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WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS OF A CLASS OF CHAOTIC NEURAL NETWORKS ON TIME SCALES

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Abstract. In this paper, by using the exponential dichotomy of linear dynamic equations on time scales, a fixed point theorem, and the theory of calculus on time scales, we obtain sufficient conditions for the existence, uniqueness, and global exponential stability of weighted pseudo-almost periodic solutions of a class of chaotic neural networks with mixed delays on time scales. Finally, we present examples and numerical simulations to illustrate the feasibility of our results.

1. Introduction

Over the last several decades, the theory of neural networks has been applied with significant success in several research fields, such as: pattern recognition, artificial intelligence, signal processing, associative memories, and so on [7, 8, 11, 16]. A great deal has been done since then to investigate the dynamics of neural networks, particularly the existence, uniqueness, and stability of almost periodic, almost automorphic, pseudo almost periodic, weighted pseudo almost periodic and weighted pseudo almost automorphic solutions (see [1, 2, 6, 19] and the references therein). Recently, in [5] Chérif studied the existence, uniqueness, and global asymptotic stability of pseudo almost-periodic solution of the following chaotic neural networks with time-varying delays in leakage terms

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g(y_j(t-\tau)) + \int_{t-\sigma}^{t} \sum_{j=1}^{n} c_{ij} h(y_j(\zeta))\Delta\zeta + J_i(t)$$

where n is the number of the neurons in the neural network, $x_i(t)$ denotes the state of the ith neural neuron at time t. $f_j(x_j)$, $g_j(x_j)$, and $h_j(x_j)$ are the activation functions of jth neuron at time t. The constants a_{ij}, b_{ij}, c_{ij} denote, respectively, the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights, of the jth neuron on the i neuron, $J_i(t)$ is the external bias on the ith neuron, d_i denotes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs. τ is the constant discrete time delay, while σ describes the distributed time delay.

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It is important to note that chaotic neural networks are non-linear dynamic systems that exhibit startling similarities to the biological neuronal networks that comprise the human brain, and this can lead to increased processing speed due to the complex spatiotemporal dynamics of neurons contained within the network.

On the other hand, the theory of time scales which was first introduced by Hilger [10] in his Ph.D. thesis in 1988 has received a lot of attention due to the fact that it is able to unite continuous and discrete systems in an equally effective manner. In other words, by selecting the time scale to be the set of real numbers, the general result produces a result for differential equations. Likewise, by selecting the time scales as a set of integers, the same general result gives a result of difference equations. Furthermore, it is also possible to extend the results over time scales to other types of equations as well. For that reason, there has been a rapid development in the theory of dynamic equations on time scales during the last years (see [9, 12, 13, 17]).

The theory of weighted pseudo almost periodicity in time scales, which is the central subject of this paper started in 2016. When Y. Li and L. Zhao [15], extended the well-known weighted almost periodic functions to time scales, then they studied the existence and global exponential stability of weighted pseudo-almost periodic solutions for a class of cellular neural networks with discrete delays on time scales. Thereafter, in [3], the author demonstrated the existence and the global exponential stability of the unique weighted pseudo-almost periodic solution of bidirectional associative memory neural networks with mixed time-varying delays and leakage time-varying delays on time-space scales. Meanwhile, X. Yu and Q. Wang [20] investigated the existence, uniqueness, and global exponential stability of weighted pseudo-almost periodic solutions for a class of Shunting Inhibitory Cellular Neural Networks with mixed delays on time scales. In 2020, S. Shen and Y. Li [18] studied the existence, and global exponential stability of weighted pseudo almost periodic solutions for a class of Clifford valued neural networks on time scales.

In the light of the above-mentioned studies, the primary purpose of this paper is to study the existence and uniqueness, global exponential stability of the weighted pseudo almost periodic of the following chaotic neural networks on time scales

$$\begin{aligned} y_i^{\Delta}(t) = -\delta_i(t)y_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(y_j(t-\eta_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t)g(y_j(t-\tau_{ij}(t))) \\ + \sum_{j=1}^{n} c_{ij}(t)\int_{t-\xi}^{t} h(y_j(\zeta))\Delta\zeta + J_i(t) \end{aligned} \tag{1.1}$$

where $i = 1, 2, ..., n$, n corresponds to the number of units in a neural network, $\mathbb{T}$ is an almost periodic time scale, $\eta(t)$, $\tau(t)$ are transmission delays at time t fulfilling $t - \eta(t) \in \mathbb{T}$ $t - \tau(t) \in \mathbb{T}$.

As far as we know, there are no published papers in the literature on the existence, and global exponential stability of weighted pseudo-almost periodic solutions on time scales for (1.1).

The organization of the paper can be summarized as follows. In section 2, As a prelude to the later sections of this paper, we introduce some notations and definitions as well as preliminary lemmas. In section 3 we study the existence and Uniqueness of weighted pseudo almost periodic solutions for a class of chaotic neural networks on time scales. In section 4, we prove that the weighted pseudo almost periodic solution obtained in the previous section is globally exponentially stable. Finally, in section5, in order to demonstrate the feasibility and effectiveness of our results obtained in the last sections, we present a few examples.

