,\\ -\ln(z_2+1) + w & \text{if } (i,w) \in \operatorname{gph}\mathscr{W} \text{ and } i = \frac{1}{3},\\ \ln(z_2+1) - w & \text{if } (i,w) \in \operatorname{gph}\mathscr{W} \text{ and } i = \frac{1}{2},\\ |z_1| - \cos w - 1 & \text{if } (i,w) \in \operatorname{gph}\mathscr{W} \text{ and } i = 1,\\ wz_2^2 & \text{if } (i,w) \in \operatorname{gph}\mathscr{W} \text{ and } i \in [0,1]\setminus\{0,\frac{1}{3},\frac{1}{2},1\}, \end{cases}$$

where, $\mathcal{Z} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_4 \end{bmatrix}$. Now, it can be verified that

$$\mathcal{F} := \left\{ \begin{bmatrix} z_1 & 0 \\ 0 & z_4 \end{bmatrix} \ :\ 0 < z_1 \le 1,\ 0 < z_4 \le 1 \right\}.$$

The feasible element $\overline{\mathcal{Z}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a local robust weak Pareto solution. Now, one can show that $(\mathcal{R}(\overline{\mathcal{Z}}))^{\leq} = \left\{ \begin{bmatrix} z_1 & 0 \\ 0 & z_4 \end{bmatrix} \ :\ z_1 \le 0,\ z_4 \le 0 \right\}$ and

$$\mathscr{T}(\overline{\mathcal{Z}},\mathcal{F}) = \left\{ \begin{bmatrix} z_1 & 0 \\ 0 & z_4 \end{bmatrix} \ :\ z_1 \le 0,\ z_4 \le 0 \right\} = (\mathcal{R}(\overline{\mathcal{Z}}))^{\leq}.$$

Therefore, ACQ holds at $\overline{\mathcal{Z}}$. On the other hand

$$\operatorname{pos}(\mathcal{R}(\overline{\mathcal{Z}})) = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, d \ge 0,\ b \in \mathbb{R} \right\},$$

which a closed set in $\mathbb{S}^2$. Moreover,

$$\partial_C\Theta_1(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\},\ \partial_C\Theta_2(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\},$$

$$\partial^z_C h_0(\overline{\mathcal{Z}},\frac{\pi}{2}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},\ \partial^z_C h_{\frac{1}{3}}(\overline{\mathcal{Z}},0) = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\},$$

$$\partial_C^z h_{\frac{1}{2}}(\overline{\mathcal{Z}},0)=\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\},\ \partial_C^z h_1(\overline{\mathcal{Z}},\frac{\pi}{2})=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}\right\},$$

$$\partial_C^z h_i(\overline{\mathcal{Z}},w)=\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}\right\},\ (i,w)\in \operatorname{gph}\mathscr{W},\ i\in[0,1]\setminus\{0,\frac{1}{3},\frac{1}{2},1\}.$$

For choosing $\alpha_1=\frac{1}{2}=\alpha_2$ and $\mu_l=1$, for all $l=0,1,\ldots,4$, one has the following inclusion:

$$\begin{aligned}0\in&\sum_{k=1}^{2}\alpha_k\partial_C\Theta_k(\mathcal{Z})+\mu_0\partial_C^z h_0(\overline{\mathcal{Z}},\frac{\pi}{2})\\&+\mu_1\partial_C^z h_{\frac{1}{3}}(\overline{\mathcal{Z}},0)+\mu_2\partial_C^z h_{\frac{1}{2}}(\overline{\mathcal{Z}},0)+\mu_3\partial_C^z h_1(\overline{\mathcal{Z}},\frac{\pi}{2})+\mu_4\partial_C^z h_i(\overline{\mathcal{Z}},w),\end{aligned}$$

and

$$\mu_l h_{i_l}(\overline{\mathcal{Z}},w_l)=0,\ \forall l=0,1,\ldots,4,$$

which are the outcomes of Theorem 3.6.

Remark 3.9. From Example 3.4 in Souza [42], it follows that the function $\Theta_1(\mathcal{Z})=-\ln\det\mathcal{Z}$ (see, Example 3.8) is not locally Lipzchitz with respect to the Euclidean metric $\langle\cdot,\cdot\rangle$. However, the aforementioned function $\Theta_1(\mathcal{Z})$ is locally Lipschitz with respect to the Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{Z}}$ (see, Example 3.8). Taking into account this observation, it follows that the outcomes obtained in this paper are more general in comparison to the Euclidean space setting (see, for instance, [8, 24, 27, 31, 46]).

The following example demonstrates that the satisfaction of ACQ is sufficient but not necessary for the existence of KKT multipliers.

Example 3.10. Consider $\mathscr{H}_1=\{w\in\mathbb{R}^2: z_1,z_2>0\}$. Then from [40, 48], $\mathscr{H}_1$ is a Riemannian manifold with the metric

$$\langle\xi_1,\xi_2\rangle_z=\langle\mathscr{G}(z)\xi_1,\xi_2\rangle\ \forall\xi_1,\xi_2\in T_z\mathscr{H}_1=\mathbb{R}^2,$$

where

$$\mathscr{G}(z)=\begin{bmatrix}\frac{1}{z_1^2}&0\\0&\frac{1}{z_2^2}\end{bmatrix},$$

and $\langle\cdot,\cdot\rangle$ is standard inner product on $\mathbb{R}^2$. The sectional curvature of $\mathscr{H}_1$ is 0 and $\mathscr{H}_1$ is a Hadamard manifold (see, for instance, [40, 48]). The Riemannian distance between any two points $x=(x_1,x_2)$ and $z=(z_1,z_2)$ is given by:

$$\operatorname{dis}(x,z)=\left\|\left(\ln\frac{x_1}{z_1},\ln\frac{x_2}{z_2}\right)\right\|.$$

For any $z\in\mathscr{H}_1$ and $p\in T_z\mathscr{H}_1$, the exponential map $\exp_z:T_z\mathscr{H}_1\to\mathscr{H}_1$ is determined in the following way (see, for instance, [40, 48])

$$\exp_z(p)=\left(z_1e^{\frac{p_1}{z_1}},z_2e^{\frac{p_2}{z_2}}\right),\ p\in T_z\mathscr{H}_1.$$

Moreover, $\exp_z^{-1}:\mathscr{H}_1\to T_z\mathscr{H}_1$ is given by:

$$\exp_z^{-1}(x)=\left(z_1\ln\frac{x_1}{z_1},z_2\ln\frac{x_2}{x_2}\right).$$

Moreover, let $\mathcal{I} = [0,1]$, and $\mathscr{W}_i = [0,i](\subseteq \mathbb{R})$ and the function $\Theta : \mathscr{H}_1 \to \mathbb{R}^2$ is defined as follows:

$$\Theta(z) = (\Theta_1(z),\ \Theta_2(z)) = \left(-\frac{z_1}{e^2},\ \frac{z_1}{e^2}\right).$$

Further, $h_i : \mathscr{H}_1 \times \mathscr{W}_i \to \mathbb{R}$ be defined by

$$h_i(z,w) = (z_1 - e)^2 + (z_2 - e)^2 - 2iw.$$

Clearly the feasible set of this problem is a singleton set, given by $\mathcal{F} = \{(e,e)\}$ and hence, $\overline{z} = (e,e)$ is the unique local robust weak Pareto solution of this problem. Moreover,

$$\partial_C\Theta_1(\overline{z}) = \{(-1,0)\},\ \partial_C\Theta_2(\overline{z}) = \{(1,0)\},\ \partial_C^z h_{i_l}(\overline{z}, w_l) = \{(0,0)\},$$

for all $(i_l, w_l) \in \mathcal{K}(\overline{z})$, $\mathscr{T}(\mathcal{F}, \overline{z}) = \{(0,0)\}$, $\mathcal{R}(\overline{z}) = \{(0,0)\}$. Hence pos$\mathcal{R}(\overline{z})$ is a closed set. However $(\mathcal{R}(\overline{z}))^{\leq} = \mathbb{R}^2 \not\subseteq \mathscr{T}(\mathcal{F}, \overline{z})$. Therefore, ACQ is not satisfied at $\overline{z}$. Now, let $\alpha_1 = \frac{1}{2} = \alpha_2$, $\mu \geq 0$ such that

$$0 \in \sum_{k=1}^{2} \alpha_k \partial_C \Theta_k(\overline{z}) + \mu \partial_C^z h_{i_l}(\overline{z}, w_l).$$

Therefore, from this interesting example, one can observe that it is possible to find KKT multipliers that satisfy the conclusions of the Theorem 3.6, while, the ACQ is not satisfied at $\overline{z}$. Therefore, we can conclude that the satisfaction of ACQ need not be necessary for the existence of KKT multipliers.

