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# SOME MULTIVALUED FIXED POINT RESULTS UNDER WEAK TOPOLOGY WITH APPLICATION TO INTEGRAL INCLUSION

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ABSTRACT. In this paper, suitable conditions are established to guarantee the existence of solutions of multivalued operator equations in WC-Banach algebras. In addition, some new hybrid fixed point theorems for multivalued operators are proved under these conditions. Finally, an application for integral inclusions is presented. Furthermore, our study extends single-valued results to multivalued for weak topology.

## 1. INTRODUCTION

In recent times many authors have been interested in the various types of operator equations in Banach spaces and Banach algebras [4–6, 8, 9, 11, 12, 17, 19–21]. Especially using the combining of Banach and Schauder fixed point theorems, Krasnoselskii [19] introduced that for a convex subset $N$ of a Banach space $A$, if (i) $S$ is a continuous and compact operator, (ii) $L$ is a contraction operator on $N$ and (iii) $Lu + Sv \in N$ for each $u, v \in N$, then equation $u = Lu + Su$ has a solution $u$ in $N$. Burton [12] improved the condition (iii) and proved that if $u = Lu + Sv$ for all $v \in N$, then $u \in N$. This result provides significant convenience in the applications of functional differential equation, integral equation and stability theory. The solution of these nonlinear single-valued equations for weak topology case was presented in [2, 5–8]. Then in [8, 9], Ben Amer *et al.* found out that the mixed types of such equations on Banach algebras have at least one fixed point.

In [5], Banas and Taoudi presented the definition of *WC*-Banach algebras and related fixed point theorems for single-valued operators on *WC*-Banach algebras. In [18], Jeribi *et al.* obtained several results on the existence of solutions of the sum and product of nonlinear single-valued operators in *WC*-Banach algebras for weak topology. In [14, 15], Dage introduced the multivalued version of Schauder's fixed point theorem and the fixed point theorems of multivalued operators in Banach algebras. Then in [10], Ben Amar *et al.* presented some fixed point theorems for multivalued operators under weak topology on Banach algebras. The work [10] is mainly based on $D$-set-Lipschitzian associated with the measure of weak noncompactness.

We establish the existence of fixed points of nonlinear operator inclusion of the form

$$u \in Lu + SuTu, \quad u \in N \tag{1.1}$$

in *WC*-Banach algebra $A$, where $L$, $S$ and $T$ are nonlinear multivalued operators.

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It is clear that the nonlinear operator inclusion (1.1) under weak topology in Banach algebras may not has a fixed point using the combination of the classic Schauder theorem and Banach fixed point theorem in [3, 14, 15].To find the fixed points of these nonlinear operator inclusions in WC-Banach algebras, suitable approaches are needed.

In this study, we investigate the existence of solutions for inclusion (1.1) in $WC$-Banach algebras. For this, we create the appropriate conditions by using the weak sequential closed graph feature of the multivalued operators and the generalized $D$-Lipschitzian structure. Then, we prove some new multivalued hybrid fixed point theorems under these conditions for weak topology. In addition, this work is based on the conditions $(H)$, which is a more general concept than weak sequentially upper semi-continuity, the generalized $D$-Lipschitzian, and the weak sequential closed graph of the multivalued operators. In the last section, as an application of our results, we investigate the existence of solutions of the nonlinear integral inclusion

$$u(t) \in X(u(t)) \int_0^1 Y(s, u(s))ds + Z(u(t)), \qquad t \in [0,1]$$

in $C([0,1], A)$ which is a reflexive WC-Banach algebra, where $X : A \to A$, $Y : [0,1] \times A \to P_{cv}(A)$, $Z : A \to A$ for $\forall t \in [0,1]$ and $u \in A$.

## 2. Preliminaries

Let $(A, \|.\|)$ be a normed linear space. The space $A$ is said to be normed algebra if $A$ is algebra and for all $u, v \in A$, $\|uv\| \le \|u\| \|v\|$. Then the normed algebra $A$ is said to be Banach algebra if the space $A$ is a Banach space.

A Banach algebra $A$ is said to be a $WC$-Banach algebra if the product of every weakly compact subset of Banach algebra $A$ is weakly compact.