2. Preliminaries

In this section, we introduce some notations and definitions and state some preliminary results.

For convenience, for any $y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$, we let $|y| = (|y_1|, |y_2|, .., |y_n|)^T$ denote the absolute-value vector, and define $\|y\| = \max_{1\leq i\leq n} |y_i|$.

Let $BC(\mathbb{T}, \mathbb{R}^n)$ denotes the set of all bounded and continued functions which go from $\mathbb{T}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{T}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space where $\|\cdot\|_\infty$ denote the sup-norm

$$\|f\|_\infty = \sup_{t\in\mathbb{T}} \|f(t)\|.$$

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},\ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},\ \text{and } \mu(t) = \sigma(t) - t.$$

In addition, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. Points that are right dense and left dense are called dense. If $\mathbb{T}$ has a left-scattered maximum M then $\mathbb{T}^k := \mathbb{T}\setminus\{M\}$; otherwise $\mathbb{T}^k := \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m then $\mathbb{T}^k := \mathbb{T}\setminus\{m\}$; otherwise $\mathbb{T}^k := \mathbb{T}$. we denote by $I_\mathbb{T} = I \cap \mathbb{T}$ each interval I of $\mathbb{R}$.

Definition 2.1 ([4])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous or rd-continuous provided it is continuous at right-dense point in and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.2 ([4])**.** For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of f at t, denoted $f^\Delta(t)$, to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|, \quad \text{for all } s \in U. \tag{2.1}$$

Lemma 2.3 ([4])**.** *Let f, g be differentiable functions at $t \in \mathbb{T}^k$. Then*

(1) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(2) *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(3) *The product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

Definition 2.4 ([12])**.** Let f be right-dense continuous. If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Lemma 2.5 ([9])**.** *Let* f *be a delta differentiable function at* $t \in T^k$*. If* f *and* f^{Δ} *are continuous, then*

$$\left(\int_a^t f(t,s)\Delta s\right)^{\Delta} = f(\sigma,t) + \int_a^t f^{\Delta}(t,s)\Delta s. \tag{2.2}$$

Definition 2.6 ([4])**.** A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0, \quad \text{for all } t \in \mathbb{T}^k.$$

The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.7 ([4])**.** We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathbb{R}(\mathbb{T}, \mathbb{R}) = \{\rho \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{R}\}$$

.

Definition 2.8 ([4])**.** If $p \in$, then for all $t, s \in \mathbb{T}$ the generalized exponential function is defined by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(t))\Delta\tau\right\},$$

where the cylinder transformation is introduced by

$$\xi_{h(z)} = \begin{cases} \dfrac{\log(1-hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Definition 2.9 ([4])**.** If $p, q \in \mathcal{R}$, then we define a circle plus addition by

$$(p \oplus q)(t) := p(t) + q(t) - p(t)q(t)\mu(t),$$

for all $t \in \mathbb{T}^k$. For $p \in \mathcal{R}$ define a circle minus p by

$$\ominus p := -\frac{p}{1+\mu p}$$

Lemma 2.10 ([4])**.** *Let* $p, q \in \mathcal{R}$*, and* $t, s, r \in \mathbb{T}$*, then*

(*i*) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*
(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t))e_p(t,s)$*;*
(*iii*) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*
(*iv*) $e_p(t,s)e_p(s,r) = e_p(t,s)$*;*
(*v*) $[e_{\ominus p}(t,s)]^{\Delta} = \ominus p(t)e_{\ominus p}(t,s)$*;*
(*vi*) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vii) If $a, b, c \in \mathbb{T}$, then

$$\int_a^b p(t)e_p(c,\sigma)\delta t = e_p(c,a) - e_p(c,b).$$

Lemma 2.11. *Assume $p \in \mathcal{R}$, and $t_0 \in \mathbb{T}$. If $1 + \mu(t)p(t) > 0$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.*

Definition 2.12 ([12])**.** A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq 0 \tag{2.3}$$

Definition 2.13 ([12])**.** Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in \mathcal{C}(\mathbb{T}, \mathbb{R}^n)$ is called almost periodic if for any given $\epsilon > 0$, the set

$$E(\epsilon, f) = \{\tau \in \Pi : \|f(t+\tau) - f(t)\| < \epsilon, \forall t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$; that is, for any given $\epsilon > 0$, there exists an l_ϵ such that every interval of length l_ϵ contains at least a number $\tau \in E(\epsilon, f)$ such that

$$\|f(t+\tau) - f(t)\| < \epsilon, \quad \forall t \in \mathbb{T}.$$

The set $E(\epsilon, f)$, is called ϵ-translation set of $f(t)$, τ is called ϵ-translation number of $f(t)$, and l_ϵ is called contain interval length of $E(\epsilon, f)$. The collection of all almost periodic functions which go from $\mathbb{T}$ to $\mathbb{R}^n$ will be denoted by $AP(\mathbb{T}, \mathbb{R}^n)$. $AP(\mathbb{T}, \mathbb{R}^n)$ equipped with the sup-norm $\|f\|_\infty = \sup_{t\in\mathbb{T}} \|f(t)\|$ is a Banach space.