The following theorem elucidates that under certain mild assumptions, the KKT conditions derived in Theorem 3.6 will be transformed into strong KKT conditions for the robust Benson-proper solution of the problem RNMSIP.

Theorem 3.11. *Let $\overline{z}$ be a local robust Benson-proper solution at which the ACQ is satisfied and pos$\mathcal{R}(\overline{z})$ is a closed set. Moreover, assume that $\Theta_k'(\overline{z}, v)$ exists for each $k = 1, 2, \ldots, r$ and $\Theta_k'(\overline{z}; v) = \Theta_k^\circ(\overline{z}; v)$. Then, in conclusion there exist multipliers $(\alpha_k)_k \in int\, \mathbb{R}_+^r$, $(\mu_l)_l \in \mathbb{R}_+^n$, and $(i_l, w_l)_l \in (gph\mathscr{W})^n$, such that*

$$0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(\overline{z}) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(\overline{z}, w_l), \tag{3.7}$$

$$\mu_l h_{i_l}(\overline{z}, w_l) = 0,\ \forall l = 1, 2, \ldots, n. \tag{3.8}$$

Proof. Since $\overline{z}$ is a local robust Benson-proper solution, then according to the Definition 3.1, there exists a neighborhood $\mathcal{U}_{\overline{z}}$ of $\overline{z}$, such that

$$\text{clcone}(\Theta(\mathcal{U}_{\overline{z}} \cap \mathcal{F}) + \mathbb{R}_+^r - \Theta(\overline{z})) \cap (-\mathbb{R}_+^r \setminus \{0\}) = \emptyset. \tag{3.9}$$

Now, for each $k = 1, 2, \ldots, r$, we assert that the following system has no solution

$$(II)\ \begin{cases} \Theta_k^\circ(\overline{z}; v) < 0, \\ \Theta_t^\circ(\overline{z}; v) \leq 0 \text{ for all } t \neq k \\ v \in \mathscr{T}(\mathcal{F}, \overline{z}). \end{cases}$$

When, $k = 1$, in a similar manner, one can show that there is no solution for the system (II). Now, for $k > 1$, suppose that there exists $v \in \mathscr{T}(\mathcal{F}, \overline{z})$ for which the system (II) has a solution.

For setting

$$a := (\Theta_1^\circ(\overline{z}; v), \Theta_2^\circ(\overline{z}; v), \ldots, \Theta_r^\circ(\overline{z}; v)),$$

one can observe that $a \in -\mathbb{R}_+^r \setminus \{0\}$. Moreover, as $v \in \mathscr{T}(\mathcal{F}, \overline{z})$, there exist $\tau_j \downarrow 0$, and $v_j \to v$ such that $\exp_{\overline{z}}(\tau_j v_j) \in \mathcal{F}$ for all $j \in \mathbb{N}$. Further, from the hypothesis $\Theta_k'(\overline{z}, v) = \Theta_k^\circ(\overline{z}; v)(k = 1, 2, \ldots, r)$, it follows that

$$\lim_{j\to\infty} \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v) - \Theta(\overline{z}))}{\tau_j} = a.$$

Now,

$$\begin{aligned} \lim_{j\to\infty} \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta(\overline{z}))}{\tau_j} &= \lim_{j\to\infty} \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta \circ \exp_{\overline{z}}(\tau_j v))}{\tau_j} \\ &\quad + \lim_{j\to\infty} \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v) - \Theta(\overline{z}))}{\tau_j}. \end{aligned} \tag{3.10}$$

From the Lipschitzness condition of the function $\Theta \circ \exp_{\overline{z}}$, there exists a $L > 0$, such that

$$\left\| \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta \circ \exp_{\overline{z}}(\tau_j v))}{\tau_j} \right\| \leq L \, \|v_j - v\|.$$

The above inequality together with the convergence of v_j to v, it follows that

$$\lim_{j\to\infty} \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta \circ \exp_{\overline{z}}(\tau_j v))}{\tau_j} = 0.$$

Therefore, from (3.10), we have

$$\lim_{j\to\infty} \frac{(\Theta \circ \exp_{\overline{z}}(\tau_j v_j) - \Theta(\overline{z}))}{\tau_j} = a.$$

This shows that,

$$a \in \operatorname{clcone}(\Theta(\mathcal{U}_{\overline{z}} \cap \mathcal{F}) + \mathbb{R}_+^r - \Theta(\overline{z})) \cap (-\mathbb{R}_+^r \setminus \{0\}),$$

which contradicts to equation (3.9). Therefore, the system (II) does not possess any solution. Moreover, the satisfaction of ACQ at $\overline{z}$ ensures that for each $k = 1, 2, \ldots, r$, there does not exist any $v \in (\mathcal{R}(\overline{z}))^{\leq}$ such that $\Theta_k^\circ(\overline{z}; v) < 0$, and $\Theta_t^\circ(\overline{z}; v) \leq 0$ for all $t \neq k$. Therefore, from Lemma 2.13, there exist $\lambda_{1t} \geq 0$ for all $t \neq k$ such that

$$\Theta_k^\circ(\overline{z}; v) + \sum_{t\in\{1,2,\ldots,r\}\setminus\{k\}} \lambda_{1t}\Theta_t^\circ(\overline{z}; v) \geq 0, \ \forall v \in (\mathcal{R}(\overline{z}))^{\leq}.$$

By summing over the values of k from 1 to r and defining

$$\alpha_k = 1 + \sum_{t\in\{1,2,\ldots,r\}\setminus\{k\}} \lambda_{1t} > 0,$$

we have

$$\sum_{k=1}^{r} \alpha_k \Theta_k^\circ(\overline{z}; v) \geq 0, \ \forall v \in (\mathcal{R}(\overline{z}))^{\leq}.$$

The remaining steps of the proof proceed in a similar fashion to those presented in Theorem 3.6. □

Remark 3.12. (1) Theorem 3.11 extends Theorem 3.2 derived in [46] from the context of Euclidean space to the framework of Hadamard manifolds.

In the following definition, we introduce the notions of geodesic and strictly geodesic convex functions for the pair of functions (Θ, h) in the framework of Hadamard manifolds.

Definition 3.13. Let, $\overline{z} \in \mathcal{F}$. Then,

(i) (Θ, h) is said to possess geodesic convexity at $\overline{z}$, if for any $z \in \mathscr{H}_1$; $\xi_k \in \partial_C \Theta_k(\overline{z})$, for $k = 1, 2, \ldots, r$; $(i_l, w_l) \in \text{gph}\mathscr{W}$, $\eta_l \in \partial_C^z h_{i_l}$, for $l = 1, 2, \ldots, n$; such that

$$\begin{cases} \Theta_k(z) - \Theta_k(\overline{z}) \geq \langle \xi_k, \exp_{\overline{z}}^{-1} z \rangle_{\overline{z}}, \ k = 1, 2, \ldots, r, \\ h_{i_l}(z, w_l) - h_{i_l}(\overline{z}, w_l) \geq \langle \eta_l, \exp_{\overline{z}}^{-1} z \rangle_{\overline{z}}, \ l = 1, 2, \ldots, n. \end{cases}$$

(ii) (Θ, h) is said to possess strictly geodesic convexity at $\overline{z}$, if for any $z \in \mathscr{H}_1 \setminus \{\overline{z}\}$; $\xi_k \in \partial_C \Theta_k(\overline{z})$, for $k = 1, 2, \ldots, r$; $(i_l, w_l) \in \text{gph}\mathscr{W}$, $\eta_l \in \partial_C^z h_{i_l}$, for $l = 1, 2, \ldots, n$; such that

$$\begin{cases} \Theta_k(z) - \Theta_k(\overline{z}) > \langle \xi_k, \exp_{\overline{z}}^{-1} z \rangle_{\overline{z}}, \ k = 1, 2, \ldots, r, \\ h_{i_l}(z, w_l) - h_{i_l}(\overline{z}, w_l) \geq \langle \eta_l, \exp_{\overline{z}}^{-1} z \rangle_{\overline{z}}, \ l = 1, 2, \ldots, n. \end{cases}$$

In the following theorem, we establish sufficient conditions for the existence of a global robust weak Pareto solution and a global robust Pareto solution for RNMSIP under the assumptions of geodesic convexity of the pair of functions (Θ, h).