Let $A$ be a Banach space with zero element $\theta$. Let

$$P(A) := \{N \subset A : N \neq \varnothing\},$$
$$P_b(A) := \{N \subset A : N \text{ is nonempty and bounded}\},$$
$$P_{cl}(A) := \{N \subset A : N \text{ is nonempty and closed}\},$$
$$P_{wcp}(A) := \{N \subset A : N \text{ is nonempty, weakly compact}\},$$
$$P_{wcl,cv}(A) := \{N \subset A : N \text{ is nonempty, weakly closed and convex}\}.$$

Assume $L : N \subseteq A \to P(N)$ is a multivalued operator, the range $R(L)$ and the graph of $L$ is determined by

$$R(L) = \bigcup_{u \in N} L(u) \quad \text{and} \quad G(L) = \{(u,v) \in N \times A : u \in N,\ v \in L(u)\}.$$

Let denote, for $M, N \subseteq A$

$$M \pm N = \{u \pm v : u \in M, v \in N\},$$
$$M.N = \{uv : u \in M, v \in N\}$$

Let $d_H(.,.)$ denote the Hausdorff-Pompeiu metric such that

$$d_H : P_{cl,b}(A) \times P_{cl,b}(A) \to \mathbb{R}^+$$

defined by

$$d_H(M,N) = \max\{d(M,N), d(N,M)\}$$

for each $M, N \in P_{cl,b}(A)$, where

$$d(M,N) = \sup_{u\in M} d(u,N) \quad \text{and} \quad d(N,M) = \sup_{v\in N} d(M,v)$$

$$d(u,N) = \inf_{v\in N} \|u-v\| \quad \text{and} \quad d(M,v) = \inf_{u\in M} \|u-v\|.$$

Hausdorff space with the metric $d_H(.,.)$ is a metric space.

In Dhage [14], for any $C \in P_{cl,b}(A)$,

$$d_H(0,C) = \|C\| = \sup\{\|c\| : c \in C\}.$$

and

$$d_H(MC, NC) \le d_H(0,C)\, d_H(M,N).$$

Also we have, for $C, D, M, N \in P_{cl,b}(M)$,

$$\begin{aligned} d(M\cup N, C\cup D) &\le \max\{d(M,C), d(N,D)\} \\ &\le \max\{d(M,C), d(C,M)\} + \max\{d(M,C), d(N,D)\} \\ &= d_H(M,C) + d_H(N,D) \end{aligned}$$

The multivalued operator $L : N \to P(N)$ is called weakly sequentially upper-semicontinuous (w.s.u.sc., for short) if for any weakly closed subset $M$ of $N$, $L^{-1}(M) = \{u \in N : L(u) \cap M \neq \emptyset\}$ is sequentially closed for weak topology in $N$. The operator $L$ is called weakly sequentially closed ( That is, has weakly sequentially closed graph (w.s.c.g., for short)) if for each $(u_n) \subset N$, $u_n \xrightarrow{w} u$ in $N$ and for each $(v_n)$ with $v_n \in L(u_n)$, $v_n \xrightarrow{w} v$ in $A$ implies $v \in L(u)$. If $L$ is weakly sequentially closed on $N$, then for every $u \in N$, $L(u)$ is a weakly sequentially closed subset of $N$. Also, $L$ is called to be weakly compact if the set $R(L)$ is weakly relatively compact in $N$.

We need to the following theorem of Agarwal and O'Regan [1, Theorem 2.3].

**Theorem 2.1.** *Let $N$ be a closed convex subset of Banach space $A$. Assume $L : N \to P(N)$ is a weakly compact and w.s.u.sc. multivalued operator. Then $L$ has a fixed point in $N$.*

We shall use the following condition for the multivalued operator $L : N \to P(N)$ on WC-Banach algebra. This condition is presented in study [17].

$$(H)\quad \begin{cases} \text{For every } (u_n) \subset N \text{ such that } u_n \xrightarrow{w} u \text{ in } N \text{ and for every} \\ \text{sequence } (v_n) \subset P(N) \text{ with } v_n \in L(u_n) \text{ for } n \in \mathbb{N}, \text{ there} \\ \text{exists a subsequence } (v_{n_k}) \text{ of } (v_n) \text{ such that } v_{n_k} \xrightarrow{w} v \in L(u). \end{cases}$$

If $L$ satisfies condition $(H)$, it is clear that $L$ is w.s.u.sc. Also if $L(N)$ is relatively weakly compact then $L$ has w.s.c.g. if and only if $L$ satisfies condition $(H)$ (see [17, Theorem 2.1]).

## 3. The Solutions of the Sum and Product of Multivalued Operators under Weak Topology

Now we research the existence of solutions of nonlinear operator inclusion (1.1) of the multivalued operators under weak topology in $WC$-Banach algebras. This inclusion is based on the generalized $D$-Lipschitzian and the weak sequential closed graph of multivalued operators. For this, we need to the following definition and a result of multivalued operators with generalized $D$-Lipschitzian contraction.

The multivalued operator $L : N \subseteq A \to P(N)$ is said to be the generalized $k$-contraction if

$$d_H(L(u), L(v)) \leq k\|u - v\|$$

for each $u, v \in N$ and fixed $k \in [0, 1)$ (see [22]).

