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ON OPTIMALITY AND DUALITY FOR APPROXIMATE SOLUTIONS IN NONSMOOTH INTERVAL-VALUED MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING

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Abstract. The paper focuses on investigating an interval-valued multiobjective semi-infinite programming problem (IVMOSIP), where the involving functions are locally Lipschitz. Initially, we introduce approximate versions of some constraint qualifications that exist in semi-infinite programming for the IVMOSIP. Using these approximate constraint qualifications, we then derive necessary optimality conditions for identifying solutions that are approximately quasi-efficient for the IVMOSIP. Under the assumptions of approximate convexity, We derive the conditions under which the KKT type necessary optimality conditions become sufficient. An approximate dual model of Mond-Weir type is used to generate primal-dual relations and to obtain duality results. The Clarke subdifferential tool from nonsmooth analysis has been used to obtain the desired results, which are well demonstrated by examples.

1. Introduction

Multiobjective semi-infinite programming (MOSIP) is a mathematical programming framework that deals with optimization problems involving multiple objectives and infinite sets of constraints. In MOSIP, the decision variables are subject to a finite number of constraints, while the objective functions are optimized over an infinite set of constraints. Kanzi and Nobakhtian [20] derived necessary and sufficient optimality conditions for nonsmooth multiobjective semi-infinite programming problems (MOSIP) involving locally Lipschitz functions using Clarke subdifferentials. Kanzi [18] proved that under suitable constraint qualification of Maeda type strong KKT optimality conditions can be derived for the MOSIP. Yu [42] investigated strict minimizers of higher order for the MOSIP. Goberna and Kanzi [11] analysed isolated efficient solutions for the MOSIP involving convex functions. Kanzi [19] extended Caristi–Ferrara–Stefanescu result for MOSIP. Tung [38] presented strong KKT optimality conditions for the MOSIP via tangential subdifferential. Rezaee [35] characterized isolated efficient solutions for MOSIP. Tung [41] explored dual models for MOSIP using tangential subdifferentials. Tung [40] studied convex MOSIP involving interval-valued functions for optimality conditions and duality results. Su and Luu [37] obtained higher-order optimality conditions for Borwein properly efficient solutions of the MOSIP.

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Interval-valued multiobjective semi-infinite programming (IVMOSIP) extends interval-valued programming to handle multiple objectives and semi-infinite constraints. It provides a framework for finding Pareto optimal solutions in the presence of uncertainty and infinite sets of constraints, allowing decision-makers to make robust and balanced decisions. Gadhi and El Idrissi [10] analysed interval-valued MOSIP (IVMOSIP) using limiting subdifferentials. Huy Hung et al. [14] derived optimality conditions and duality results for approximate quasi Pareto solutions in MOSIP involving interval-valued functions. Jennane et al. [17] dveloped the KKT type optimality conditions by using Abadie's constraint qualification and convexificators for semi-infinite programs, where the multiobjective function and constraints both are interval-valued but need not be locally Lipschitz. Tung [39] established the KKT optimality conditions and investigated the duality problems for the semi-infinite programming with multiple interval-valued objective functions. Antczak et al. [7] studied the class of nondifferentiable semi-infinite vector optimization problems with both objective and constraints are interval-valued functions under appropriate invexity hypothesis.

IVMOSIP is a promising area of research that deals with optimization problems involving multiple objectives and semi-infinite constraints. While there have been some notable contributions in IVMOSIP in previous works [7, 10, 14, 17, 39], analogous results to those achieved in [18, 20] have not yet been obtained, particularly regarding approximate solutions as presented in [14]. Therefore, our research aims to fill this gap by exploring MOSIP with locally Lipschitz interval-valued approximately convex functions. Our objective is to derive necessary and sufficient optimality conditions and duality results, employing suitable constraint qualifications, in order to identify approximate solutions. By addressing these aspects, we seek to advance the understanding and practical applicability of IVMOSIP.

The structure of the paper is as follows: in Section 2 some preliminary results regarding MOSIP are given. In Section 3, we introduce approximate variants of some constraint qualifications given in [18, 20] for IVMOSIP to derive the approximate KKT type necessary optimality condition and strong approximate KKT optimality condition for IVMOSIP to identify approximate quasi weakly efficient solution of the MOSIP. Under the generalized approximate convexity assumptions, the sufficient optimality condition is derived in Section 4. In Section 5, approximate duality results in terms of Clarke subdifferentials are developed. The conclusions and future research possibilities are matter of the last Section 6.

2. Preliminaries

Consider a nonempty subset A of n-dimensional Euclidean space, denoted as $\mathbb{R}^n$. We represent the closure of A, convex hull of A, and convex cone (which contains the origin) generated by A as $cl(A)$, $conv(A)$, and $cone(A)$, respectively. Additionally, we define the *polar cone* and the *strict polar cone* of A as follows:

$$A^- := \{d \in \mathbb{R}^n \mid \langle x, d\rangle \leq 0, \ \forall x \in A\}$$

and

$$A^s := \{d \in \mathbb{R}^n \mid \langle x, d\rangle < 0, \ \forall x \in A\},$$

respectively, where $\langle ., . \rangle$ indicates the standard inner product in $\mathbb{R}^n$.

The *contingent cone* and the *Clarke tangent cone* to A at $\bar{x} \in cl(A)$ are defined by

$$\Gamma(A,\bar{x}) := \{d \in \mathbb{R}^n \mid \exists\{(t_k,d_k)\} \to (0^+,d) : \bar{x}+t_kd_k \in A, \forall k \in \mathbb{N}\},$$

and

$$T(A,\bar{x}) := \{d \in \mathbb{R}^n \mid \forall\{(t_k,x_k)\} \to (0^+,\bar{x})\ \exists\ d_k \to d : \bar{x}_k+t_kd_k \in A, \forall k \in \mathbb{N}\},$$

respectively. Notice that $\Gamma(A,\bar{x})$ is generally a nonconvex closed cone in $\mathbb{R}^n$.

Let $\bar{x}$ be an element of $\mathbb{R}^n$, and consider a function $h : \mathbb{R}^n \to \mathbb{R}$, which is locally Lipschitz at $\bar{x}$. This means that there exists a number $l > 0$ such that for all x and y in a neighborhood of $\bar{x}$, we have the inequality $\|h(x) - h(y)\| \leq l\|x - y\|$. If a function h satisfies this property for every point in a subset $A \subset \mathbb{R}^n$, we say that h is locally Lipschitz on A. We refer to [34] for more details of locally Lipschitz functions and their application in optimization. The *Clarke directional derivative* of h at $\bar{x}$ in the direction $v \in \mathbb{R}^n$ is defined by

$$h^o(\bar{x};v) := \limsup_{y\to\bar{x},t\downarrow 0} \frac{h(y+tv)-h(y)}{t}$$

and the *Clarke subdifferential* of h at $\bar{x}$ is defined by

$$\partial^0 h(\bar{x}) := \{\bar{x}^* \in \mathbb{R}^n | \langle \bar{x}^*, v\rangle \leq h^o(\bar{x};v), \forall v \in \mathbb{R}^n\},$$

If the function h is continuously differentiable at $\bar{x}$, then $\partial^0 h(\bar{x}) = \{\nabla h(\bar{x})\}$. Moreover, if the function h is convex, then the Clarke subdifferential $\partial^0 h(\bar{x})$ coincides with the subdifferential $\partial h(\bar{x})$ in the sense of convex analysis given by

$$\partial h(\bar{x}) := \{\bar{x}^* \in \mathbb{R}^n \mid h(x) \geq h(\bar{x}) + \langle \bar{x}^*, x-\bar{x}\rangle\ \ \forall x \in \mathbb{R}^n\}.$$

Proposition 2.1 ([9])**.** *Let both ξ and η be locally Lipschitz functions defined from $\mathbb{R}^n$ to $\mathbb{R}$ and let $\bar{x} \in dom(\xi) \cap dom(\eta)$. Then*

$$\partial^0(\xi+\eta)(\bar{x}) \subseteq \partial^0\xi(\bar{x}) + \partial^0\eta(\bar{x}).$$

Consider a multiobjective semi-infinite programming problem as follows:

$$\text{(MOSIP)}\qquad \begin{aligned} &\min f(x) := (f_1(x),\ldots,f_m(x)) \\ &\text{s.t. } x \in \mathcal{F} := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \forall j \in J\}, \end{aligned}$$

where $f_i, i \in I := \{1,2,...,m\}$, and $g_j, j \in J$ are locally Lipschitz real-valued functions from $\mathbb{R}^n$ to $\mathbb{R}$ with an index set J which is arbitrary nonempty not necessarily finite.