Theorem 3.14. *Let $\overline{z} \in \mathcal{F}$ and there exist multipliers $(\alpha_k)_k \in \mathbb{R}^r_+$, $(\mu_l)_l \in \mathbb{R}^n_+$, and $(i_l, w_l)_l \in (gph\mathscr{W})^n$, such that (3.1) and (3.2) in Theorem 3.6 are satisfied.*

- (i) *If (Θ, h) is geodesic convex at $\overline{z}$, then $\overline{z}$ is a global robust weak Pareto solution of RNMSIP.*
- (ii) *If $(\alpha_k)_k \in \mathbb{R}^r_+ \setminus \{0\}$ and (Θ, h) is strictly geodesic convex at $\overline{z}$, then $\overline{z}$ is a global robust Pareto solution of RNMSIP.*

Proof. (i) On the contrary, we assume that $\overline{z}$ is not a global robust weak Pareto solution of RNMSIP. Hence, there is $z \in \mathcal{F}$, satisfying

$$\Theta(z) \prec \Theta(\overline{z}).$$

This implies that

$$\Theta_k(z) < \Theta_k(\overline{z}), \ \forall k = 1, 2, \ldots, r. \tag{3.11}$$

From the given hypotheses, it follows that (3.1) and (3.2) are satisfied. Therefore, there exist multipliers $(\alpha_k)_k \in \mathbb{R}^r_+$, $(\mu_l)_l \in \mathbb{R}^n_+$, and $(i_l, w_l)_l \in (\text{gph}\mathscr{W})^n$, $\xi_k \in \partial_C \Theta_k(\overline{z})$, for $k = 1, 2, \ldots, r$ and $\eta_l \in \partial_C^z h_{i_l}(\overline{z}, w_l)$ for $l = 1, 2, \ldots, n$, such that

$$\textstyle\sum_{k=1}^r \alpha_k \xi_k + \sum_{l=1}^n \mu_l \eta_l = 0.$$

Employing the geodesic convexity assumption of the pair of functions (Θ, h) at $\overline{z}$, one has

$$\begin{aligned}
0 &= \sum_{k=1}^{r} \alpha_k \langle \xi_k, \exp_{\overline{z}}^{-1} z\rangle_{\overline{z}} + \sum_{l=1}^{n} \mu_l \langle \eta_l, \exp_{\overline{z}}^{-1} z\rangle_{\overline{z}} \\
&\leq \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(\overline{z})) + \sum_{l=1}^{n} \mu_l (h_{i_l}(z, w_l) - h_{i_l}(\overline{z}, w_l)), \\
&\leq \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(\overline{z})) - \sum_{l=1}^{n} \mu_l h_{i_l}(\overline{z}, w_l),
\end{aligned}$$

where the last inequality follows from the fact, $z \in \mathcal{F}$. Moreover, from condition (3.2), it follows that $\mu_l h_{i_l}(\overline{z}, w_l) = 0$, for all $l = 1, 2, \ldots, n$. Thus, we have

$$\textstyle\sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(\overline{z})) \geq 0.$$

The last inequality along with the condition $(\alpha_k)_k \in \mathbb{R}^r_+$, implies the existence of $\widetilde{k} \in \{1, 2, \ldots, r\}$ such that

$$\Theta_{\widetilde{k}}(z) \geq \Theta_{\widetilde{k}}(\overline{z}),$$

which contradicts equation (3.11). This completes the proof of (i).

(ii) On the contrary, suppose that there is $z \in \mathcal{F}$ such that $\Theta(z) \preceq \Theta(\overline{z})$, this implies that

$$\begin{cases} \Theta_{\widetilde{k}}(z) < \Theta_{\widetilde{k}}(\overline{z}) \\ \Theta_k(z) \leq \Theta_k(\overline{z}), \ \forall k \in \{1, 2, \ldots, r\} \setminus \{\widetilde{k}\}. \end{cases} \tag{3.12}$$

According to the hypotheses of the theorem, there exist multipliers $(\alpha_k)_k \in \mathbb{R}^r_+$, $(\mu_l)_l \in \mathbb{R}^n_+$, and $(i_l, w_l)_l \in (\mathrm{gph}\mathscr{W})^n$, such that (3.1) and (3.2) hold true. Then, there are $\xi_k \in \partial_C \Theta_k(\overline{z})$ for $k = 1, 2, \ldots, r$ and $\eta_l \in \partial_C^z h_{i_l}(\overline{z}, w_l)$ for $l = 1, 2, \ldots, n$, such that

$$\textstyle\sum_{k=1}^{r} \alpha_k \xi_k + \sum_{l=1}^{n} \mu_l \eta_l = 0.$$

Based on the strict geodesic convexity assumption of (Θ, h) at $\overline{z}$, we have the following:

$$\begin{aligned}
0 &= \sum_{k=1}^{r} \alpha_k \langle \xi_k, \exp_{\overline{z}}^{-1} z\rangle_{\overline{z}} + \sum_{l=1}^{n} \mu_l \langle \eta_l, \exp_{\overline{z}}^{-1} z\rangle_{\overline{z}} \\
&< \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(\overline{z})) + \sum_{l=1}^{n} \mu_l (h_{i_l}(z, w_l) - h_{i_l}(\overline{z}, w_l)), \\
&\leq \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(\overline{z})) - \sum_{l=1}^{n} \mu_l h_{i_l}(\overline{z}, w_l),
\end{aligned}$$

where the last inequality follows from the fact that $z \in \mathcal{F}$. Moreover, from condition (3.2), it follows that $\mu_l h_{i_l}(\overline{z}, w_l) = 0$, for all $l = 1, 2, \ldots, n$. Consequently, we have

$$\sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(\overline{z})) > 0.$$

Since $(\alpha_k)_k \in \mathbb{R}^r_+ \setminus \{0\}$, therefore there exists $\widetilde{k} \in \{1, 2, \ldots, r\}$ satisfying $\Theta_{\widetilde{k}}(z) > \Theta_{\widetilde{k}}(\overline{z})$, which contradicts (3.12). This completes the proof of (ii). □

Remark 3.15. (1) Theorem 3.14 extends Theorem 4.1 derived by Kanzi and Nobakhtian [27] from single objective nonsmooth SIPs to the more general optimization problems, namely, RNMSIP. In addition, Theorem 3.14 generalizes Corollary 3.8 from the context of Euclidean spaces to the framework of Hadamard manifolds.

(2) If $\mathscr{H}_1 = \mathbb{R}^{n_1}$, then Theorem 3.11 extends Theorem 4.3 derived by Kanzi and Nobakhtian [24] in the domain of robust optimization.

(3) Theorem 3.14 (i) and (ii) generalize Theorems 3.3 and 3.4, respectively, derived in [46], from the context of Euclidean space to the setting of Hadamard manifolds.

(4) If $\mathscr{H}_1 = \mathbb{R}^{n_1}$, Theorem 3.14 (i) extends Theorem 2 derived in Lee and Lee [31] from smooth MSIPs to non-smooth MSIPs in the face of data uncertainty.

(5) Theorem 3.14 (i) and (ii) extend Theorem 3.11 derived in [8] from multiobjective optimization problems to MSIPs and generalizes it from Euclidean spaces to Hadamard manifolds.

(6) In the Hadamard manifold setting, Theorem 3.14 extends Proposition 3.4 in [48] from smooth MSIPs to nonsmooth MSIPs. Moreover, Theorem 3.14 extends Proposition 3.4 derived in [48] in the domain of robust optimization.

(7) In the Hadamard manifold setting, Theorem 3.14 extends Theorem 3.7 deduced in [59] from nonsmooth MSIPs to more general optimization problems, namely, RNMSIP.

The following example illustrates the significance of Theorem 3.14 and provides a framework for finding a global robust weak Pareto solution within the class of problems falling under the category of RNMSIP.

Example 3.16. Consider $\mathscr{H}_1 = \{z \in \mathbb{R}^2 : z_2 > 0\}$. Then, $\mathscr{H}_1$ is a Riemannian manifold (see, for instance [30, 48, 49]), with Riemannian metric

$$\langle \xi_1, \xi_2 \rangle_z = \langle \mathscr{G}(z)\xi_1, \xi_2 \rangle_z, \ \forall \xi_1, \xi_2 \in T_z\mathscr{H}_1 = \mathbb{R}^2,$$

where $\mathscr{G}(z) = \begin{bmatrix} \frac{1}{z_2^2} & 0 \\ 0 & \frac{1}{z_2^2} \end{bmatrix}$. Moreover, from [30, 48, 49], it follows that, the sectional curvature of $\mathscr{H}_1$ is -1 and $\mathscr{H}_1$ is a Hadamard manifold.