**Definition 3.1.** The multivalued operator $L : N \subseteq A \to P(N)$ is said to be the generalized $D$-Lipschitzian if there is a continuous nondecreasing function $\tau_L : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\tau_L(0) = 0$, for any $r > 0$, $\tau_L(r) < r$ and

$$d_H(L(u), L(v)) \leq \tau_L(\|u - v\|)$$

for all $u, v \in N$.

The generalized $k$-contraction is a generalized $D$-Lipschitzian, but the converse is generally not true.

The following example can easily explain this statement.

**Example 3.2.** Let $A = C(\mathbb{R}, [0, 1])$ and assume that $L : A \to P(A)$ and $h : \mathbb{R} \times [0, 1] \to [0, 1]$ are defined by

$$L(u)(s) = \{h(s, u(s))\} = \left\{u(s) - u^3(s) + \sin^2(s)\right\}$$

for every $u \in A$ and $s \in \mathbb{R}$.

As the some main results of our work we give the following theorems.

The proof of the following theorem emerges immediately from Temel [21] and Theorem 9.A of Zeidler [22].

**Theorem 3.3.** *Let $N$ be a nonempty weakly closed and bounded convex subset of WC-Banach algebra $A$. Suppose that the multivalued operator $L : N \to P_{wcl,cv}(N)$ is a generalized $D$-Lipschitzian and has w.s.c.g. Then there exists a $u \in N$ such that $u \in L(u)$.*

**Theorem 3.4.** *Suppose $A$ is a WC-Banach algebra and $N \subset A$ is a nonempty weakly closed convex subset. $S : N \to P(N)$ is a regular multivalued operator and $L, T : N \to P_{wcl,cv}(N)$ are multivalued operators such that*

*(i) $\left(\frac{I-L}{S}\right)^{-1}$ exists on $T(N)$,*

*(ii) $\left(\frac{I-L}{S}\right)^{-1} T$ has w.s.c.g.,*

*(iii) $\left(\frac{I-L}{S}\right)^{-1} T(N)$ is relatively weakly compact,*

*(iv) For each $v \in N$, $u \in L(u) + S(u)T(v)$ implies $u \in N$,*

*(v) $\left(\frac{I-L}{S}\right)^{-1} T(N) \in P_{wcl,cv}(N)$.*

*Then there exists a $u \in N$ such that $u \in L(u) + S(u)T(u)$.*

*Proof.* By assumption $(i)$, there exists an $u = u_v$ in $A$ such that for each $v \in N$ and for some $z \in T(v)$,

$$z \in \left(\frac{I-L}{S}\right)(u_v)$$

and hence using the assumption $(iv)$, we have

$$u_v \in L(u_v) + S(u_v)T(v).$$

Taking the conditions $(i)$ and $(v)$ into consideration, multivalued operator $F : N \to P_{wcl,cv}(N)$ can be defined by

$$F(v) := \left(\frac{I-L}{S}\right)^{-1} T(v)$$

for any $v \in N$. Again using assumption $(iv)$, we have $u_v \in N$. Therefore $F(N) \subset N$. By assumption $(iii)$, $F(N)$ is relatively weakly compact. Therefore if we take into account assumption $(ii)$, then $F$ is a w.s.u.sc. multivalued operator. Thus Theorem 2.1 implies that there is a $u \in F(u)$ for some $u \in N$. It follows that there is a $u \in L(u) + S(u)T(u)$ for some $u \in N$. $\square$

**Theorem 3.5.** *Suppose $A$ is a WC-Banach algebra and $N \subset A$ is a nonempty weakly closed convex subset. Assume $S : N \to P_{wcl,cv}(N)$ is a regular multivalued operator and $L, T : N \to P_{wcl,cv}(N)$ are multivalued operators such that*

*$(i)$ $\left(\frac{I-L}{S}\right)^{-1}$ exists on $T(N)$,*
*$(ii)$ $\left(\frac{I-L}{S}\right)^{-1} T$ has w.s.c.g.,*
*$(iii)$ $\left(\frac{I-L}{S}\right)^{-1}$ is weakly compact on $T(N)$,*
*$(iv)$ For each $v \in N$, $u \in L(u) + S(u)T(v)$ implies $u \in N$,*
*$(v)$ $\left(\frac{I-L}{S}\right)^{-1} T(N) \in P_{wcl,cv}(N)$.*
*Then there exists a $u \in N$ such that $u \in L(u) + S(u)T(u)$.*

*Proof.* If we take into consideration the claim $(iii)$, we can obtain that $\left(\frac{I-L}{S}\right)^{-1} T(N)$ is relatively weakly compact. Briefly, by virtue of preceding result, we immediately reach the desired result. $\square$