A point $\bar{x} \in \mathcal{F}$ is said to be an *efficient* (or a *weak efficient*) solution of the MOSIP, iff for any $x \in \mathcal{F}$, one has

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\} (\text{or} f(x) - f(\bar{x}) \notin -int\mathbb{R}^m_+).$$

The set of all efficient (or weakly efficient) solutions of the MOSIP is denoted by $\mathcal{F}^E$ (or $\mathcal{F}^{WE}$). The index set of all active constraints at $\bar{x} \in \mathcal{F}$ is given by

$$J(\bar{x}) := \{j \in J : g_j(\bar{x}) = 0\}.$$

For each $\bar{x} \in \mathcal{F}$ and $k \in I$, define

$$\mathcal{A}(\bar{x}) := \bigcup_{i\in I} \partial^0 f_i(\bar{x})$$

$$\mathcal{B}(\bar{x}) := \bigcup_{j \in J(\bar{x})} \partial^0 g_j(\bar{x})$$
$$\mathcal{A}_k(\bar{x}) := \bigcup_{i \in I, i \neq k} \partial^0 f_i(\bar{x}).$$

Definition 2.2. The MOSIP satisfies

(a) [20, Definition 3.2 (c)] the *regular constraint qualification* (RCQ) at $\bar{x} \in \mathcal{F}$, iff
$$(\mathcal{A}(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^- \subseteq \Gamma(\mathcal{F}, \bar{x}).$$

(b) [18] the *constraint qualification* (CQ) at $\bar{x} \in \mathcal{F}$, iff
$$(\mathcal{A}_k(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^s \neq \emptyset, \forall k \in I.$$

Let $\mathbb{R}_+^{|J|}$ denotes the set of all functions $\beta : J \to \mathbb{R}_+$ taking values $\beta_j := \beta(j) = 0$ for all $j \in J$ except for finitely many points.

Theorem 2.3 ([20, Theorem 3.4](KKT necessary optimality conditions))**.** *Let $\bar{x} \in \mathcal{F}^{WE}$. If RCQ holds at $\bar{x}$ and the cone $(\mathcal{B}(\bar{x}))$ is a closed cone, then there exist $\alpha_i \geq 0$ $(i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*
$$0 \in \sum_{i \in I} \alpha_i \partial^0 f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}), \quad \sum_{i \in I} \alpha_i = 1.$$

Assumption 2.4 (A)**.** The index set J is a nonempty compact subset of $\mathbb{R}^l$, the function $(x, j) \to g_j(x)$ is upper semicontinuous on $\mathbb{R}^n \times J$ and for each x, $\partial^0 g_j(x)$ is an upper semi-continuous mapping in j.

Theorem 2.5 ([18, Theorem 5](Strong KKT necessary condition))**.** *If CQ is satisfied at $\bar{x} \in \mathcal{F}^{WE}$ and the Assumption 2.4(A) holds, then there exist $\alpha_i > 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*
$$0 \in \sum_{i=1}^{m} \alpha_i \partial^0 f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}).$$

Let $\mathcal{K}_c := \{[p^L, p^U] : p^L, p^U \in \mathbb{R},\ p^L \leq p^U\}$ be the class of all closed and bounded intervals in $\mathbb{R}$. Let $\mathcal{P} := [p^L, p^U]$ and $\mathcal{Q} := [q^L, q^U]$ be two intervals in $\mathcal{K}_c$. Then

(a) $\mathcal{P} + \mathcal{Q} := \{p + q : p \in \mathcal{P}, q \in \mathcal{Q}\} = [p^L + q^L, p^U + q^U]$;
(b) $\mathcal{P} - \mathcal{Q} := \{p - q : p \in \mathcal{P}, q \in \mathcal{Q}\} = [p^L - q^U, p^U - q^L]$;
(c) For each $k \in \mathbb{R}$,
$$k\mathcal{P} := \{kp : p \in \mathcal{P}\} = \begin{cases} [kp^L, kp^U], & \text{if } k \geq 0 \\ [kp^U, kp^L], & \text{if } k < 0. \end{cases}$$

If $p^L = p^U$, then $\mathcal{P} = [p, p] = p$ which is a real number.

Definition 2.6 ([40, Definition 3])**.** Let $\mathcal{P} = [p^L, p^U]$ and $\mathcal{Q} = [q^L, q^U]$ be two intervals in $\mathcal{K}_c$. we say that:

(i) $\mathcal{P} \leq_{LU} \mathcal{Q}$ iff $p^L \leq q^L$ and $p^U \leq q^U$.

(ii) $\mathcal{P} <_{LU} \mathcal{Q}$ iff $\mathcal{P} \leq_{LU} \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$, or, equivalently, $\mathcal{P} <_{LU} \mathcal{Q}$ iff

$$\begin{cases} p^L < q^L \\ p^U \leq q^U \end{cases} or \quad \begin{cases} p^L \leq q^L \\ p^U < q^U \end{cases} or \quad \begin{cases} p^L < q^L \\ p^U < q^U. \end{cases}$$

(iii) $\mathcal{P} <^s_{LU} \mathcal{Q}$ iff $p^L < q^L$ and $p^U < q^U$.

Consider an interval-valued multiobjective semi-infinite programming problem as follows:

$$\text{(IVMOSIP)} \qquad \begin{aligned} LU-\min F(x) &:= (F_1(x), ..., F_m(x)) \\ \text{s.t. } x \in \mathcal{F} &:= \{x \in \mathbb{R}^n : g_j(x) \leq 0, j \in J\}, \end{aligned}$$

where $F_i\colon \mathbb{R}^n \to \mathcal{K}_c, i \in I := \{1, \dots, m\}$ are interval-valued functions defined by $F_i(x) = [f_i^L(x), f_i^U(x)]$, $f_i^L, f_i^U\colon \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions satisfying $f_i^L(x) \leq f_i^U(x)$ for all $x \in \mathbb{R}^n$, and $g_j(x) : \mathbb{R}^n \to \mathbb{R}, j \in J$ are locally Lipschitz functions. This problem has been studied by Gadhi et al. [10], Tung [40], Hung et al. [14], Jennane et al. [17], Antczak et al. [7] etc.

Hung et al. [14] introduced approximate solutions of (IVMOSIP) with respect to LU interval order relation.

Definition 2.7 ([14, Definition 3.1])**.** Let $\mathcal{E}_i^L$, $\mathcal{E}_i^U, i \in I$ be real-numbers satisfying $0 \leq \mathcal{E}_i^L \leq \mathcal{E}_i^U$ with $\mathcal{E}_i := [\mathcal{E}_i^L, \mathcal{E}_i^U]$ for all $i \in I$ and let $\mathcal{E} := (\mathcal{E}_1, \dots, \mathcal{E}_m)$. Then, $\bar{x} \in \mathcal{F}$ is a

(a) *type-1* $\mathcal{E}$-quasi Pareto solution of (IVMOSIP), denoted by $\bar{x} \in \mathcal{E} - \mathcal{F}_1^q$ $(IVMOSIP)$, iff there is no $x \in \mathcal{F}$ such that

$$F_i(x) + \mathcal{E}_i\|x - \bar{x}\| \leq_{LU} F_i(\bar{x}), \forall i \in I,$$

and

$$F_k(x) + \mathcal{E}_k\|x - \bar{x}\| <_{LU} F_k(\bar{x}), \text{ for at least one } k \in I;$$

(b) *type-2* $\mathcal{E}$-quasi Pareto solution of (IVMOSIP), denoted by $\bar{x} \in \mathcal{E} - \mathcal{F}_2^q$ $(IVMOSIP)$, iff there is no $x \in \mathcal{F}$ such that

$$F_i(x) + \mathcal{E}_i\|x - \bar{x}\| \leq_{LU} F_i(\bar{x}), \forall i \in I,$$

and

$$F_k(x) + \mathcal{E}_k\|x - \bar{x}\| <^s_{LU} F_k(\bar{x}), \text{ for at least one } k \in I;$$

(c) *type-1* $\mathcal{E}$-quasi-weakly Pareto solution of (IVMOSIP), denoted by $\bar{x} \in \mathcal{E} - \mathcal{F}_1^{qw}(IVMOSIP)$, iff there is no $x \in \mathcal{F}$ such that

$$F_i(x) + \mathcal{E}_i\|x - \bar{x}\| <_{LU} F_i(\bar{x}), \forall i \in I;$$

(d) *type-2* $\mathcal{E}$-quasi-weakly Pareto solution of (IVMOSIP), denoted by $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$, iff there is no $x \in \mathcal{F}$ such that

$$F_i(x) + \mathcal{E}_i\|x - \bar{x}\| <^s_{LU} F_i(\bar{x}), \forall i \in I.$$

Remark 2.8. If $\mathcal{E}_i = 0$, i.e. for any $i \in I$, $\mathcal{E}_i^L = \mathcal{E}_i^U = 0$, then the concepts of a type-1 $\mathcal{E}$-quasi Pareto solution, a type-2 $\mathcal{E}$-quasi Pareto solution coincides with a type-1 Pareto solution, a type-2 Pareto solution, respectively, and a type-1 $\mathcal{E}$-quasi-weakly Pareto solution and a type-2 $\mathcal{E}$-quasi-weakly Pareto solution coincides with a type-1 weakly Pareto solution and a type-2 weakly Pareto solution, respectively (see, e.g. [40]). The following inclusion relationships exist:

(a) $\mathcal{E}-\mathcal{F}_1^q(IVMOSIP) \subset \mathcal{E}-\mathcal{F}_2^q(IVMOSIP) \subset \mathcal{E}-\mathcal{F}_2^{qw}(IVMOSIP)$.
(b) $\mathcal{E}-\mathcal{F}_1^q(IVMOSIP) \subset \mathcal{E}-\mathcal{F}_1^{qw}(IVMOSIP) \subset \mathcal{E}-\mathcal{F}_2^{qw}(IVMOSIP)$.