The Riemannian distance between any two points $x = (x_1, x_2)$ and $z = (z_1, z_2)$ is given by

$$\operatorname{dis}(x, z) = \begin{cases} \left| \ln \frac{z_2}{x_2} \right| & \text{if } x_1 = z_1, \\ \left| \ln \frac{x_1 - n + m}{z_1 - n + m} \cdot \frac{z_2}{x_2} \right|, & \text{if } x_1 \neq z_1, \end{cases}$$

where

$$m = \sqrt{(z_1 - n)^2 + z_2^2}, \ n = \frac{z_1^2 + z_2^2 - (x_1^2 + x_2^2)}{2(z_1 - x_1)}.$$

For any $z \in \mathscr{H}_1$ and $p \in T_z\mathscr{H}_1$, the exponential map $\exp_z : T_z\mathscr{H}_1 \to \mathscr{H}_1$ is defined as follows (see, for instance, [30, 48]):

For $p_1 = 0$,

$$\exp_z(p) = (z_1, z_2 e^{\frac{p_2}{z_2}}),$$

and for $p_1 \neq 0$,

$$\exp_z(p)$$
$$= \left(z_1 + \frac{p_2}{p_1} + \sqrt{1+\left(\frac{p_2}{p_1}\right)^2}\tanh(a_{p_1,p_2}(1)), z_2\sqrt{1+\left(\frac{p_2}{p_1}\right)^2}\frac{1}{\cosh(b_{p_1,p_2}(1))} \right),$$

where

$$a_{p_1,p_2}(s) = \begin{cases} s\sqrt{p_1^2+p_2^2} - \sinh^{-1}\frac{p_2}{p_1} & \text{if } p_1 > 0, \\ -s\sqrt{p_1^2+p_2^2} - \sinh^{-1}\frac{p_2}{p_1} & \text{if } p_1 < 0, \end{cases}$$
$$b_{p_1,p_2}(s) = \begin{cases} s\frac{\sqrt{p_1^2+p_2^2}}{z_2} - \sinh^{-1}\frac{p_2}{p_1} & \text{if } p_1 > 0, \\ -s\frac{\sqrt{p_1^2+p_2^2}}{z_2} - \sinh^{-1}\frac{p_2}{p_1} & \text{if } p_1 < 0. \end{cases}$$

Furthermore, $\exp_z(sp) = \Gamma_{z,p}(s)$ and $\exp_z(p) = \Gamma_{z,p}(1)$, where $\Gamma_{z,p}(s)$ is defined in the following way:

For $p_1 = 0$,
$$\Gamma_{z,p}(s) = (z_1, z_2 e^{\frac{p_2}{z_2}s}),$$
and for $p_1 \neq 0$,
$$\Gamma_{z,p}(s) = \left(z_1 + \frac{p_2}{p_1} + \sqrt{1+\left(\frac{p_2}{p_1}\right)^2}\tanh(a_{p_1,p_2}(s)), z_2\sqrt{1+\left(\frac{p_2}{p_1}\right)^2}\frac{1}{\cosh(b_{p_1,p_2}(s))} \right),$$
with $\Gamma_{z,p}(0) = z$ and $\Gamma'_{z,p}(0) = p$. Moreover, from [30, 48], the inverse exponential map $\exp_z^{-1} : \mathscr{H}_1 \to T_z\mathscr{H}_1$ is given by:
$$\exp_z^{-1}(x) = \begin{cases} \left(0, z_2 \ln\frac{x_2}{z_2}\right) & \text{if } z_1 = x_1, \\ \frac{z_2}{m}\left(\tanh^{-1}\frac{n-z_1}{m} - \tanh^{-1}\frac{n-x_1}{m}\right)(z_2, n-x_1) & \text{if } z_1 \neq x_1, \end{cases}$$
where
$$m = \sqrt{(z_1-n)^2 + z_2^2},\ n = \frac{z_1^2+z_2^2-(x_1^2+x_2^2)}{2(z_1-x_1)}.$$
Let us define $\mathcal{S} := \{z \in \mathscr{H}_1 |\ z_1 = 0,\ z_2 \geq \frac{1}{4}\} \subset \mathscr{H}_1$, which is a geodesic convex subset of $\mathscr{H}_1$. Moreover, let $\mathcal{I} = [0,1]$ and $\mathscr{W}_i = [i,2] (\subset \mathbb{R})$, $\forall i \in \mathcal{I}$ and the function $\Theta : \mathcal{S} \to \mathbb{R}^2$ is defined by:
$$\Theta(z) = (\Theta_1(z),\ \Theta_2(z)) = \left(|z_2|,\ |z_2 - \tfrac{1}{2}| + z_2\right).$$
Further, $h_i(\cdot, w) : \mathcal{S} \to \mathbb{R}$ is defined by:
$$h_i(z,w) = \frac{i}{z_2} - i - w,\ i \in \mathcal{I},\ w \in \mathscr{W}_i.$$
The feasible set of this problem is given by
$$\mathcal{F} := \{z \in \mathscr{H}_1 |\ z_1 = 0,\ z_2 \geq \frac{1}{2}\}.$$
Choose $\overline{z} = (0, \frac{1}{2}) \in \mathcal{F}$. Then, $\mathcal{K}(\overline{z}) = \mathcal{I}$. Furthermore, one can show that,
$$T(\mathcal{F}, \overline{z}) = \{\nu \in \mathbb{R}^2 |\ v_2 \geq 0\}.$$
Now,
$$\partial_C\Theta_1(\overline{z}) = \{(0,4)\},\ \partial_C\Theta_2(\overline{z}) = \text{co}\{(0,0), (0, \tfrac{1}{2})\},$$

$$\partial_C^z h_{i_l}(\overline{z}, w_l) = \{(0, -i_l)\},\ \forall (i_l, w_l) \in \mathcal{K}(\overline{z}).$$

Moreover, $\text{pos}\mathcal{R}(\overline{z}) = \{\nu \in T_{\overline{z}}\mathscr{H}_1 |\ \nu_1 = 0, \nu_2 \leq 0\}$. Hence $\text{pos}\mathcal{R}(\overline{z})$ is a closed set and $(\mathcal{R}(\overline{z}))^{\leq} = \{\nu \in T_{\overline{z}}\mathscr{H}_1 |\ \nu_2 \geq 0\} \subset T(\mathcal{F}, \overline{z})$. Therefore, ACQ is satisfied at $\overline{z}$. Further, it can be inferred that (Θ, h) is geodesic convex on $\mathcal{S}$.

Now, for choosing, $\alpha_1 = \frac{1}{2} = \alpha_2,\ l = 1,\ \mu_1 = 1,\ i_l = \frac{3}{8} = w_l$, the conditions (3.1) and (3.2) in Theorem 3.6 are both hold true simultaneously. Therefore, from the conclusion of Theorem 3.14 (i), it follows that $\overline{z}$ is a global robust weak Pareto solution to this problem.

4. Robust duality

In this section, we formulate Mond-Weir and Wolfe type dual problems for the considered problem RNMSIP. Moreover, we derive weak, strong, direct, and converse duality results that establish the relationship between the primal problem RMSIP and its corresponding dual problems by exploiting geodesic convexity assumptions of the pair of functions (Θ, h).

4.1. **Mond-Weir duality.** For $x \in \mathscr{H}_1$, $\alpha = (\alpha_k)_k \in \mathbb{R}^r_+$, $\mu = (\mu_l)_l \in \mathbb{R}^n_+$, and $\omega = (i_l, w_l)_l \in (\text{gph}\mathscr{W})^n$, we use $\Theta_{MW}(x, \omega, \alpha, \mu)$ to denote the objective function of the Mond-Weir type dual problem. The Mond-Weir type dual problem RNMSID_{MW} for the primal problem RNMSIP can be formulated in the following manner:

$$\begin{aligned} \text{RNMSID}_{MW}\text{: } \max\ & \Theta_{MW}(x, \omega, \alpha, \mu) := (\Theta_1(x), \Theta_2(x), ..., \Theta_r(x)) \\ & \text{subject to } 0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(x) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(x, w_l), \\ & \sum_{k=1}^{r} \alpha_k = 1,\ \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \geq 0, \\ & x \in \mathscr{H}_1,\ \alpha \in \mathbb{R}^r_+,\ \mu \in \mathbb{R}^n_+,\ \omega \in (\text{gph}\mathscr{W})^n \end{aligned}$$

Let $\mathcal{F}_{MW}$ be the set of all feasible points of RNMSID_{MW}, then

$$\begin{aligned} \mathcal{F}_{MW} := \Big\{ & (x,\omega, \alpha, \mu) \in \mathscr{H}_1 \times (\text{gph}\mathscr{W})^n \times \mathbb{R}^r_+ \times \mathbb{R}^n_+ : \sum_{k=1}^{r} \alpha_k = 1, \\ & 0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(x) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(x, w_l),\ \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \geq 0 \Big\}. \end{aligned}$$

Remark 4.1.
(1) Given any $(x, \omega, \alpha, \mu) \in \mathcal{F}_{MW}$, if $\Theta_{MW}(\widetilde{x}, \widetilde{\omega}, \widetilde{\alpha}, \widetilde{\mu}) \not\prec \Theta_{MW}(x, \omega, \alpha, \mu)$, then $(\widetilde{x}, \widetilde{\omega}, \widetilde{\alpha}, \widetilde{\mu}) \in \mathcal{F}_{MW}$ is said to be robust weak Pareto solution of RNMSID_{MW}.
(2) Given any $(x, \omega, \alpha, \mu) \in \mathcal{F}_{MW}$, if $\Theta_{MW}(\widetilde{x}, \widetilde{\omega}, \widetilde{\alpha}, \widetilde{\mu}) \not\preceq \Theta_{MW}(x, \omega, \alpha, \mu)$, then $(\widetilde{x}, \widetilde{\omega}, \widetilde{\alpha}, \widetilde{\mu}) \in \mathcal{F}_{MW}$ is said to be robust Pareto solution of RNMSID_{MW}.