**Theorem 3.6.** *Suppose $A$ is a WC-Banach algebra and $N \subset A$ is a nonempty weakly closed convex subset. Assume $S : N \to P_{wcl,cv}(N)$ is a regular multivalued operator and $L, T : N \to P_{wcl,cv}(N)$ are multivalued operators such that*

*$(i)$ $\left(\frac{I-L}{S}\right)^{-1}$ exists on $T(N)$,*
*$(ii)$ $\left(\frac{I-L}{S}\right)^{-1} T$ satisfies the condition $(H)$ and has weakly compact range,*
*$(iii)$ For each $v \in N$, $u \in L(u) + S(u)T(v)$ implies $u \in N$,*
*$(iv)$ $\left(\frac{I-L}{S}\right)^{-1} T(N) \in P_{wcl,cv}(N)$.*
*Then there exists a $u \in N$ such that $u \in L(u) + S(u)T(u)$.*

*Proof.* By the conjecture $(iv)$, a multivalued operator $F : N \to P_{wcl,cv}(N)$ can be defined by

$$F(v) := \left(\frac{I-L}{S}\right)^{-1} T(v).$$

So the multivalued operator $F$ is well defined. Since the multivalued operator $F$ satisfies the condition $(H)$, then the condition $(H)$ requires the w.s.u.sc. multivalued operator of $F$. By [17, Theorem 2.1], it can be obtained that a w.s.u.sc. multivalued operator $F$ also has w.s.c.g. At the same time, $F(N)$ is relatively weakly compact because $F$ has weakly compact range. Hence we take that there exists a $u \in N$ such that $u \in F(u)$ using Theorem 3.4. This shows that for $u \in N$, $u \in L(u) + S(u)T(u)$. □

**Lemma 3.7.** *Let $A$ is a WC-Banach algebra and $N \subset A$ is a nonempty weakly closed convex subset. Let $S : N \to P_{wcl,cv}(N)$ be a regular multivalued operator and $L : N \to P_{wcl,cv}(N)$ and $T : N \to P_{wcl,b,cv}(N)$ are multivalued operators such that*

- *$(i)$ Operators $L$ and $S$ are generalized D-Lipschitzian with functions $\tau_L$ and $\tau_S$,*
- *$(ii)$ $T$ is a bounded on $N$ with bound $K = \|T(v)\|$ for each $v \in N$,*
- *$(iii)$ for all $u \in N$, $L(u)$, $S(u)$ and $T(u)$ are relatively weakly compact,*
- *$(iv)$ $\left(\frac{I-L}{S}\right) : N \to P_{wcl,cv}(N)$ is defined by $\left(\frac{I-L}{S}\right)(u) = \frac{u-L(u)}{S(u)}$ for $u \in N$.*

*Then $\left(\frac{I-L}{S}\right)^{-1}$ exists on $T(N)$ if $\tau_L(r) + K\tau_S(r) < r$ for each $r > 0$.*

*Proof.* Let choose a fixed element $v$ in $N$ and define operator

$$\begin{cases} \Gamma_v : N \to P_{wcl,b,cv}(N), \\ \quad u \to \Gamma_v(u) = L(u) + S(u)T(v) \quad \text{for } u \in N. \end{cases}$$

It clearly shows that $\Gamma_v(u)$ is a weakly closed convex subset of $N$. For this reason, this multivalued operator is well defined. By assumption $(i)$ and $(ii)$, we have, for each $u_1, u_2 \in A$

$$\begin{aligned}
& \|\Gamma_v(u_1) - \Gamma_v(u_2)\| \\
\leq\ & d_H(\Gamma_v(u_1), \Gamma_v(u_2)) \\
=\ & d_H(L(u_1) + S(u_1)T(v), L(u_2) + S(u_2)T(v)) \\
=\ & \max\left\{ \begin{array}{c} d(L(u_1) + S(u_1)T(v), L(u_2) + S(u_2)T(v)), \\ d(L(u_2) + S(u_2)T(v), L(u_1) + S(u_1)T(v)) \end{array} \right\} \\
\leq\ & \max\left\{ \begin{array}{c} \max\{d(L(u_1), L(u_2)), d(S(u_1)T(v), S(u_2)T(v))\}, \\ \max\{d(L(u_2), L(u_1)), d(S(u_2)T(v), S(u_1)T(v))\} \end{array} \right\} \\
\leq\ & \max\{d(L(u_1), L(u_2)), d(L(u_2), L(u_1))\} \\
& + \max\{d(S(u_1)T(v), S(u_2)T(v)), d(S(u_2)T(v), S(u_1)T(v))\} \\
=\ & d_H(L(u_1), L(u_2)) + d_H(S(u_1)T(v), S(u_2)T(v)) \\
\leq\ & d_H(L(u_1), L(u_2)) + d_H(S(u_1), S(u_2)) d_H(0, T(v))
\end{aligned}$$

$$\leq \tau_L(\|u_1 - u_2\|) + K\tau_S(\|u_1 - u_2\|). \tag{3.1}$$

Therefore $\Gamma_v$ is a generalized $D$-Lipschitzian with $\tau_L(r) + K\tau_S(r) < r$ for each $r > 0$.