Tung [40] proved a relationship between the IVMOSIP and the following MOSIP:

$$\text{(MOSIP)} \qquad \begin{array}{c} \mathbb{R}_+^{2m}-\min(f_1^L(x),\ldots,f_m^L(x),f_1^U(x),\ldots,f_m^U(x)) \\ \text{subject to } x\in\mathcal{F}. \end{array}$$

Lemma 2.9 ([40, Lemma 4])**.** *A feasible point $\bar{x}\in\mathcal{F}$ is a type−2 weakly Pareto solution of the IVMOSIP if and only if $\bar{x}$ is a weakly efficient solution of the MOSIP.*

3. KKT optimality condition

Kanzi and Nobakhtian [20, Theorem 3.4] provided the KKT necessary condition for a point to be a weakly efficient solution of MOSIP under some constraint qualification and Kanzi [18, Theorem 5] extended this result for strong case. Now we extend these results for MOSIP when objective function is interval-valued and solution is aprroximate weakly pareto optimal rather than weakly pareto optimal. For this we introduce approximate variants of some constraint qualifications for IVMOSIP which will be useful to derive KKT optimality conditions. Let $\mathcal{E}_i^L$, $\mathcal{E}_i^U, i\in I$ be real-numbers satisfying $0\le\mathcal{E}_i^L\le\mathcal{E}_i^U$ with $\mathcal{E}_i:=[\mathcal{E}_i^L,\mathcal{E}_i^U]$ for all $i\in I$ and let $\mathcal{E}:=(\mathcal{E}_1,\ldots,\mathcal{E}_m)$. For each $x\in\mathcal{F}$, define

$$\mathcal{A}_{\mathcal{E}}^L(x):=\bigcup_{i\in I}\partial^0(f_i^L+\mathcal{E}_i^L||.-\bar{x}||)(x);$$
$$\mathcal{A}_{\mathcal{E}}^U(x):=\bigcup_{i\in I}\partial^0(f_i^U+\mathcal{E}_i^U||.-\bar{x}||)(x);$$
$$\mathcal{A}_{k,\mathcal{E}}^L(x):=\bigcup_{i\in I,i\neq k}\partial^0(f_i^L+\mathcal{E}_i^L||.-\bar{x}||)(x);$$
$$\mathcal{A}_{k,\mathcal{E}}^U(x):=\bigcup_{i\in I,i\neq k}\partial^0(f_i^U+\mathcal{E}_i^U||.-\bar{x}||)(x);$$
$$\mathcal{A}_{\mathcal{E}}(x):=\mathcal{A}_{\mathcal{E}}^L(x)\cup\mathcal{A}_{\mathcal{E}}^U(x).$$

Definition 3.1. The IVMOSIP satisfies

(a) the *$\mathcal{E}$-regular constraint qualification*, denoted by $\mathcal{E}$-IV-RCQ, at $\bar{x}\in\mathcal{F}$, iff
$$(\mathcal{A}_{\mathcal{E}}(\bar{x}))^s\cap(\mathcal{B}(\bar{x}))^-\subseteq\Gamma(\mathcal{F},\bar{x});$$
(b) the *$\mathcal{E}$-constraint qualification*, denoted by $\mathcal{E}$-IV-CQ at $\bar{x}\in\mathcal{F}$, iff
$$(\mathcal{A}_{k,\mathcal{E}}^L(\bar{x})\cup\mathcal{A}_{\mathcal{E}}^U(\bar{x}))^s\cap(\mathcal{B}(\bar{x}))^s\neq\emptyset,\forall k\in I,$$
and
$$(\mathcal{A}_{\mathcal{E}}^L(\bar{x})\cup\mathcal{A}_{k,\mathcal{E}}^U(\bar{x}))^s\cap(\mathcal{B}(\bar{x}))^s\neq\emptyset,\forall k\in I.$$

Remark 3.2. For $\mathcal{E}_i^L=\mathcal{E}_i^U=0$ for all $i\in I$, we say that the IVMOSIP satisfies IV-RCQ and IV-CQ at $\bar{x}\in\mathcal{F}$, respectively.

Theorem 3.3 (Approximate KKT necessary optimality condition for IVMOSIP). *Let $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$. If $\mathcal{E}$-IV-RCQ holds at $\bar{x}$ and the cone $(\mathcal{B}(\bar{x}))$ is a closed cone, then there exist $\alpha_i^L \geq 0 (i \in I), \alpha_i^U \geq 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*

$$0 \in \sum_{i=1}^m \alpha_i^L \partial^0 f_i^L(\bar{x}) + \sum_{i=1}^m \alpha_i^U \partial^0 f_i^U(\bar{x})$$

$$+ \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}) + \sum_{i=1}^m (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \mathbb{B}_{\mathbb{R}^n}, \tag{3.1}$$

$$\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1. \tag{3.2}$$

Proof. Since $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$, therefore $\bar{x} \in \mathcal{F}_2^{qw}(\mathcal{E} - IVMOSIP)$, where $\mathcal{E} - IVMOSIP$ is given by

$$(\mathcal{E}\text{-IVMOSIP}) \quad \begin{aligned} & LU - \min F(x) + \mathcal{E}\|x - \bar{x}\| \\ & := (F_1(x) + \mathcal{E}_1\|x - \bar{x}\|, ..., F_m(x) + \mathcal{E}_m\|x - \bar{x}\|) \\ & \text{subject to } \ x \in \mathcal{F}. \end{aligned}$$

According to Lemma 2.9, we can conclude that the solution $\bar{x} \in \mathcal{F}$ is a weakly efficient solution of the $\mathcal{E}-$MOSIP given by

$$(\mathcal{E}\text{-MOSIP}) \quad \begin{aligned} & \mathbb{R}_+^{2m} - \min(f_1^L(x) + \mathcal{E}_1^L\|x - \bar{x}\|, ..., f_m^L(x) + \mathcal{E}_m^L\|x - \bar{x}\|, \\ & \quad f_1^U(x) + \mathcal{E}_1^U\|x - \bar{x}\|, ..., f_m^U(x) + \mathcal{E}_m^U\|x - \bar{x}\|) \\ & \text{subject to } \ x \in \mathcal{F}. \end{aligned}$$

Since $\mathcal{E}$-IV-RCQ holds at $\bar{x}$ and the cone $(\mathcal{B}(\bar{x}))$ is a closed cone, therefore by Theorem 2.3, there exist $\alpha_i^L \geq 0 (i \in I), \alpha_i^U \geq 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that

$$\begin{aligned} 0 \in & \sum_{i=1}^m \alpha_i^L \partial^0 (f_i^L + \mathcal{E}_i^L ||. - \bar{x}||)(\bar{x}) \\ & + \sum_{i=1}^m \alpha_i^U \partial^0 (f_i^U + \mathcal{E}_i^U ||. - \bar{x}||)(\bar{x}) \\ & + \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}), \\ & \sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1. \end{aligned}$$

By the property of the Clarke subdifferentials in Proposition 2.1, one has $\partial^0(f_i^L + \mathcal{E}_i^L||. - \bar{x}||)(\bar{x}) \subseteq \partial^0 f_i^L(\bar{x}) + \mathcal{E}_i^L \partial^0 ||. - \bar{x}||(\bar{x})$ and $\partial^0(f_i^U + \mathcal{E}_i^U||. - \bar{x}||)(\bar{x}) \subseteq \partial^0 f_i^U(\bar{x}) + \mathcal{E}_i^U \partial^0||. - \bar{x}||(\bar{x})$.

Since the Clarke subdifferential of the norm function $\partial^0||. - \bar{x}||(\bar{x}) = \mathbb{B}_{\mathbb{R}^n}$ (see [15, Example 4, p. 198]), we have the required result. □

We have the following corollary based on the above result for $\mathcal{E}_i^L = \mathcal{E}_i^U = 0$ for all $i \in I$.