In the following theorem, we derive weak duality relations relating to the primal problem of RNMSIP and its associated dual problem RNMSID_{MW}.

Theorem 4.2 (Weak duality)**.** *Let $z \in \mathcal{F}$ and $(x, \omega, \alpha, \mu) \in \mathcal{F}_{MW}$. Moreover, if (Θ, h) is geodesic, then $\Theta(z) \not\prec \Theta_{MW}(x, \omega, \alpha, \mu)$. Furthermore, if (Θ, h) is strictly geodesic convex, then $\Theta(z) \not\preceq \Theta_{MW}(x, \omega, \alpha, \mu)$.*

Proof. To begin with, we assume that (Θ, h) is geodesic convex and we show that $\Theta(z) \not\prec \Theta_{MW}(x,\omega,\alpha,\mu)$. On the contrary, suppose that $\Theta(z) \prec \Theta_{MW}(x,\omega,\alpha,\mu)$. This implies that

$$\Theta_k(z) < \Theta_k(x), \ \forall k \in \{1, 2, \ldots, r\}. \tag{4.1}$$

From the feasibility condition of RNMSID_{MW}, it follows that $(\alpha_k)_k \in \mathbb{R}^r_+$ and there exists at least one $\hat{k} \in \{1, 2, \ldots, r\}$ such that $\alpha_{\hat{k}} \neq 0$. Multiplying both sides of Inequality 4.1 by $\alpha_k,\ k \in \{1, 2, \ldots, r\}$ and summing from 1 to r, we get

$$\sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) < 0.$$

Since (x,ω,α,μ) is a feasible element of RNMSID_{MW}, there are $\xi_k \in \partial_C\Theta_k(x)$ for $k = 1, 2, \ldots, r$, and $\eta_l \in \partial h_{i_l}(x, w_l)$ for $l = 1, 2, \ldots, n$ such that

$$\sum_{k=1}^{r} \alpha_k\xi_k + \sum_{l=1}^{n} \mu_l\eta_l = 0.$$

The geodesic convexity assumption of (Θ, h) yields

$$\begin{aligned}
0 &= \sum_{k=1}^{r} \alpha_k\langle\xi_k, \exp_x^{-1} z\rangle_x + \sum_{l=1}^{n} \mu_l\langle\eta_l, \exp_x^{-1} z\rangle_x \\
&\leq \sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) + \sum_{l=1}^{n} \mu_l(h_{i_l}(z, w_l) - h_{i_l}(x, w_l)) \\
&\leq \sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) - \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \\
&\leq \sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) < 0,
\end{aligned}$$

which is a contradiction. This completes the proof of the first part of this theorem.

Now, we assume that (Θ, h) is strictly geodesic convex and we show that

$$\Theta \not\preceq \Theta_{MW}(x,\omega,\alpha,\mu).$$

Suppose on the contrary, let $\Theta(z) \preceq \Theta_{MW}(x,\omega,\alpha,\mu) = \Theta(x)$. In a similar process, one can show that

$$\sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) \leq 0.$$

Since (x,ω,α,μ) is a feasible element of RNMSID_{MW}, there are $\xi_k \in \partial_C\Theta_k(x)$ for $k = 1, 2, \ldots, r$, and $\eta_l \in \partial h_{i_l}(x, w_l)$ for $l = 1, 2, \ldots, n$ such that

$$\sum_{k=1}^{r} \alpha_k\xi_k + \sum_{l=1}^{n} \mu_l\eta_l = 0.$$

From the strict geodesic convexity assumption of (Θ, h), we have

$$\begin{aligned} 0 &= \sum_{k=1}^{r} \alpha_k \langle \xi_k, \exp_x^{-1} z \rangle_x + \sum_{l=1}^{n} \mu_l \langle \eta_l, \exp_x^{-1} z \rangle_x \\ &< \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(x)) + \sum_{l=1}^{n} \mu_l (h_{i_l}(z, w_l) - h_{i_l}(x, w_l)) \\ &\leq \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(x)) - \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \\ &\leq \sum_{k=1}^{r} \alpha_k (\Theta_k(z) - \Theta_k(x)) \leq 0 \end{aligned},$$

which is a contradiction. This completes the proof of the second part of this theorem. □

Remark 4.3. (1) The weak duality results derived in Theorem 4.2 generalize the analogous weak duality results in Theorem 4.4 derived in [46] from the Euclidean space to the Hadamard manifold setting.

(2) In the Hadamard manifold setting, Theorem 4.2 extends Proposition 3.3 in [48] from smooth MSIPs to nonsmooth MSIPs. Moreover, Theorem 4.2 extends Proposition 3.3 in the domain of robust optimization.

(3) In the Hadamard manifold setting, Theorem 4.2 extends Theorem 4.1 deduced in [59] from nonsmooth MSIPs to more general optimization problems, namely, RNMSIP.

By employing the notions of geodesic convexity of the pair of functions (Θ, h), in the following theorem, we deduce strong duality relations relating to the primal problem of RNMSIP and its corresponding dual problem RNMSID$_{MW}$.

Theorem 4.4 (Strong duality). *Let $z \in \mathcal{F}$ and $(x, \omega, \alpha, \mu) \in \mathcal{F}_{MW}$. Moreover, assume that $\Theta(z) = \Theta_{MW}(x, \omega, \alpha, \mu)$. Furthermore,*

(i) If (Θ, h) is geodesic convex at x, then z and (x, ω, α, μ) are robust weak Pareto solutions of RNMSIP and RNMSID$_{MW}$, respectively.
(ii) If (Θ, h) is strictly geodesic convex at x, then z and (x, ω, α, μ) are robust Pareto solutions of RNMSIP and RNMSID$_{MW}$, repectively.

Proof. (i) Let z' and $(x', \omega', \alpha', \mu')$ be any feasible elements of RNMSIP and RNMSIP$_{MW}$, respectively. From Theorem 4.2, one can observe that

$$\Theta(z') \not< \Theta_{MW}(x, \omega, \alpha, \mu) \quad \text{and} \quad \Theta(z) \not< \Theta_{MW}(x', \omega', \alpha', \mu').$$

According to the given condition we have,

$$\Theta(z') \not< \Theta_{MW}(x, \omega, \alpha, \mu) = \Theta(z),$$

and

$$\Theta_{MW}(x, \omega, \alpha, \mu) = \Theta(z) \not< \Theta_{MW}(x', \omega', \alpha', \mu').$$

Therefore, z and (x, ω, α, μ) are robust weak Pareto solutions of RNMSIP and RNMSID$_{MW}$, respectively.

(ii) Let z' and $(x', \omega', \alpha', \mu')$ be any feasible elements of RNMSIP and RNMSIP$_{MW}$, respectively. Employing Theorem 4.2, we have

$$\Theta(z') \not\preceq \Theta_{MW}(x, \omega, \alpha, \mu) \text{ and } \Theta(z) \not\preceq \Theta_{MW}(x', \omega', \alpha', \mu').$$

According to the given condition, we get

$$\Theta(z') \not\preceq \Theta_{MW}(x, \omega, \alpha, \mu) = \Theta(z),$$

and

$$\Theta_{MW}(x, \omega, \alpha, \mu) = \Theta(z) \not\preceq \Theta_{MW}(x', \omega', \alpha', \mu').$$

Therefore, z and (x, ω, α, μ) are robust Pareto solutions of RNMSIP and RNMSID$_{MW}$, respectively. □

Remark 4.5. (1) Theorem 4.4 generalizes Theorem 4.5 derived in [46] from the Euclidean space to the Hadamard manifold setting.

Under certain assumptions, the following theorem illustrates that a robust (weak) Pareto solution of the primal problem RNMSIP will also serve as a robust (weak) Pareto solution of the dual problem RNMSID$_{MW}$.