Now we show that $\Gamma_v$ has w.s.c.g.: Since $A$ is a $WC$-Banach algebra, by assumption $(iii)$, for all $u \in N$, $S(u)T(u)$ is relatively weakly compact and so is $\Gamma_v(u)$. By the Eberlein-Smulian Theorem [13], we get that for all $u \in N$, $\Gamma_v(u)$ is sequentially weakly closed, which means that $\Gamma_v$ has w.s.c.g..

Hence, from Theorem 3.3, there is an element $u$ in $M$ such that $u \in \Gamma_v(u)$. Therefore if $\left(\frac{I-L}{S}\right)$ is well defined by assumption $(iv)$, then $u$ verifies $c \in \left(\frac{I-L}{S}\right)(u)$ for any $c \in T(v)$ and also

$$u \in L(u) + S(u)\,c.$$

It follows that

$$\left(\frac{I-L}{S}\right)(u) \cap T(v) \neq \varnothing.$$

Hence $\left(\frac{I-L}{S}\right)^{-1}$exists and well defined on $T(N)$. □

**Theorem 3.8.** *Let $N$ be a weakly closed convex subset of WC-Banach algebra $A$. Supposes the multivalued operator $S : N \to P_{wcl,cv}(N)$ is a regular operator, the multivalued operator $L : N \to P_{wcl,cv}(N)$ and the multivalued operator $T : N \to P_{wcl,b,cv}(N)$ such that*

*$(i)$ operators $L$ and $S$ are generalized D-Lipschitzian with functions $\tau_L$ and $\tau_S$, with $\tau_L(r) + K\tau_S(r) < r$ for each $r > 0$, where $K = \|T(N)\| < \infty$,*

*$(ii)$ $\left(\frac{I-L}{S}\right) : N \to P_{wcl,cv}(N)$ is defined by $\left(\frac{I-L}{S}\right)(u) = \frac{u-L(u)}{S(u)}$ for $u \in N$,*

*$(iii)$ for all $u \in N$, $L(u)$, $S(u)$ and $T(u)$ are relatively weakly compact,*

*$(iv)$ $\left(\frac{I-L}{S}\right)^{-1}$ is weakly compact on $T(N)$,*

*$(v)$ for each $v \in N$, $u \in L(u) + S(u)T(v)$ implies $u \in N$,*

*$(vi)$ $\left(\frac{I-L}{S}\right)^{-1} T(N) \in P_{wcl,cv}(N)$.*

*Then there exists a $u \in N$ such that $u \in L(u) + S(u)T(u)$.*

*Proof.* Considering Lemma 3.7, we have the mapping $\left(\frac{I-L}{S}\right)^{-1}$ exists on $T(N)$. And also by assumption $(vi)$, we can define a multivalued operator

$$\begin{cases} F : N \to P_{wcl,cv}(N), \\ \quad v \to F(v) = \left(\frac{I-L}{S}\right)^{-1} T(v) \quad \text{for } v \in N. \end{cases}$$

Therefore $F$ is well defined, and $F$ is weakly closed convex subset of $N$. By assumptions $(ii)$ and $(v)$, there is an $u = u_v$ in $N$ such that for each $v \in N$ and for some $z \in T(v)$,

$$z \in \left(\frac{I-L}{S}\right)(u_v).$$

This mains that

$$u_v \in L(u_v) + S(u_v)T(v). \tag{3.2}$$

Since $\left(\frac{I-L}{S}\right)^{-1}$is weakly compact on $T(N)$, then $F(N)$ is relatively weakly compact. Also, since $A$ is a $WC$-Banach algebra, take into consideration assumption $(iii)$, by inclusion (3.2) for all $v \in N$, $F(v)$ is relatively weakly compact thus, $F$ has w.s.c.g. as mentioned in the proof of Lemma 3.7. Consequently, by Theorem 3.4, it follows that there is a $u \in N$ such that $u \in L(u) + S(u)T(u)$. □

**Corollary 3.9.** *Let $N$ be a weakly closed convex subset of WC-Banach algebra $A$. Suppose $L, S : N \to P_{wcl,cv}(N)$ and $T : N \to P_{wcl,b,cv}(N)$ are w.s.u.sc. multivalued operators such that*