Corollary 3.4. *(KKT necessary optimality condition for IVMOSIP) Let $\bar{x} \in \mathcal{F}_2^{qw}$ (IVMOSIP). If IV-RCQ holds at $\bar{x}$ and the cone $(\mathcal{B}(\bar{x}))$ is a closed cone, then there exist $\alpha_i^L \geq 0 (i \in I), \alpha_i^U \geq 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i^L \partial^0 f_i^L(\bar{x}) + \sum_{i=1}^{m} \alpha_i^U \partial^0 f_i^U(\bar{x}) + \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}),$$

$$\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1.$$

Similarly, by using Theorem 2.5 and Proposition 2.1, we can derive strong KKT necessary optimality conditions to identify approximate efficient solutions of the IVMOSIP.

Theorem 3.5 (Strong approximate KKT necessary optimality conditions for IVMOSIP)**.** *Let $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$. If $\mathcal{E}$-IV-CQ holds at $\bar{x}$ and Assumption 2.4(A) is satisfied, then there exist $\alpha_i^L > 0 (i \in I), \alpha_i^U > 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i^L \partial^0 f_i^L(\bar{x}) + \sum_{i=1}^{m} \alpha_i^U \partial^0 f_i^U(\bar{x})$$

$$+ \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}) + \sum_{i=1}^{m} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \mathbb{B}_{\mathbb{R}^n}. \tag{3.3}$$

Proof. Since $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$, therefore $\bar{x} \in \mathcal{F}_2^{qw}(\mathcal{E} - IVMOSIP)$, where $\mathcal{E} - IVMOSIP$ is given by

$$\begin{aligned} & LU - \min F(x) + \mathcal{E}\|x - \bar{x}\| \\ (\mathcal{E}\text{-IVMOSIP}) \qquad & := (F_1(x) + \mathcal{E}_1\|x - \bar{x}\|, ..., F_m(x) + \mathcal{E}_m\|x - \bar{x}\|) \\ & \text{subject to } \ x \in \mathcal{F}. \end{aligned}$$

According to Lemma 2.9, we can conclude that the solution $\bar{x} \in \mathcal{F}$ is a weakly efficient solution of the $\mathcal{E}-$MOSIP given by

$$\begin{aligned} & \mathbb{R}_+^{2m} - \min(f_1^L(x) + \mathcal{E}_1^L\|x - \bar{x}\|, ..., f_m^L(x) + \mathcal{E}_m^L\|x - \bar{x}\|, \\ (\mathcal{E}\text{-MOSIP}) \qquad & f_1^U(x) + \mathcal{E}_1^U\|x - \bar{x}\|, ..., f_m^U(x) + \mathcal{E}_m^U\|x - \bar{x}\|) \\ & \text{subject to } \ x \in \mathcal{F}. \end{aligned}$$

Since $\mathcal{E}$-IV-CQ holds at $\bar{x}$ and Assumption 2.4(A) is satisfied, therefore by Theorem 2.5, there exist $\alpha_i^L > 0 (i \in I), \alpha_i^U > 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that

$$0 \in \sum_{i=1}^{m} \alpha_i^L \partial^0 (f_i^L + \mathcal{E}_i^L ||. - \bar{x}||)(\bar{x})$$

$$+ \sum_{i=1}^{m} \alpha_i^U \partial^0 (f_i^U + \mathcal{E}_i^U ||. - \bar{x}||)(\bar{x}) + \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}),$$

By the property of the Clarke subdifferentials in Proposition 2.1, one has $\partial^0(f_i^L + \mathcal{E}_i^L||. - \bar{x}||)(\bar{x}) \subseteq \partial^0 f_i^L(\bar{x}) + \mathcal{E}_i^L \partial^0||. - \bar{x}||(\bar{x})$ and $\partial^0(f_i^U + \mathcal{E}_i^U||. - \bar{x}||)(\bar{x}) \subseteq \partial^0 f_i^U(\bar{x}) +$

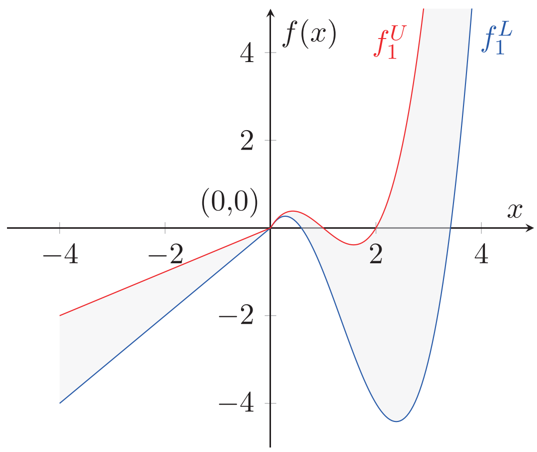

FIGURE 1. Plot of $F_1(x) := [f_1^L(x), f_1^U(x)]$.

$\mathcal{E}_i^U \partial^0 ||. - \bar{x}||(\bar{x})$.
Since the Clarke subdifferential of the norm function $\partial^0 ||. - \bar{x}||(\bar{x}) = \mathbb{B}_{\mathbb{R}^n}$ (see [15, Example 4, p. 198]), we have the required result. □

A corollary of the above result for $\mathcal{E}_i^L = \mathcal{E}_i^U = 0$ for every $i \in I$ is given as follows:

Corollary 3.6. *(Strong KKT necessary optimality conditions for IVMOSIP) Let $\bar{x} \in \mathcal{F}_2^{qw}(IVMOSIP)$. If IV-CQ holds at $\bar{x}$ and Assumption 2.4(A) is satisfied, then there exist $\alpha_i^L > 0 (i \in I), \alpha_i^U > 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i^L \partial^0 f_i^L(\bar{x}) + \sum_{i=1}^{m} \alpha_i^U \partial^0 f_i^U(\bar{x}) + \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}).$$

The following example illustrates the above results.

Example 3.7. Consider an IVMOSIP as follows:

$$\text{(P)} \qquad \begin{aligned} & LU-\min F(x) := (F_1(x), F_2(x)) \\ & \quad \text{s.t. } x \in \mathcal{F} = \{x \in \mathbb{R} : g_j(x) \leq 0, j \in J\}, \end{aligned}$$

where the index set $J = [5, 10], F_1(x) := [f_1^L(x), f_1^U(x)], F_2(x) := [f_2^L(x), f_2^U(x)]$ and $g_j(x)$ are given by

$$f_1^L(x) = \begin{cases} x^3 - 4x^2 + 2x & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}, \quad f_1^U(x) = \begin{cases} x^3 - 3x^2 + 2x & \text{if } x \geq 0 \\ \dfrac{x}{2} & \text{if } x < 0 \end{cases}$$

$$f_2^L(x) = \begin{cases} 2x^3 - 5x^2 + x & \text{if } x \geq 0 \\ 3x & \text{if } x < 0 \end{cases}, \quad f_2^U(x) = \begin{cases} 2x^3 - 4x^2 + x & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

and

$$g_j(x) = \begin{cases} x^2(x - j) & \text{if } x \geq 0 \\ jx^2 & \text{if } x < 0 \end{cases}$$

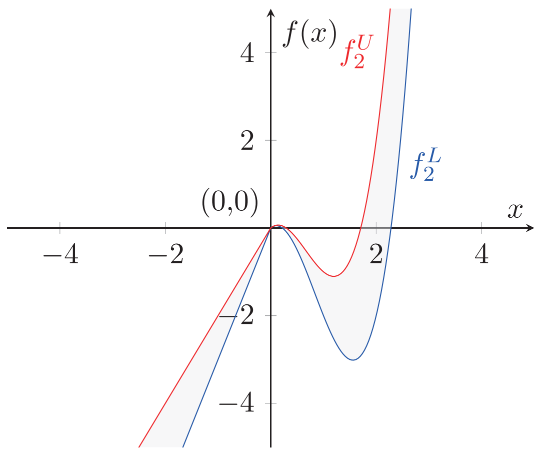

FIGURE 2. Plot of $F_2(x) := [f_2^L(x), f_2^U(x)]$.