Theorem 4.6 (Direct duality). *Let us assume that $\overline{z} \in \mathcal{F}$ at which the ACQ is satisfied. Moreover, assume that pos$(\mathcal{R}(\overline{z}))$ is a closed set.*

(i) *If (Θ, h) is geodesic convex and $\overline{z}$ is a robust weak Pareto solution of RNMSIP, then there exist $\overline{\omega} \in (gph\mathscr{W})^n, \overline{\alpha} = (\overline{\alpha}_k)_k \in \mathbb{R}^r_+$, and $\overline{\mu} \in \mathbb{R}^n_+$, such that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a robust weak Pareto solution of RNMSID$_{MW}$ and $\Theta(\overline{z}) = \Theta_{MW}(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$.*

(ii *If (Θ, h) is strictly geodesic convex and $\overline{z}$ is a robust Pareto solution of RNMSIP, then there exist $\overline{\omega} \in (gph\mathscr{W})^n, \overline{\alpha} = (\overline{\alpha}_k)_k \in \mathbb{R}^r_+$, and $\overline{\mu} \in \mathbb{R}^n_+$, such that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a robust Pareto solution of RNMSID$_{MW}$ and $\Theta(\overline{z}) = \Theta_{MW}(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$.*

Proof. (i) Since $\overline{z}$ is a robust weak Pareto solution of RNMSIP, therefore by employing Theorem 3.6, we have there exist $\overline{\omega} = (\overline{i}, \overline{w}_l)_l \in (\text{gph}\mathscr{W})f^n, \overline{\alpha} = (\overline{\alpha}_k)_k \in \mathbb{R}^r_+ \setminus \{0\}$, with $\sum_{k=1}^r \alpha_k = 1$, and $\overline{\mu} = (\overline{\mu}_l)_l \in \mathbb{R}^n_+$, so that (3.1) and (3.2) are satisfied. Hence, $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu}) \in \mathcal{F}_{MW}$ and $\Theta(\overline{z}) = \Theta_{MW}(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$. From, Theorem 4.4, it follows that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a robust weak Pareto solution of RNMSID$_{MW}$.

(ii) By employing analogous arguments to those utilized in part (i) of the proof, we can effectively conclude and thereby complete the entire proof. □

Remark 4.7. (1) Theorem 4.6 generalizes Theorem 4.6 in [46] from the Euclidean space to the Hadamard manifold setting. the setting of Euclidean space.

(2) In the Hadamard manifold setting, Theorem 4.6 extends Proposition 4.3 in [48] from smooth MSIPs to nonsmooth MSIPs. Moreover, Theorem 4.6 extends Proposition 4.3 derived in [48] in the domain of robust optimization.

(3) In the Hadamard manifold setting, Theorem 4.6 extends Theorem 4.2 deduced in [59] from nonsmooth MSIPs to more general optimization problems, namely, RNMSIP.

Under certain assumptions, the following theorem deduces converse duality results, demonstrating that a robust (weak) Pareto solution of the dual problem RNMSID_{MW} will also serve as a robust (weak) Pareto solution of the primal problem RNMSIP.

Theorem 4.8 (Converse duality)**.** *Let us assume that pos($\mathcal{R}(x)$) is closed and ACQ is satisfied at $x \in \mathcal{F}$. Then the following assertions are true:*

(i) If (Θ, h) is geodesic convex at x and (x, ω, α, μ) is a robust weak Pareto solution of RNMSID$_{MW}$, then x is a robust weak Pareto solution of RNMSIP;
(ii) If (Θ, h) is strictly geodesic convex at x and (x, ω, α, μ) is a robust Pareto solution of RNMSID$_{MW}$, then x is a robust Pareto solution of RNMSIP.

Proof. (i) From the feasibility conditions of RNMSID_{MW}, we have

$$0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(x) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(x, w_l), \ \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \geq 0 \tag{4.2}$$

Moreover, from the satisfaction of the ACQ at x and the closeness of the set pos($\mathcal{R}(x)$) together with the fact that $x \in \mathcal{F}$, we have $\sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \leq 0$. Combining this with the inequality in (4.2), we get

$$\sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) = 0. \tag{4.3}$$

Furthermore as, $x \in \mathcal{F}$, pos($\mathcal{R}(x)$) is a closed set, and ACQ is satisfied at x, hence $\mu_l h_{i_l}(x, w_l) \leq 0, \ \forall l = 1, 2, \ldots, n$. Hence, combining (4.3) with the last inequality, we get $\mu_l h_{i_l}(x, w_l) = 0, \ \forall l = 1, 2, \ldots, n$. Therefore, Theorem 3.14 (i) ensures that x is a robust weak Pareto solution of RNMSIP.

(ii) Utilizing a similar approach to that employed in part (i), along with Theorem 3.14 (ii), we get the desired result. ☐

Remark 4.9. (1) Theorem 4.8 generalizes the Theorem 4.7 in Tung and Duy [46] from the context of Euclidean space to the framework of Hadamard manifolds.

4.2. **Wolfe duality.** For $x \in \mathscr{H}_1$, $\alpha = (\alpha_k)_k \in \mathbb{R}^r_+$, $\mu = (\mu_l)_l \in \mathbb{R}^n_+$, $\omega = (i_l, w_l)_l \in (\text{gph}\mathscr{W})^n$, and $e = (1, 1, \ldots, 1) \in \mathbb{R}^r$, we use $\Theta_W(x, \omega, \alpha, \mu) := \Theta(x) + \left(\sum_{l=1}^{n} \mu_l h_{i_l}(x, \omega_l)\right) e$, to denote the objective function of the Wolfe type dual problem. The Wolfe type dual model of the primal problem RNMSIP can be formulated in the following manner:

$$\begin{aligned} \text{RNMSID}_W\text{:} \ & \max \ \Theta_W(x, \omega, \alpha, \mu) \\ & \text{subject to } 0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(x) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(x, w_l), \\ & \sum_{k=1}^{r} \alpha_k = 1, \ x \in \mathscr{H}_1, \ \alpha \in \mathbb{R}^r_+, \ \mu \in \mathbb{R}^n_+, \ \omega \in (\text{gph}\mathscr{W})^n. \end{aligned}$$

Let $\mathcal{F}_W$ be the set of all feasible points of (RNMSID$_W$), then

$$\mathcal{F}_W := \Big\{(x,\omega,\alpha,\mu) \in \mathscr{H}_1 \times (\mathrm{gph}\mathscr{W})^n \times \mathbb{R}^r_+ \times \mathbb{R}^n_+ : \sum_{k=1}^r \alpha_k = 1,$$
$$0 \in \sum_{k=1}^r \alpha_k \partial_C \Theta_k(x) + \sum_{l=1}^n \mu_l \partial_C^z h_{i_l}(x,w_l)\Big\}.$$

Remark 4.10. (1) Given any $(x,\omega,\alpha,\mu) \in \mathcal{F}_W$, if $\Theta_W(\widetilde{x},\widetilde{\omega},\widetilde{\alpha},\widetilde{\mu}) \not\prec \Theta_W(x,\omega,\alpha,\mu)$, then $(\widetilde{x},\widetilde{\omega},\widetilde{\alpha},\widetilde{\mu}) \in \mathcal{F}_W$ is said to be robust weak Pareto solution of RNMSID$_W$.

(2) Given any $(x,\omega,\alpha,\mu) \in \mathcal{F}_W$, if $\Theta_W(\widetilde{x},\widetilde{\omega},\widetilde{\alpha},\widetilde{\mu}) \not\preceq \Theta_W(x,\omega,\alpha,\mu)$, then $(\widetilde{x},\widetilde{\omega},\widetilde{\alpha},\widetilde{\mu}) \in \mathcal{F}_W$ is said to be robust Pareto solution of RNMSID$_W$.

In the following theorem, we deduce weak duality relations relating to the primal problem of RNMSIP and its corresponding dual problem RNMSID$_W$.

Theorem 4.11 (Weak duality)**.** *Let $z \in \mathcal{F}$ and $(x,\omega,\alpha,\mu) \in \mathcal{F}_W$. Moreover, if (Θ,h) is geodesic, then $\Theta(z) \not\prec \Theta_W(x,\omega,\alpha,\mu)$. Furthermore, if (Θ,h) is strictly geodesic convex, then $\Theta(z) \not\preceq \Theta_W(x,\omega,\alpha,\mu)$.*

Proof. Let us assume that (Θ,h) is geodesic convex and we show that $\Theta(z) \not\prec \Theta_W(x,\omega,\alpha,\mu)$. Let us suppose that $\Theta(z) \prec \Theta_W(x,\omega,\alpha,\mu)$. This implies that

$$\Theta_k(z) < \Theta_k(x) + \sum_{l=1}^n \mu_l h_{i_l}(x,\omega_l) \ \forall k \in \{1,2,\dots,r\}. \tag{4.4}$$

From the feasibility condition of RNMSID$_W$, it follows that $(\alpha_k)_k \in \mathbb{R}^r_+$ and there exists at least one $\hat{k} \in \{1,2,\dots,r\}$ such that $\alpha_{\hat{k}} \neq 0$. Multiplying both sides of Inequality (4.4) by α_k, $k \in \{1,2,\dots,r\}$ and summing from 1 to r, we get

$$\sum_{k=1}^r \alpha_k(\Theta_k(z) - \Theta_k(x)) - \sum_{l=1}^n \mu_l h_{i_l}(x,\omega_l) < 0.$$