($i$) *Operators $L$ and $S$ are generalized D-Lipschitzian with functions $\tau_L$ and $\tau_S$, with $\tau_L(r) + K\tau_S(r) < r$ for each $r > 0$, where $K = \|T(N)\| < \infty$,*
($ii$) $\left(\frac{I-L}{S}\right) : N \to P_{wcl,cv}(N)$ *is defined by* $\left(\frac{I-L}{S}\right)(u) = \frac{u-L(u)}{S(u)}$ *for $u \in N$,*
($iii$) $\left(\frac{I-L}{S}\right)^{-1}$*satisfies condition $(H)$ and $T(N)$ is relatively weakly compact,*
($iv$) *For each $v \in N$, $u \in L(u) + S(u)T(v)$ implies $u \in N$,*
($v$) $\left(\frac{I-L}{S}\right)^{-1} T(N) \in P_{wcl,cv}(N)$.
*Then there exists a $u \in N$ such that $u \in L(u) + S(u)T(u)$.*

**Theorem 3.10.** *Suppose $A$ is a WC-Banach algebra and $N \subset A$ is a weakly closed convex set. Suppose that multivalued operators $L, T : N \to P_{wcl,cv}(N)$ and $S : N \to P_{wcl,cv}(N)$ is regular such that*
($i$) *Operators $L$ and $S$ are generalized D-Lipschitzian with functions $\tau_L$ and $\tau_S$, with $\tau_L(r) + K\tau_S(r) < r$ for each $r > 0$, where $K = \|T(N)\| < \infty$,*
($ii$) *The operator $L$ satisfy the condition $(H)$,*
($iii$) *for all $u \in N$, $L(u)$, $S(u)$ and $T(u)$ are relatively weakly compact,*
($iv$) $\left(\frac{I-L}{S}\right) : N \to P_{wcl,cv}(N)$ *is defined by* $\left(\frac{I-L}{S}\right)(u) = \frac{u-L(u)}{S(u)}$ *for $u \in N$,*
($v$) *for each $v \in N$, $u \in L(u) + S(u)T(v)$ implies $u \in N$,*
($vi$) *for each $u \in N$, $L(u) + S(u)T(u)$ is a weakly closed convex subset of $N$.*
*Then there exists a $u \in N$ such that $u \in L(u) + S(u)T(u)$.*

*Proof.* By assumption ($v$), for $v \in N$ and for each $u \in N$, we have $\Gamma_v : N \to P(N)$ by

$$\Gamma_v = L(u) + S(u)T(v),$$

Also from assumption ($iii$), $\Gamma_v$ has w.s.c.g. as mentioned in the proof of Lemma 3.7. By means of Theorem 3.3, there is an $u = u_v$ in $N$ such that $u_v \in \Gamma_v(u_v)$. In view of Lemma 3.7, mapping $\left(\frac{I-L}{S}\right)^{-1}$ exists on $T(N)$. Therefore the mapping $\left(\frac{I-L}{S}\right)^{-1}$ is well defined on $T(N)$. Let for any $v \in N$ and

$$y \in \left(\frac{I-L}{S}\right)^{-1} (T(v)),$$

then there is an $x \in T(v)$ such that if

$$y = \left(\frac{I-L}{S}\right)^{-1} (x),$$

then

$$y = L(y) + S(y)x \subseteq L(y) + S(y)T(v).$$

Therefore if we take into account the assumption ($v$), then there is a $u_v$ in $N$ such that

$$u_v \in L(u_v) + S(u_v)T(v). \tag{3.3}$$

Thus multivalued operator $F : N \to P(N)$ can be defined by

$$F(v) := \left(\frac{I-L}{S}\right)^{-1} T(v).$$

Hence the multivalued operator $F$ is well defined on $N$. By assumptions $(i)$, $(v)$ and $(vi)$, for each $v \in N$, $F(v)$ is a weakly closed convex subset of $N$.

Let $v \in N$ and choose $y \in F(v)$, then there is a $w \in T(v)$ such that

$$y = \left(\frac{I-L}{S}\right)^{-1}(w) \quad \text{and} \quad y = L(y) + S(y)\,w.$$

By inclusion (3.3), for each $v \in N$, there exists a $u_v$ such that

$$u_v \in L(u_v) + S(u_v)\,w \subseteq L(u_v) + S(u_v)\,T(v).$$

By the assumption $(v)$, $u_v \in N$. Hence we have $F(N) \subset N$.