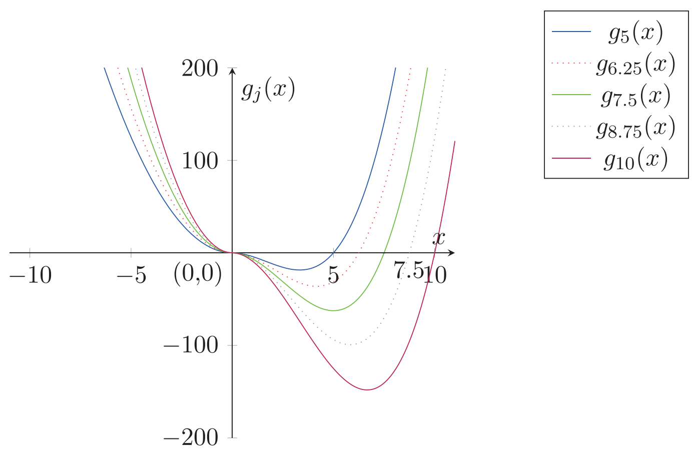

FIGURE 3. Plot of $g_j(x)$ for some values of $j \in [5, 10]$.

as shown in Figures 1, 2 and 3, respectively. Observe that f_1^L, f_1^U, f_2^L and f_2^U are locally Lipschitz at $\bar{x} = 0$ with Lipschitz constants 3, 2, 3.2 and 2, respectively. It is easy to see that $f_1^L(x) \leq f_1^U(x)$ and $f_2^L(x) \leq f_2^U(x)$ for every $x \in \mathcal{F} = [0, 5]$. Moreover, for $\bar{x} = 0$, one has

$$f_1^L(x) - f_1^L(0) < 0, \quad \forall x \in]2 - \sqrt{2}, 2 + \sqrt{2}[,$$
$$f_1^U(x) - f_1^U(0) < 0, \quad \forall x \in]1, 2[,$$
$$f_2^L(x) - f_2^L(0) < 0, \quad \forall x \in \left] \frac{5 - \sqrt{17}}{4}, \frac{5 + \sqrt{17}}{4} \right[,$$
$$f_2^U(x) - f_2^U(0) < 0, \quad \forall x \in \left] 1 - \sqrt{\frac{1}{2}}, 1 + \sqrt{\frac{1}{2}} \right[,$$

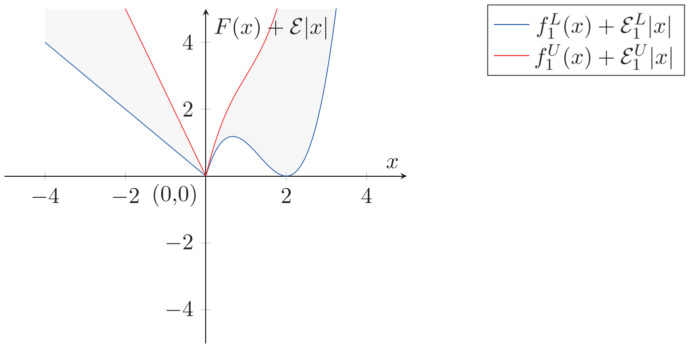

FIGURE 4. Plot of $F_1(x) + \mathcal{E}_1|x|$.

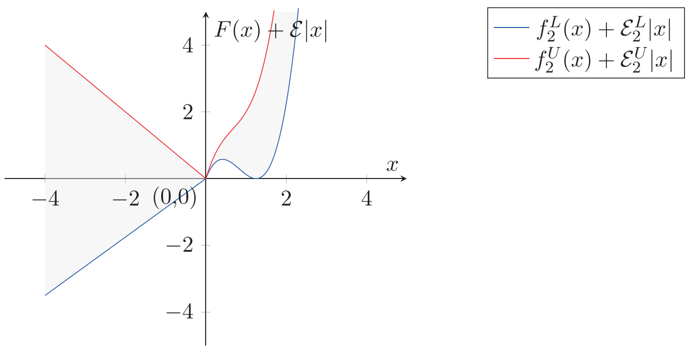

FIGURE 5. Plot of $F_2(x) + \mathcal{E}_2|x|$.

which gives $F_1(x) <_{LU}^s F_1(\bar{x})$ and $F_2(x) <_{LU}^s F_2(\bar{x})$ for every $x \in \left]1, 1+\sqrt{\frac{1}{2}}\right[$. Thus, $\bar{x} = 0 \in \mathcal{F}$ is not a *type-2* weakly Pareto solution of (P).

Now, for $\mathcal{E}_1^L = 2, \mathcal{E}_1^U = 3, \mathcal{E}_2^L = \frac{17}{8}, \mathcal{E}_2^U = 3$ and for any $x \in \mathcal{F}$, one has

$$f_1^L(x) - f_1^L(0) + \mathcal{E}_1^L|x-0| \geq 0, f_1^U(x) - f_1^U(0) + \mathcal{E}_1^U|x-0| \geq 0,$$
$$f_2^L(x) - f_2^L(0) + \mathcal{E}_2^L|x-0| \geq 0, f_2^U(x) - f_2^U(0) + \mathcal{E}_2^U|x-0| \geq 0,$$

as shown in the Figures 4 and 5, which implies that $\bar{x} = 0 \in \mathcal{F}$ is a *type-2* $\mathcal{E}$-quasi weakly Pareto solution of (P).

The Clarke subdifferentials of $f_1^L, f_1^U, f_2^L, f_2^U$ and $g_j, j \in J$ at $\bar{x} = 0$ are given by $\partial^0 f_1^L(\bar{x}) = [1,2]$, $\partial^0 f_1^U(\bar{x}) = \left[\frac{1}{2}, 2\right]$, $\partial^0 f_2^L(\bar{x}) = [1,3]$, $\partial^0 f_2^U(\bar{x}) = [1,2]$, and

$\partial^0 g_j(\bar{x}) = \{0\}, j \in J$, respectively, which gives

$$\begin{aligned}
&\mathcal{A}(\bar{x}) = \left[\frac{1}{2}, 3\right], \mathcal{B}(\bar{x}) = \{0\},\\
&\mathcal{A}_1^L(\bar{x}) = [1,3], \mathcal{A}_2^L(\bar{x}) = [1,2], \mathcal{A}^L(\bar{x}) = \mathcal{A}_1^L(\bar{x}) \cup \mathcal{A}_2^L(\bar{x}) = [1,3],\\
&\mathcal{A}_1^U(\bar{x}) = [1,2], \mathcal{A}_2^U(\bar{x}) = \left[\frac{1}{2}, 2\right], \mathcal{A}^U(\bar{x}) = \mathcal{A}_1^U(\bar{x}) \cup \mathcal{A}_2^U(\bar{x}) = \left[\frac{1}{2}, 2\right],\\
&(\mathcal{A}(\bar{x}))^s = -\mathbb{R}_+ \setminus \{0\}, (\mathcal{B}(\bar{x}))^- = \mathbb{R}, (\mathcal{B}(\bar{x}))^s = \emptyset,\\
&(\mathcal{A}(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^- = -\mathbb{R}_+ \setminus \{0\},\\
&\Gamma(\mathcal{F}, \bar{x}) = \mathbb{R}_+.
\end{aligned}$$

Since $(\mathcal{A}(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^- \nsubseteq \Gamma(\mathcal{F}, \bar{x})$, therefore (P) doesn't satisfy the IV-RCQ at $\bar{x} = 0$. Moreover, since

$$\begin{aligned}
(A_i^L(\bar{x}) \cup A^U(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^s = \emptyset, i = 1, 2,\\
(A^L(\bar{x}) \cup A_i^U(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^s = \emptyset, i = 1, 2,
\end{aligned}$$

therefore (P) doesn't satisfy the IV-CQ at $\bar{x} = 0$.

Now, for $\bar{x} = 0$, one has

$$\begin{aligned}
\partial^0(f_1^L + \mathcal{E}_1^L||.-0||)(\bar{x}) &= [-1, 4],\\
\partial^0(f_2^L + \mathcal{E}_2^L||.-0||)(\bar{x}) &= \left[\frac{7}{8}, \frac{25}{8}\right],\\
\partial^0(f_1^U + \mathcal{E}_1^U||.-0||)(\bar{x}) &= \left[\frac{-5}{2}, 5\right],\\
\partial^0(f_2^U + \mathcal{E}_2^U||.-0||)(\bar{x}) &= [-1, 4],
\end{aligned}$$

which gives

$$\begin{aligned}
\mathcal{A}_{\mathcal{E}}(\bar{x}) &= \left[\frac{-5}{2}, 5\right],\\
(\mathcal{A}_{\mathcal{E}}(\bar{x}))^s &= \emptyset,\\
(\mathcal{A}_{\mathcal{E}}(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^- &= \emptyset.
\end{aligned}$$

Since $(\mathcal{A}_{\mathcal{E}}(\bar{x}))^s \cap (\mathcal{B}(\bar{x}))^- \subseteq \Gamma(\mathcal{F}, \bar{x})$, therefore (P) satisfies the $\mathcal{E}$-IV-RCQ at $\bar{x} = 0$. Moreover, it is easy to observe that (P) does not satisfy $\mathcal{E}$-IV-CQ at $\bar{x} = 0$ as $(\mathcal{B}(\bar{x}))^s = \emptyset$.