Since (x,ω,α,μ) is a feasible element of RNMSID$_W$, there are $\xi_k \in \partial_C \Theta_k(x)$ for $k = 1,2,\dots,r$, and $\eta_l \in \partial h_{i_l}(x,w_l)$ for $l = 1,2,\dots,n$ such that

$$\sum_{k=1}^r \alpha_k \xi_k + \sum_{l=1}^n \mu_l \eta_l = 0.$$

From the geodesic convexity assumption of (Θ,h) we have

$$\begin{aligned}
0 &= \sum_{k=1}^r \alpha_k \langle \xi_k, \exp_x^{-1} z\rangle_x + \sum_{l=1}^n \mu_l \langle \eta_l, \exp_x^{-1} z\rangle_x \\
&\leq \sum_{k=1}^r \alpha_k(\Theta_k(z) - \Theta_k(x)) + \sum_{l=1}^n \mu_l(h_{i_l}(z,w_l) - h_{i_l}(x,w_l)), \\
&\leq \sum_{k=1}^r \alpha_k(\Theta_k(z) - \Theta_k(x)) - \sum_{l=1}^n \mu_l h_{i_l}(x,w_l) < 0
\end{aligned}$$

which is a contradiction.

Now, we assume that (Θ, h) is strictly geodesic convex and we show that $\Theta \not\preceq \Theta_W(x,\omega,\alpha,\mu)$. Suppose on the contrary, let $\Theta(z) \preceq \Theta_W(x,\omega,\alpha,\mu) = \Theta(x) + \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l)e$. In a similar manner, one can show that

$$\sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) - \sum_{l=1}^{n} \mu_l h_{i_l}(x, \omega_l) \leq 0.$$

Since (x,ω,α,μ) is a feasible element of RNMSID_W, there are $\xi_k \in \partial_C \Theta_k(x)$ for $k = 1, 2, \ldots, r$, and $\eta_l \in \partial h_{i_l}(x, w_l)$ for $l = 1, 2, \ldots, n$ such that

$$\sum_{k=1}^{r} \alpha_k \xi_k + \sum_{l=1}^{n} \mu_l \eta_l = 0.$$

From the strict geodesic convexity assumption of (Θ, h) we have

$$\begin{aligned}
0 &= \sum_{k=1}^{r} \alpha_k \langle \xi_k, \exp_x^{-1} z\rangle_x + \sum_{l=1}^{n} \mu_l \langle \eta_l, \exp_x^{-1} z\rangle_x \\
&< \sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) + \sum_{l=1}^{n} \mu_l (h_{i_l}(z, w_l) - h_{i_l}(x, w_l)), \\
&\leq \sum_{k=1}^{r} \alpha_k(\Theta_k(z) - \Theta_k(x)) - \sum_{l=1}^{n} \mu_l h_{i_l}(x, w_l) \leq 0
\end{aligned}$$

which is a contradiction. □

Remark 4.12. (1) The weak duality results derived in Theorem 4.11 generalizes Theorem 4.1 derived in [46], from the context of Euclidean space to the framework of Hadamard manifold.

(2) If $\mathscr{H}_1 = \mathbb{R}^{n_1}$, the weak duality results derived in Theorem 4.11 generalize Theorem 5 established in [31] from smooth MSIP to nonsmooth MSIP.

(3) Theorem 4.11 extends Theorem 4.1 derived in [8] from multiobjective optimization problems to MSIPs and generalizes it from Euclidean spaces to Hadamard manifolds.

(4) In the Hadamard manifold setting, Theorem 4.11 extends Proposition 4.5 in [48] from smooth MSIPs to nonsmooth MSIPs. Moreover, Theorem 4.11 extends Proposition 4.5 derived in [48] in the domain of robust optimization.

(5) In the Hadamard manifold setting, Theorem 4.11 extends Theorem 4.6 deduced in [59] from nonsmooth MSIPs to more general optimization problems, namely, RNMSIP.

By employing the notions of geodesic convexity of the pair of functions (Θ, h), in the following theorem, we deduce strong duality relations relating to the primal the problem of RNMSIP and its corresponding dual problem RNMSID_W.

Theorem 4.13 (Strong duality). *Let $z \in \mathcal{F}$ and $(x,\omega,\alpha,\mu) \in \mathcal{F}_{MW}$. Moreover, assume that $\Theta(z) = \Theta_W(x,\omega,\alpha,\mu)$. Furthermore,*

(i) If (Θ, h) is geodesic convex at x, then z is a robust weak Pareto solution of (RNMSIP) and (x,ω,α,μ) is a robust weak Pareto solution of $RNMSID_W$.

(ii) If (Θ, h) is strictly geodesic convex at x, then z is a robust Pareto solution of (RNMSIP) and (x, ω, α, μ) is a robust Pareto solution of RNMSID$_W$.

Proof. (i) Let z' and $(x', \omega', \alpha', \mu')$ be any feasible elements of RNMSIP and RNMSIP$_W$, respectively, it follows from Theorem 4.11 that

$$\Theta(z') \not\prec \Theta_W(x, \omega, \alpha, \mu) \text{ and } \Theta(z) \not\prec \Theta_W(x', \omega', \alpha', \mu').$$

Employing the given condition, one has,

$$\Theta(z') \not\prec \Theta_W(x, \omega, \alpha, \mu) = \Theta(z),$$

and

$$\Theta_W(x, \omega, \alpha, \mu) = \Theta(z) \not\prec \Theta_W(x', \omega', \alpha', \mu').$$

Therefore, z and (x, ω, α, μ) are weak Pareto solutions of RNMSIP and RNMSID$_W$, respectively.

(ii) Let z' and $(x', \omega', \alpha', \mu')$ be any feasible elements of RNMSIP and RNMSIP$_W$, respectively, it follows from Theorem 4.11 that

$$\Theta(z') \not\preceq \Theta_W(x, \omega, \alpha, \mu) \text{ and } \Theta(z) \not\preceq \Theta_W(x', \omega', \alpha', \mu').$$

From the given condition, we have

$$\Theta(z') \not\preceq \Theta_W(x, \omega, \alpha, \mu) = \Theta(z),$$

and

$$\Theta_W(x, \omega, \alpha, \mu) = \Theta(z) \not\preceq \Theta_W(x', \omega', \alpha', \mu').$$

Therefore, z and (x, ω, α, μ) are Pareto solutions of RNMSIP and RNMSID$_W$, respectively. □

Remark 4.14. (1) Theorem 4.13 generalizes Theorem 4.2 derived in [46], from Euclidean space setting the framework of Hadamard manifolds.

(2) If $\mathscr{H}_1 = \mathbb{R}^{n_1}$, the strong duality results derived in Theorem 4.13 generalize Theorem 6 derived by Lee and Lee [31] from smooth MSIPs to nonsmooth MSIPs.

(3) Theorem 4.13 extends Theorem 4.3 derived by Chuong [8] from multiobjective optimization problems to MSIPs and generalizes it from Euclidean spaces to Hadamard manifolds.

Under certain assumptions, the following theorem illustrates that a robust (weak) Pareto solution of the primal problem RNMSIP will also serve as a robust (weak) Pareto solution of the dual problem RNMSID$_W$.

Theorem 4.15 (Direct Duality)**.** *Let $\overline{z} \in \mathcal{F}$ at which the ACQ is satisfied. Moreover, let pos$(\mathcal{R}(\overline{z}))$ is a closed set.*

(i) If (Θ, h) is geodesic convex and $\overline{z}$ is a robust weak Pareto solution of RNMSIP, then there exist $\overline{\omega} \in (gph\mathscr{W})^n, \overline{\alpha} = (\overline{\alpha}_k)_k \in \mathbb{R}^r_+$, and $\overline{\mu} \in \mathbb{R}^n_+$, such that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a robust weak Pareto solution of RNMSID$_W$ and $\Theta(\overline{z}) = \Theta_W(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$.

(ii) If (Θ, h) is strictly geodesic convex and $\overline{z}$ is a robust Pareto solution of RNMSIP, then there exist $\overline{\omega} \in (gph\mathscr{W})^n, \overline{\alpha} = (\overline{\alpha}_k)_k \in \mathbb{R}^r_+$, and $\overline{\mu} \in \mathbb{R}^n_+$, such that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a robust Pareto solution of RNMSID$_W$ and $\Theta(\overline{z}) = \Theta_W(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$.