In addition, from assumption $(iii)$, we have $L(N)$, $S(N)$, $T(N)$ are relatively weakly compact and since $A$ is a $WC$-Banach algebra, notice that inclusion (3.3), then $F(N)$ is relatively weakly compact.

Using assumption $(iii)$ and definition of $WC$-Banach algebra, we reach that for all $u \in N$, $S(u)\,T(u)$ is relatively weakly compact and by Eberlein-Smulian Theorem [13] sequentially weakly closed, this implies that $ST$ has w.s.c.g. According to [17, Theorem 2.1] $ST$ satisfies condition $(H)$ and since $L$ satisfies condition $(H)$, by inclusion (3.3) it is obtained that multivalued operator $F$ satisfies condition $(H)$. Therefore Theorem 3.6 imply that there exists a $u \in N$ such that $u \in F(u)$. This means that there exists a $u \in N$ such that $u \in L(u) + S(u)\,T(u)$. □

## 4. Application to the nonlinear integral inclusion

In this section, we will present an application under certain conditions. Assume that $A$ is a real reflexive WC-Banach algebra and $C([0,1], A)$ is a reflexive WC-Banach algebra [5, Remark 4.1]. Now we consider the nonlinear integral inclusion

$$u(t) \in X(u(t)) \int_0^1 Y(s, u(s))ds + Z(u(t)), \qquad t \in [0,1] \tag{4.1}$$

where $X : A \to A$, $Y : [0,1] \times A \to P_{cv}(A)$, $Z : A \to A$ for $\forall t \in [0,1]$ and $u \in A$,

$$\|Y(t,u)\| = \sup\{|y|\,,\ y \in Y(t,u)\}.$$

Let us consider the following hypotheses, which are similar to the conditions in [10, Theorem 5.2]:

($H_1$) $X$ and $Z$ are $D-$Lipschitzian with $D-$functions $\tau_X$ and $\tau_Z$, and $\|X\| \le \psi$, $\|Z\| \le \varphi$ for $\psi, \varphi > 0$.

($H_2$) There is a weakly continuous, Pettis integrable function $k : [0,1] \to A$ with $k(t) \in Y(t, u(t))$, $t \in [0,1]$ and $u : [0,1] \to A$ is a continuous function,

($H_3$) for each $t \in [0,1]$, $X(u(t))$, $Y(u(t))$ and $Z(u(t))$ are relatively weakly compact,

($H_4$) $X(u(t))$ has a w.s.c.g.,

(H$_5$) For any $q > 0$, $t \in [0,1]$ and $v \in A$ with $|v| \leq q$, there exist $\phi_q \in L^1([0,1])$ such that $\|Y(t,v)\| \leq \phi_q(t)$. Also, assume that

$$\int_0^1 \phi_q(s)ds < q.$$

(H$_6$) There is $\tau_X(r) + q\tau_Z(r) < r$, $\forall r > 0$.

**Theorem 4.1.** *(4.1) has a solution in $C([0,1],A)$ under hypotheses (H$_1$)-(H$_6$).*

*Proof.* Define a subset $N$ of $C([0,1],A)$ by

$$N = \{u \in C([0,1],A) : \|u\| \leq M\}$$

it is clear that $N$ is a closed, convex and bounded set. Then, with [conway, 1.5 Corallary], it is concluded that $N$ is weakly closed. Now we introduce the multivalued operators $L$, $T$ and $S$:
$L : C([0,1],A) \to P_{wcl,cv}(C([0,1],A))$, $Lu(t) = \{X(u(t))\}$
$S : C([0,1],A) \to P_{wcl,cv}(C([0,1],A))$,

$$Su(t) = \left\{ a(t) = \int_0^1 k(s)ds, \quad k(t) \in Y(t,u(t)) \text{ and } k \text{ is Pettis integrable} \right\}$$

$T : C([0,1],A) \to P_{wcl,cv}(C([0,1],A))$, $Tu(t) = \{Z(u(t))\}$.
We can write the nonlinear integral inclusion (4.1) as follows:

$$u(t) \in Lu(t)Su(t) + Tu(t). \tag{4.2}$$

For the proof, we must show that the operators $L$, $S$ and $T$ satisfy all the conditions of Theorem 3.10.

Since $X$ and $Z$ are single valued mappings on $A$ then $Xu(t)$ and $Zu(t)$ are single point sets and are weakly closed by hypothesis (H$_3$), consequently $L, T : C([0,1],A) \to P_{wcl,cv}(C([0,1],A))$ are multivalued operators. Also, by hypotheses (H$_2$) and (H$_6$), $S$ is well defined. Taking into consideration the definition of mappings $L, T$ and hypothesis (H$_1$), it can be obtained that $L$ and $T$ are $D$-Lipschitzian.with $D-$functions $\tau_L$ and $\tau_T$ such that $\tau_L(r) + q\tau_T(r) < r$, $\forall r > 0$.