Since all the conditions of Theorem 3.3 are satisfied, therefore there exist $\alpha_i^L \geq 0 (i \in I), \alpha_i^U \geq 0 (i \in I)$ with $\sum_{i=1}^{2}(\alpha_i^L + \alpha_i^U) = 1$ and $\beta \in \mathbb{R}_+^{|J|}$ such that

$$0 \in \sum_{i=1}^{2} \alpha_i^L \partial^0 f_i^L(0) + \sum_{i=1}^{2} \alpha_i^U \partial^0 f_i^U(0) + \sum_{j \in J} \beta_j \partial^0 g_j(0) + \sum_{i=1}^{2}(\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)[-1,1]$$

$$= \left[\alpha_1^L + \alpha_2^L, 2\alpha_1^L + 3\alpha_2^L\right] + \left[\frac{1}{2}\alpha_1^U + \alpha_2^U, 2\alpha_1^U + 2\alpha_2^U\right] + \{0\}$$

$$+ (2\alpha_1^L + 3\alpha_1^U + \frac{17}{8}\alpha_2^L + 3\alpha_2^U)[-1,1]$$

$$= \left[-\alpha_1^L - \frac{9}{8}\alpha_2^L - \frac{5}{2}\alpha_1^U - 2\alpha_2^U, 4\alpha_1^L + 5\alpha_1^U + \frac{41}{8}\alpha_2^L + 5\alpha_2^U\right]$$

which is true for any $\alpha_i^L \geq 0 (i \in I), \alpha_i^U \geq 0 (i \in I)$ with $\sum_{i=1}^{2}(\alpha_i^L + \alpha_i^U) = 1$ and for any $\beta \in \mathbb{R}_+^{|J|}$. Since, for $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}$(IVMOSIP), $\mathcal{E}$-RCQ holds at $\bar{x}$ and the cone $(\mathcal{B}(\bar{x}))$ is a closed cone then for any $\alpha_i^L \geq 0$ and $\alpha_i^U \geq 0$ the equation(1) and equation(2) given in Theorem 3.3 are satisfied for Problem(P) i.e. approximate KKT necessary optimality condition for Problem(P) is satisfied. Hence, Theorem 3.3 is verified .

4. Sufficient optimality conditions

The following weaker versions of generalized approximate convexity on the lines of Gupta et al. [12] and Bhatia et al. [8] will be used to derive sufficient optimality conditions.

Definition 4.1. Let $\epsilon \geq 0$. A locally Lipschitz function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be

(a) $\epsilon - \partial^0-$convex at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$h(x) - h(\bar{x}) \geq < \bar{x}^*, x - \bar{x} > -\epsilon||x - \bar{x}||, \forall \bar{x}^* \in \partial^0 h(\bar{x});$$

(b) $\epsilon - \partial^0-$quasiconvex at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$\forall x \in K : h(x) - h(\bar{x}) \leq 0 \implies \forall \bar{x}^* \in \partial^0 h(\bar{x}) :< \bar{x}^*, x - \bar{x} > -\epsilon||x - \bar{x}|| \leq 0;$$

(c) $\epsilon - \partial^0-$pseudoconvex at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$\forall x \in K : h(x) - h(\bar{x}) + \epsilon||x - \bar{x}|| < 0 \implies \forall \bar{x}^* \in \partial^0 h(\bar{x}) :< \bar{x}^*, x - \bar{x} > +\frac{\epsilon}{2}||x - \bar{x}|| < 0;$$

(d) $\epsilon - \partial^0-$pseudoconvex of type I at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$\forall x \in K : h(x) - h(\bar{x}) + \epsilon||x - \bar{x}|| < 0 \implies \forall \bar{x}^* \in \partial^0 h(\bar{x}) :< \bar{x}^*, x - \bar{x} >< 0;$$

(e) $\epsilon - \partial^0-$pseudoconvex of type II at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$\forall x \in K : h(x) - h(\bar{x}) < 0 \implies \forall \bar{x}^* \in \partial^0 h(\bar{x}) :< \bar{x}^*, x - \bar{x} > +\epsilon||x - \bar{x}|| < 0;$$

(f) $\epsilon - \partial^0-$quasiconvex of type I at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$\forall x \in K : h(x) - h(\bar{x}) \leq 0 \implies \forall \bar{x}^* \in \partial^0 h(\bar{x}) :< \bar{x}^*, x - \bar{x} > -\epsilon||x - \bar{x}|| \leq 0;$$

(g) $\epsilon - \partial^0-$quasiconvex of type II at $\bar{x} \in K \subset \mathbb{R}^n$ over K, iff

$$\forall x \in K : h(x) - h(\bar{x}) - \epsilon||x - \bar{x}|| \leq 0 \implies \forall \bar{x}^* \in \partial^0 h(\bar{x}) :< \bar{x}^*, x - \bar{x} >\leq 0.$$

Remark 4.2. For $\epsilon = 0$, the above concepts reduce to ∂^0-convexity, ∂^0-pseudoconvexity and ∂^0-quasiconvexity, respectively (see, e.g. [20, Definition 4.1]).

Define the following index sets $I^+ := \{i \in I : \alpha_i^L > 0 \text{ or } \alpha_i^U > 0\}$ and $J^+(\bar{x}) := \{j \in J(\bar{x}) : \beta_j > 0\}$.

Now, we are going to derive the sufficient optimality condition for a feasible solution to be an approximate weakly pareto optimal solution of the IVMOSIP under suitable generalised convexity assumptions.

Theorem 4.3 (Approximate sufficient optimality condition for IVMOSIP)**.** *Let $\mathcal{E}_i^L$, $\mathcal{E}_i^U, i \in I$ be real-numbers satisfying $0 \le \mathcal{E}_i^L \le \mathcal{E}_i^U$ for all $i \in I$, let $\mathcal{E} := (\mathcal{E}_1, \ldots, \mathcal{E}_m)$ with $\mathcal{E}_i := [\mathcal{E}_i^L, \mathcal{E}_i^U]$, and let $\bar{x} \in \mathcal{F}$. Assume that there exist $\alpha_i^L \ge 0 (i \in I), \alpha_i^U \ge 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i^L \partial^0 f_i^L(\bar{x}) + \sum_{i=1}^{m} \alpha_i^U \partial^0 f_i^U(\bar{x})$$

$$+ \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}) + \sum_{i=1}^{m} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \mathbb{B}_{\mathbb{R}^n}, \tag{4.1}$$

$$\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1. \tag{4.2}$$

(a) *If $f_i^L(i \in I^+), f_i^U(i \in I^+)$, and $g_j(j \in J^+)$ are $\mathcal{E}_i^L-$pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at $\bar{x}$ over $\mathcal{F}$, then $\bar{x} \in 2\mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$.*

(b) *If $f_i^L(i \in I^+), f_i^U(i \in I^+)$, and $g_j(j \in J^+)$ are $\mathcal{E}_i^L-$pseudoconvex of type I, $\mathcal{E}_i^U-$pseudoconvex of type I, and $(\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)-$pseudoconvex of type II, respectively, at $\bar{x}$ over $\mathcal{F}$, then $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$.*

Proof. (a) Suppose to the contrary that $\bar{x} \notin 2\mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$. Then, there exists $\tilde{x} \in \mathcal{F}$ such that

$$f_i^L(\tilde{x}) - f_i^L(\bar{x}) + 2\mathcal{E}_i^L ||\tilde{x} - \bar{x}|| < 0, \forall i \in I,$$

and

$$f_i^U(\tilde{x}) - f_i^U(\bar{x}) + 2\mathcal{E}_i^U ||\tilde{x} - \bar{x}|| < 0, \forall i \in I.$$

Also, $g_j(\tilde{x}) \le 0 = g_j(\bar{x})$ for every $j \in J(\bar{x})$. Since $f_i^L(i \in I^+), f_i^U(i \in I^+)$, and $g_j(j \in J^+)$ are $\mathcal{E}_i^L-$pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at $\bar{x}$ over $\mathcal{F}$, therefore

$$\langle \bar{x}^*_{f_i^L}, \tilde{x} - \bar{x} \rangle + \mathcal{E}_i^L \|\tilde{x} - \bar{x}\| < 0, \forall \bar{x}^*_{f_i^L} \in \partial^0 f_i^L(\bar{x}), \forall i \in I^+, \tag{4.3}$$

$$\langle \bar{x}^*_{f_i^U}, \tilde{x} - \bar{x} \rangle + \mathcal{E}_i^U \|\tilde{x} - \bar{x}\| < 0, \forall \bar{x}^*_{f_i^U} \in \partial^0 f_i^U(\bar{x}), i \in I^+, \tag{4.4}$$

$$\langle \bar{x}^*_{g_j}, \tilde{x} - \bar{x} \rangle \le 0, \forall \bar{x}^*_{g_j} \in \partial^0 g_j(\bar{x}), j \in J^+(\bar{x}). \tag{4.5}$$

Multiplying (4.3), (4.4) and (4.5) by $\alpha_i^L (i \in I^+)$, $\alpha_i^U (i \in I^+)$, and $\beta_j (j \in J^+(\bar{x}))$, respectively, and adding, one has

$$\left\langle \sum_{i \in I^+} \alpha_i^L \bar{x}^*_{f_i^L} + \sum_{i \in I^+} \alpha_i^U \bar{x}^*_{f_i^U} + \sum_{j \in J^+(\bar{x})} \beta_j \bar{x}^*_{g_j}, \tilde{x} - \bar{x} \right\rangle + \sum_{i \in I^+} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \|\tilde{x} - \bar{x}\| < 0, \tag{4.6}$$

$$\forall \bar{x}^*_{f_i^L} \in \partial^0 f_i^L(\bar{x})(I^+), \forall \bar{x}^*_{f_i^U} \in \partial^0 f_i^U(\bar{x})(I^+), \forall \bar{x}^*_{g_j} \in \partial^0 g_j(\bar{x})(j \in J^+(\bar{x})).$$

By Cauchy-Schwartz inequality, for any $b \in \mathbb{B}_{\mathbb{R}^n}$, one has

$$\langle b, \tilde{x} - \bar{x} \rangle \leq \|b\| \|\tilde{x} - \bar{x}\| \leq \|\tilde{x} - \bar{x}\|,$$

which implies that

$$\left\langle \sum_{i=1}^{m} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) b, \tilde{x} - \bar{x} \right\rangle \leq \sum_{i=1}^{m} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \|\tilde{x} - \bar{x}\|. \tag{4.7}$$

Adding (4.6) with (4.7) and using (4.1), we arrive at a contradiction and hence the result.