Proof. (i) Since $\overline{z}$ is a robust weak Pareto solution of RNMSIP, by employing Theorem 3.6 we have, there exist $\overline{\omega} = (\bar{i}, \overline{w}_l)_l \in (\text{gph}\mathscr{W})^n, \overline{\alpha} = (\overline{\alpha}_k)_k \in \mathbb{R}^r_+ \setminus \{0\}$, with $\sum_{k=1}^{r} \alpha_k = 1$, and $\overline{\mu} = (\overline{\mu}_l)_l \in \mathbb{R}^n_+$, so that (3.1) and (3.2) are satisfied. Hence, $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu}) \in \mathcal{F}_W$, and $\Theta(\overline{z}) = \Theta(\overline{z}) + \sum_{l=1}^{n} \overline{\mu}_l h_{\bar{i}_l}(\overline{x}, \overline{w}_l)e = \Theta_W(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$. From, Theorem 4.13, it follows that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a robust weak Pareto solution of RNMSID$_W$.

(ii) By employing analogous arguments to those utilized in part (i) of the proof, we can effectively conclude and thereby complete the entire proof. □

Remark 4.16. (1) Theorem 4.15 generalizes Theorem 4.3 in [46] from the Euclidean space to the Hadamard manifold setting.

(2) In the Hadamard manifold setting, Theorem 4.15 extends Proposition 4.6 in [48] from smooth MSIPs to nonsmooth MSIPs. Moreover, Theorem 4.15 extends Proposition 4.6 derived in [48] in the domain of robust optimization.

(3) In the Hadamard manifold setting, Theorem 4.15 extends Theorem 4.7 deduced in [59] from nonsmooth MSIPs to more general optimization problems, namely, RNMSIP.

Under certain assumptions, the following theorem deduces converse duality results, demonstrating that a robust (weak) Pareto solution of the dual problem RNMSID$_W$ will also serve as a robust (weak) Pareto solution of the primal problem RNMSIP.

Theorem 4.17 (Converse duality)**.** *Let $x \in \mathcal{F}$ and $\mu_l h_{i_l}(x, w_l) = 0,\ \forall l = 1, 2, \ldots, n$. Then the following assertions are true:*

(i) *If (Θ, h) is geodesic convex at x and (x, ω, α, μ) is a robust weak Pareto solution of RNMSID$_W$, then x is a robust weak Pareto solution of RNMSIP;*

(ii) *If (Θ, h) is strictly geodesic convex at x and (x, ω, α, μ) is a robust Pareto solution of RNMSID$_W$, then x is a robust Pareto solution of RNMSIP.*

Proof. (i) From the feasibility conditions of RNMSID$_W$, we have

$$0 \in \sum_{k=1}^{r} \alpha_k \partial_C \Theta_k(x) + \sum_{l=1}^{n} \mu_l \partial_C^z h_{i_l}(x, w_l),$$

and from the given condition, it follows that

$$\mu_l h_{i_l}(x, w_l) = 0,\ \forall l = 1, 2, \ldots, n.$$

Therefore, Theorem 3.14 (ii) ensures that x is a robust weak Pareto solution of RNMSIP.

(ii) Utilizing a similar approach to that employed in part (i), along with Theorem 3.14 (ii), we get the desired result. □

The following example demonstrates the results for Wolfe duality in Theorem 4.13 and Theorem 4.15.

Example 4.18. Consider the same two-dimensional manifold $\mathscr{H}_1$ as considered in Example 3.16. Moreover, consider problem, which is the form RNMSIP as follows:

$$\text{P1: } \min\ \Theta(z) := (\Theta_1(z), \Theta_2(z)) = \Big(\frac{z_2 - e}{2}, \frac{|z_2 - e|}{2}\Big)$$
$$\text{subject to } h_i(z, w) := w - (1 - i)\ln z_1 - i \ln z_2 - i,$$

for all $i \in \mathcal{I} = [0, 1]$, $w \in \mathscr{W}_i = [0, 1 + i](\subseteq \mathbb{R})$ and $z \in \mathcal{S} := \{z \in \mathscr{H}_1 |\ z_1, z_2 \geq \frac{e}{2}\}$. Here, $\mathcal{S}$ is a geodesic convex subset of $\mathscr{H}_1$. The feasible set of RNMSIP is given by

$$\mathcal{F} := \{z \in \mathscr{H}_1 |\ z_1 \geq e,\ z_2 \geq e\}.$$

For the primal problem P1, the associated Wolfe type dual problem can be formulated as

$$\text{P1}_W :\ \max\ \Theta_W(x, \omega, \alpha, \mu)$$
$$= (\Theta_1(x) + \mu(w - (1 - i)\ln x_1 - i \ln x_2 - i), \Theta_2(x) + \mu(w - (1 - i)\ln x_1 - i \ln x_2 - i))$$

$$\text{subject to }\ (\#) \begin{cases} 0 \in \sum_{k=1}^{2} \alpha_k \partial_C \Theta_k(x) + \mu(-(1 - i)x_1 - ix_2),\ \alpha_1 + \alpha_2 = 1, \\ x \in \mathscr{H}_1,\ \alpha_1,\ \alpha_2,\ \mu \in \mathbb{R}_+,\ i \in \mathcal{I},\ w \in [0, 1 + i] \end{cases}$$

It can be verified that $\overline{z} = (e, e)$ is a robust weak Pareto solution of P1. Moreover, it can be inferred that $\mathscr{T}(\mathcal{F}, \overline{z}) = \{v \in \mathbb{R}^2 |\ v_1 \geq 0,\ v_2 \geq 0\}$ and $\text{pos}\mathcal{R}(\overline{z}) = \{v \in T_{\overline{z}}\mathscr{H}_1 |\ v_1 \leq 0,\ v_2 \leq 0\}$. Therefore, $\text{pos}\mathcal{R}(\overline{z})$ is a closed set and

$$(\mathcal{R}(\overline{z}))^{\leq} = \{v \in T_{\overline{z}}\mathscr{H}_1 |\ v_1 \geq 0,\ v_2 \geq 0\} \subseteq \mathscr{T}(\mathcal{F}, \overline{z}).$$

This implies that ACQ holds at $\overline{z}$. Moreover, we can show that (Θ, h) is geodesic convex on $\mathcal{S}$. By employing Theorem 4.15, we can conclude that $x = \overline{z} = (e, e)$ is a feasible point of problem P1_W. Furthermore, $\partial_C \Theta_1(\overline{z}) = \{(0, \frac{e^2}{2})\}$, $\partial_C \Theta_2(\overline{z}) = \text{co}\{(0, \frac{e^2}{2}), (0, -\frac{e^2}{2})\}$. Hence, we can see that $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu}) = ((e, e), (1, 0), (\frac{1}{2}, \frac{1}{2}), 0)$ is a solution of the system $(\#)$. Moreover, $\Theta(\overline{z}) = \Theta_W(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu}) = (0, 0)$. Then, by invoking Theorem 4.13, $(\overline{z}, \overline{\omega}, \overline{\alpha}, \overline{\mu})$ is a weak Pareto solution of (P1_W).

5. Conclusions and future directions

In this paper, we have considered an uncertain non-smooth multiobjective semi-infinite programming problem and its associated robust counterpart, namely, a robust nonsmooth multiobjective semi-infinite programming problem in the framework of Hadamard manifolds. The ACQ at the feasible point of RNMSIP is introduced in terms of Clarke subdifferential to derive KKT-type necessary optimality criteria for a local robust weak Pareto solution and a local robust Benson-proper solution to the problem RNMSIP. Moreover, we have established robust sufficient optimality conditions under the assumptions of geodesic convexity. The Mond-Weir and Wolfe type dual models are formulated for the primal problem RNMSIP, and the concept of geodesic convexity is employed to establish several duality results. Non-trivial examples have been provided in manifold settings to demonstrate the significance of the results established in this paper.

The outcomes in this paper expand the number of significant results from the literature beyond the context of Euclidean space to that of a broader space, namely the Hadamard manifold, and extend them in the area of robust optimization. In particular, the established results of this paper generalize the analogous results studied in [8, 24, 27, 31, 46] from the Euclidean space setting to the framework of Hadamard manifolds. Moreover, the findings of this paper extend the corresponding findings derived in [24, 27, 48, 59] in the domain of robust optimization. Additionally, the outcomes in this paper extend the analogous outcomes in [27] from single objective nonsmooth SIPs to nonsmooth MSIPs within the framework of robust optimization. Furthermore, the results derived in this paper extend the analogous results derived in [31] (considering data uncertainty in the Euclidean space setting), [48] (regarding the absence of data uncertainty in the Hadamard manifold setting) from smooth MSIPs to non-smooth MSIPs. In addition, our paper extends the results derived in [8] from multiobjective optimization problems to MSIPs.

The research presented in this paper serves several opportunities for future investigation. Based on the work of [11, 34, 35], the results derived in this paper could be further generalized for robust nonsmooth multiobjective semi-infinite programming problems on Hadamard manifolds, utilizing the notion of limiting subdifferentials.

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