We will show that $Su \in P_{wcl,cv}(C([0,1],A))$, for $\forall u \in N$. For this $a_1, a_2 \in Su$ such that

$$a_1(t) = \int_0^t k_1(s)ds \quad \text{and} \quad a_2(t) = \int_0^t k_2(s)ds, \tag{4.3}$$

by definition of $S$, $k_1, k_2$ are Pettis integrable and $k_1, k_2 \in Y(t,u(t))$. For $\forall \lambda \in (0,1)$, it can be write

$$\lambda a_1(t) + (1-\lambda)a_2(t) = \int_0^t \lambda(k_1(s) + (1-\lambda)k_2(s))ds$$

from $Y(t,u(t)) \in P_{cv}(A)$, we get $\lambda a_1 + (1-\lambda)a_2 \in Su$ thus for $\forall u \in N$, $Su$ is convex.

Let $u_n \in N$, $u_n \rightharpoonup u$ and $z_n \in Su_n$ with $z_n \rightharpoonup z$. We take a sequence $(k_n) : [0,1] \to A$ such that $k_n$ is Pettis integrable, $k_n(s) \in Y(s, u_n(s))$ with $u_n(s) \rightharpoonup u(s)$ and for each $s, t \in [0,1]$ there is

$$z_n(t) = \int_0^t k_n(s)ds.$$

By hypothesis (H$_4$), there is $k(s) \in Y(s, u(s))$ for $\forall s \in [0,1]$ and

$$z(t) = \int_0^t k(s)ds \in Su(t).$$

In conclusion, for $\forall u \in N$, $Su$ is weakly closed.

On the other hand, if we consider hypotheses (H$_3$) and (H$_4$), $L\left(C([0,1], A)\right)$ is relatively weakly compact and L has w.s.c.g., by [17, Theorem 2.1] we reach that $L$ satisfies the condition (H).

Let us show that for $\forall v \in N$, $L(u) + S(u)T(v) \subset N$. Let $a(t) \in Su(t)$ and assume that relation $w(t) = L(u(t)) + a(t)T(v(t))$ for $t \in [0,1]$. Then, using hypothesis (H$_5$), it can be write

$$\begin{aligned}
\|w(t)\| &\leq \|L(u(t))\| + \|T(v(t))\| \int_0^1 \|a(s)\|\, ds \\
&\leq \psi + \varphi \int_0^1 \phi_q(s)ds \\
&\leq \psi + \varphi q = M,
\end{aligned}$$

if take the supremum in the last inequality, we get $\|w\| \leq M$ so $w \in N$.

Finally, we see that for each $u \in N$, $L(u) + S(u)T(u)$ is a weakly closed, convex subset of $N$. Since $A$ is a reflexive WC-Banach algebra, by hypothesis (H$_3$) for $\forall u \in N$, $L(u) + S(u)T(u)$ is a weakly closed. Then let $v_1, v_2 \in L(u) + S(u)T(u)$; $a_1, a_2 \in Su$ as in the (4.3) such that

$$v_1(t) = L(u) + T(u) \int_0^t k_1(s)ds \quad \text{and} \quad v_2(t) = L(u) + T(u) \int_0^t k_2(s)ds,$$

where $k_1, k_2$ are Pettis integrable, $k_1(s), k_2(s) \in Y(s, u(s))$, for $\forall \lambda \in (0,1)$, it can be write

$$\lambda v_1(t) + (1-\lambda)v_2(t) = L(u) + T(u) \int_0^t (\lambda k_1(s) + (1-\lambda)k_2(s))ds.$$

Since $Y(t, u(t))$ is a convex subset of $A$, we get

$$\lambda v_1 + (1-\lambda)v_2 \in L(u) + S(u)T(u).$$

As a result, since all conditions of the Theorem 3.10 are satisfied, (4.2) has a solution in $C([0,1], A)$. $\square$

## 5. CONCLUSION

It is clear that the nonlinear operator inclusions (1.1) may not have a fixed point in $WC$-Banach algebra $C([0,1], A)$. In particular, such inclusions under weak topology in the transport equation (the growing cell population or the hereditary systems) may not have a fixed point in $WC$-Banach algebras. Therefore suitable conditions were established to guarantee the existence of solution for nonlinear inclusion (1.1) on $WC$-Banach algebras. In addition $WC$-Banach algebra has been used specically for single-valued operator equations in the study [5,18]. Because this algebraic structure is based on the product of the weak compactness of sets, it is also extremely useful for the outcomes of multivalued operator equations. Further, we use our results to prove the existence of the solution for integral inclusions.

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