(b) The proof is similar to the part (a) above.

□

Example 4.4. Let us consider the problem (P) from Example 3.7. Now, if we take $\mathcal{E}_1^L = 1$, $\mathcal{E}_1^U = \frac{3}{2}$, $\mathcal{E}_2^L = \frac{17}{16}$, $\mathcal{E}_2^U = \frac{3}{2}$, then it is easy to see that f_i^L, f_i^U and g_j are $\mathcal{E}_i^L-$ pseudoconvex, $\mathcal{E}_i^U-$ pseudoconvex and quasiconvex, respectively, at $\bar{x} = 0$ over $\mathcal{F}$ for $i = 1, 2$ and $j \in J$.

Moreover, one has

$$\begin{aligned}
&\sum_{i=1}^{2} \alpha_i^L \partial^0 f_i^L(0) + \sum_{i=1}^{2} \alpha_i^U \partial^0 f_i^U(0) + \sum_{j \in J} \beta_j \partial^0 g_j(0) + \sum_{i=1}^{2} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)[-1, 1] \\
=& \left[\alpha_1^L + \alpha_2^L, 2\alpha_1^L + 3\alpha_2^L\right] + \left[\frac{1}{2}\alpha_1^U + \alpha_2^U, 2\alpha_1^U + 2\alpha_2^U\right] + \{0\} \\
&+ \left(1\alpha_1^L + \frac{3}{2}\alpha_1^U + \frac{17}{16}\alpha_2^L + \frac{3}{2}\alpha_2^U\right)[-1, 1]. \\
=& \left[-\frac{1}{16}\alpha_2^L - \alpha_1^U - \frac{1}{2}\alpha_2^U, 3\alpha_1^L + \frac{7}{2}\alpha_1^U + \frac{65}{16}\alpha_2^L + \frac{7}{2}\alpha_2^U\right]
\end{aligned}$$

which contains the origin in particular for $\alpha_1^L = 0$, $\alpha_2^L = 0$, $\alpha_1^U = \frac{1}{2}$, $\alpha_2^U = \frac{1}{2}$ and for any $\beta \in \mathbb{R}_+^{|J|}$. Since, all the conditions of Theorem 4.3 are satisfied, therefore $\bar{x} = 0 \in \mathcal{F}$ is $2\mathcal{E}-$ quasi weakly *type-2* Pareto solution of (P) as already verified in Example 3.7.

Similarly, we can derive sufficient optimality conditions for an approximate strong KKT point to be an quasi efficient solution of the IVMOSIP.

Theorem 4.5. *Let $\mathcal{E}_i^L$, $\mathcal{E}_i^U, i \in I$ be real-numbers satisfying $0 \le \mathcal{E}_i^L \le \mathcal{E}_i^U$ for all $i \in I$, let $\mathcal{E} := (\mathcal{E}_1, \dots, \mathcal{E}_m)$ with $\mathcal{E}_i := [\mathcal{E}_i^L, \mathcal{E}_i^U]$, and let $\bar{x} \in \mathcal{F}$. Assume that there exist $\alpha_i^L > 0 (i \in I), \alpha_i^U > 0 (i \in I)$, and $\beta \in \mathbb{R}_+^{|J|}$, such that*

$$0 \in \sum_{i=1}^m \alpha_i^L \partial^0 f_i^L(\bar{x}) + \sum_{i=1}^m \alpha_i^U \partial^0 f_i^U(\bar{x})$$

$$+ \sum_{j \in J(\bar{x})} \beta_j \partial^0 g_j(\bar{x}) + \sum_{i=1}^m (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \mathbb{B}_{\mathbb{R}^n}, \tag{4.8}$$

$$\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1. \tag{4.9}$$

(a) *If $f_i^L (i \in I), f_i^U (i \in I)$, and $g_j (j \in J^+(\bar{x}))$ are $\mathcal{E}_i^L-$pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at $\bar{x}$ over $\mathcal{F}$, then $\bar{x} \in 2\mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$.*

(b) *If $f_i^L (i \in I), f_i^U (i \in I)$, and $g_j (j \in J^+(\bar{x}))$ are $\mathcal{E}_i^L-$pseudoconvex of type I, $\mathcal{E}_i^U-$pseudoconvex of type I, and $(\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)-$pseudoconvex of type II, respectively, at $\bar{x}$ over $\mathcal{F}$, then $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$.*

5. Approximate dual models

Let $\mathcal{X} := (X_1, ..., X_m)$ and $\mathcal{Y} = (Y_1, ..., Y_m)$, where $X_i, Y_i, i \in I$, are intervals in $\mathcal{K}_c$. In what follows, we use the following notations for convenience.

$$\mathcal{X} \preceq_{LU} \mathcal{Y} \iff \begin{Bmatrix} X_i \le_{LU} Y_i, & \forall\, i \in I, \\ X_k <_{LU} Y_k & \text{for at least one } k \in I. \end{Bmatrix}.$$

$$\mathcal{X} \prec_{LU}^s \mathcal{Y} \iff X_i <_{LU}^s Y_i \ \ \forall i \in I.$$

$\mathcal{X} \not\preceq_{LU} \mathcal{Y}$ and $\mathcal{X} \not\prec_{LU}^s \mathcal{Y}$ are the negations of $\mathcal{X} \preceq_{LU} \mathcal{Y}$ and $\mathcal{X} \prec_{LU} \mathcal{Y}$, respectively.

Define

$$\mathcal{L}(y, \alpha^L, \alpha^U, \beta) := F(y) = ([f_1^L(y), f_1^U(y)], ..., [f_m^L(y), f_m^U(y)]),$$

for any $y \in \mathbb{R}^n, (\alpha^L, \alpha^U) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \setminus \{0_{\mathbb{R}^{2m}}\}$, and $\beta \in \mathbb{R}_+^{|J|}$.

The $\mathcal{E}-$Mond-Weir dual associated with the primal (IVMOSIP) is given as follows:

$$(\mathcal{E} - IVMOSIP_{MWD}) \qquad \max \mathcal{L}(y, \alpha^L, \alpha^U, \beta) \ \text{ s.t. } (y, \alpha^L, \alpha^U, \beta) \in \mathcal{F}_{MW},$$

where the feasible set is given by

$$\mathcal{F}_{MW} := \Bigg\{ (y, \alpha^L, \alpha^U, \beta) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^{|J|} : 0 \in \sum_{i=1}^m \alpha_i^L \partial^0 f_i^L(y) + \sum_{i=1}^m \alpha_i^U \partial^0 f_i^U(y) + \sum_{j \in J} \beta_j \partial^0 g_j(y) + \sum_{i=1}^m (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) \mathbb{B}_{\mathbb{R}^n},$$

$$\beta_j g_j(y) \ge 0, j \in J, \sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1 \Bigg\}.$$

The projection of $\mathcal{F}_{MW}$ on $\mathbb{R}^n$ is denoted by $pr\mathcal{F}_{MW}$. A feasible solution $(\bar{y}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}) \in \mathcal{F}_{MW}$ is a *type-2 $\mathcal{E}$-quasi weakly Pareto solution* of $(\mathcal{E} - IVMOSIP_{MWD})$, iff

$$\mathcal{L}(\bar{y}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}) + \mathcal{E}\|y - \bar{y}\| \not\prec_{LU}^{s} \mathcal{L}(y, \alpha^L, \alpha^U, \beta), \forall (y, \alpha^L, \alpha^U, \beta) \in \mathcal{F}_{MW}.$$

The following theorem describes weak duality relations for approximate quasi Pareto solutions between the primal (IVMOSIP) and the dual problem $(\mathcal{E} - IVMOSIP_{MWD})$.

Theorem 5.1 ($\mathcal{E}$-weak duality)**.** *Let $x \in \mathcal{F}$ and $(y, \alpha^L, \alpha^U, \beta) \in \mathcal{F}_{MW}$.*

(a) *If $f_i^L (i \in I^+), f_i^U (i \in I^+)$, and $g_j (j \in J^+)$ are $\mathcal{E}_i^L-$pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at y over $\mathcal{F} \cup pr\mathcal{F}_{MW}$, then*

$$F(x) + 2\mathcal{E}\|x - y\| \not\prec_{LU}^{s} \mathcal{L}(y, \alpha^L, \alpha^U, \beta).$$

(b) *If $f_i^L (i \in I^+), f_i^U (i \in I^+)$, and $g_j (j \in J^+)$ are $\mathcal{E}_i^L-$pseudoconvex of type I, $\mathcal{E}_i^U-$pseudoconvex of type I, and $(\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)-$pseudoconvex of type II, respectively, at $\bar{x}$ over $\mathcal{F} \cup pr\mathcal{F}_{MW}$, then*

$$f(x) + \mathcal{E}\|x - y\| \not\prec_{LU}^{s} \mathcal{L}(y, \alpha^L, \alpha^U, \beta).$$

Proof. (a) Let $x \in \mathcal{F}$ and $(y, \alpha^L, \alpha^U, \beta) \in \mathcal{F}_{MW}$, then there exist $y^*_{f_i^L} \in \partial^0 f_i^L(y)$, $y^*_{f_i^U} \in \partial^0 f_i^U(y)$, $i \in I$, $y^*_{g_j} \in \partial^0 g_j(y)$, $j \in J$, $b \in \mathbb{B}_{\mathbb{R}_n}$ such that

$$\sum_{i=1}^{m} \alpha_i^L y^*_{f_i^L} + \sum_{i=1}^{m} \alpha_i^U y^*_{f_i^U} + \sum_{j \in J} \beta_j y^*_{g_j} + \sum_{i=1}^{m} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) b = 0 \tag{5.1}$$

and

$$g_j(x) \le 0, \quad \beta_j g_j(y) \ge 0, \quad \forall j \in J. \tag{5.2}$$

Assume to the contrary that

$$F(x) + 2\mathcal{E}\|x - y\| \prec_{LU}^{s} \mathcal{L}(y, \alpha^L, \alpha^U, \beta),$$

that is

$$F_i(x) + 2\mathcal{E}_i\|x - y\| <_{LU}^{s} \mathcal{L}_i(y, \alpha^L, \alpha^U, \beta), \quad \forall i \in I,$$

that is,

$$\begin{cases} f_i^L(x) + 2\mathcal{E}_i^L ||x - y|| < f_i^L(y), \\ f_i^U(x) + 2\mathcal{E}_i^U ||x - y|| < f_i^U(y), \end{cases}$$

for all $i \in I$. By (5.2), it follows that

$$g_j(x) \le 0 \le g_j(y), \forall j \in J^+.$$

Since $f_i^L (i \in I^+), f_i^U (i \in I^+)$, and $g_j (j \in J^+)$ are $\mathcal{E}_i^L-$ pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at y over $\mathcal{F} \cup pr\mathcal{F}_{MW}$, therefore

$$\langle y^*_{f_i^L}, x - y \rangle + \mathcal{E}_i^L \|x - y\| < 0, \forall y^*_{f_i^L} \in \partial^0 f_i^L(y), \forall i \in I^+, \tag{5.3}$$

$$\langle y^*_{f_i^U}, x - y \rangle + \mathcal{E}_i^U \|x - y\| < 0, \forall y^*_{f_i^U} \in \partial^0 f_i^U(y), i \in I^+, \tag{5.4}$$

$$\langle y^*_{g_j}, x - y \rangle \le 0, \forall y^*_{g_j} \in \partial^0 g_j(y), j \in J^+. \tag{5.5}$$

Multiplying (5.3), (5.4) and (5.5) by $\alpha_i^L(i \in I^+)$, $\alpha_i^U(i \in I^+)$, and $\beta_j(j \in J^+)$, respectively, and adding, one has

$$\left\langle \sum_{i\in I^+} \alpha_i^L y^*_{f_i^L} + \sum_{i\in I^+} \alpha_i^U y^*_{f_i^U} + \sum_{j\in J^+(\beta)} \beta_j y^*_{g_j}, x-y \right\rangle + \sum_{i\in I^+} (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)\|x-y\| < 0, \tag{5.6}$$

$$\forall y^*_{f_i^L} \in \partial^0 f_i^L(y)(I^+), \forall\ y^*_{f_i^U} \in \partial^0 f_i^U(y)(I^+), \forall y^*_{g_j} \in \partial^0 g_j(y)(j \in J^+).$$

By Cauchy-Schwartz inequality, for any $b \in \mathbb{B}_{\mathbb{R}^n}$, one has

$$\langle b, x-y\rangle \le \|b\|\|x-y\| \le \|x-y\|,$$

which implies that

$$\left\langle \sum_{i=1}^m (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U) b, x-y \right\rangle \le \sum_{i=1}^m (\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)\|x-y\|. \tag{5.7}$$

Adding (5.6) with (5.7) and using (5.1), we arrive at a contradiction and hence the result.

(b) The proof is similar to the part (a) above.

□

Theorem 5.2 ($\mathcal{E}-$ strong duality). *Let $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP)$ such that $\mathcal{E} - IV - RCQ$ is satisfied at $\bar{x}$ and the cone$\mathcal{B}(\bar{x})$ is closed. Then, there exist $\bar{\alpha}_i^L \ge 0(i \in I), \bar{\alpha}_i^U \ge 0(i \in I)$, and $\bar{\beta} \in \mathbb{R}_+^{|J|}$ such that $(\bar{x}, \bar{\alpha}_L, \bar{\alpha}_U, \bar{\beta}) \in \mathcal{F}_{MW}$ and $F(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\alpha}_L, \bar{\alpha}_U, \bar{\beta})$. Furthermore,*

(a) *If $f_i^L(i \in I), f_i^U(i \in I)$, and $g_j(j \in J^+(\bar{x}))$ are $\mathcal{E}_i^L-$pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at $\bar{x}$ over $\mathcal{F}$, then $\bar{x} \in 2\mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP_{MWD})$.*

(b) *If $f_i^L(i \in I), f_i^U(i \in I)$, and $g_j(j \in J^+(\bar{x}))$ are $\mathcal{E}_i^L-$pseudoconvex of type I, $\mathcal{E}_i^U-$pseudoconvex of type I, and $(\alpha_i^L \mathcal{E}_i^L + \alpha_i^U \mathcal{E}_i^U)-$pseudoconvex of type II, respectively, at $\bar{x}$ over $\mathcal{F}$, then $\bar{x} \in \mathcal{E} - \mathcal{F}_2^{qw}(IVMOSIP_{MWD})$.*

Proof. By the assumptions in the theorem, it follows from Theorem 3.3 that, there exist $\bar{\alpha}_i^L \ge 0(i \in I), \bar{\alpha}_i^U \ge 0(i \in I)$, and $\bar{\beta} \in \mathbb{R}_+^{|J|}$ such that $(\bar{x}, \bar{\alpha}_L, \bar{\alpha}_U, \bar{\beta}) \in \mathcal{F}_{MW}$ and $F(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\alpha}_L, \bar{\alpha}_U, \bar{\beta})$.

(a) Since $f_i^L(i \in I^+), f_i^U(i \in I^+)$, and $g_j(j \in J^+)$ are $\mathcal{E}_i^L-$pseudoconvex, $\mathcal{E}_i^U$-pseudoconvex, and quasiconvex, respectively, at $\bar{x}$ over $\mathcal{F} \cup pr\mathcal{F}_{MW}$, therefore by Theorem 5.1, it follows that

$$\mathcal{L}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}) + 2\mathcal{E}\|x - \bar{x}\| \not\prec^s_{LU} \mathcal{L}(y, \alpha^L, \alpha^U, \beta),$$

for any $(y, \alpha^L, \alpha^U, \beta) \in \mathcal{F}_{MW}$. Hence, $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\beta}) \in 2\mathcal{E} - \mathcal{F}_2^{qw}$ $(IVMOSIP_{MWD})$.

(b) The proof of (b) is similar to that of (a).

□

6. Conclusions

We have introduced approximate versions of the regular constraint qualification [20] and the Cottle constraint qualification [18] for an IVMOSIP. We have derived KKT necessary optimality conditions to identify approximate quasi weakly efficient solutions [14] for IVMOSIP. The sufficient optimality conditions are verified under generalized approximate convexity assumptions [8, 12]. Dual models are also developed and duality results are derived. The results are illustrated with examples.

The results of this paper may be extended under additional assumptions of saddle point criterion [22, 31], equilibrium constraints [25, 32, 33], vanishing constraints [13,23,29,30] etc. Moreover, we may extend these results for some other aprroximate solution concepts (see, e.g. [6, 24, 27, 28]) and constraint qualification under the assumption of some additional generalized convexity [16,26,36] using convexificators [21].

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