Definition 2.14 ([14])**.** A function $f \in \mathcal{C}(\mathbb{T}, \mathbb{R}^n)$ is called pseudo-almost periodic if $f = g + h$, where $g \in AP(\mathbb{T}, \mathbb{R}^n)$ and $h \in PAP_0(\mathbb{T}, \mathbb{R}^n)$, where

$$PAP_0(\mathbb{T}, \mathbb{R}^n)$$
$$:= \left\{ \phi \in BC(\mathbb{T}, \mathbb{R}^n) : \phi \text{ is } \Delta\text{-mesurable such that } \lim_{t\to+\infty} \frac{1}{2r} \int_{\bar{t}-r}^{\bar{t}+r} \|\phi(s)\| \Delta s = 0 \right\},$$

for each $\bar{t} \in \mathbb{T}, r \in \Pi$.

Definition 2.15 ([12])**.** Let $y \in \mathbb{R}^n$ and let $A(t)$ be a $n \times n$ continuous matrix defined on $\mathbb{T}$. The linear system

$$y^\Delta(t) = A(t)y(t) \tag{2.4}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k, α, projection P, and the fundamental solution matrix $Y(t)$ of (2.4), satisfying

$$\left\|Y(t)PY^{-1}(\sigma(s))\right\|_0 \le ke_{\ominus\alpha}(t, \sigma(s)), \quad s, t \in \mathbb{T},\ t \ge \sigma(s),$$
$$\left\|Y(t)(I-P)Y^{-1}(\sigma(s))\right\|_0 \le ke_{\ominus\alpha}(t, \sigma(s)), \quad s, t \in \mathbb{T},\ t \le \sigma(s),$$

where $\|\cdot\|_0$ is a matrix norm on $\mathbb{T}$.

Consider the following almost periodic system

$$y^\Delta(t) = A(t)y(t) + f(t), \quad t \in \mathbb{T}. \tag{2.5}$$

where $A(t)$ is an almost periodic matrix function, $f(t)$ is an almost periodic vector function.

Lemma 2.16 ([12]). *If the linear system* (2.4) *admits exponential dichotomy, then system* (2.5) *has a bounded solution* $y(t)$ *as follows:*

$$y(t) = \int_{-\infty}^{t} Y(t)PY^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} Y(t)(I-P)Y^{-1}(\sigma(s))f(s)\Delta s, \tag{2.6}$$

where $Y(t)$ *is the fundamental solution matrix of* (2.4).

Lemma 2.17 ([12]). *Let* $\delta_i(t)$ *be an almost periodic function on* $\mathbb{T}$, *where* $\delta_i(t) > 0$, $-\delta_i(t) \in \mathbb{R}^+, \forall t \in \mathbb{T}$ *and* $\min_{1\le i\le n}\left\{\inf_{t\in\mathbb{T}} \delta_i(t)\right\} = m > 0$ *then the linear system*

$$y^{\Delta}(t) = \operatorname{diag} -\delta_1(t), -\delta_2(t), ..., -\delta_n(t)y(t) \tag{2.7}$$

admits an exponential dichotomy on $\mathbb{T}$.

Let Λ the collection of functions (weights) $\nu : \mathbb{T} \to (0,\infty)$, that are locally integrable over $\mathbb{T}$ such that $\nu > 0$ almost everywhere. Let $\nu \in \Lambda$, for $r \in \Pi$ with $r > 0$, we denote

$$\nu(Q_r) := \int_{Q_r} \nu(x)\Delta x,$$

where $Q_r := [\bar{t}-r, \bar{t}+r]_{\mathbb{T}}$ ($\bar{t} = \min\{[0,\infty)_{\mathbb{T}}\}$). If $\nu(x) = 1$ for each $x \in \mathbb{T}$, then $\lim_{r\to\infty} \nu(Q_r) = \infty$. Consequently, we define the space of weights Λ_∞ by

$$\Lambda_\infty := \left\{\nu \in \Lambda : \inf_{t\in\Lambda} \nu(t) = \nu_0 > 0, \lim_{r\to\infty} \nu(Q_r) = \infty\right\}.$$

In addition we define the set of weight Λ_B by

$$\Lambda_B := \left\{\nu \in \Lambda_\infty : \sup_{t\in\mathbb{T}} < \infty\right\}$$

and denote

$$BCU(\mathbb{T},\mathbb{R}^n) = \{f \in BC(\mathbb{T},\mathbb{R}^n) : \ f \text{ is uniformly continuous}\}$$

Definition 2.18 ([15]). Fix $\nu \in \Lambda_\infty$. A continuous function $f : \mathbb{T} \to \mathbb{R}^n$ is called weighted pseudo-almost periodic if it can be written as $f = h+\phi$ with $h \in AP(\mathbb{T},\mathbb{R}^n)$ and $\phi \in PAP_0(\mathbb{T},\mathbb{R}^n,\nu)$ where the space $PAP_0(\mathbb{T},\mathbb{R}^n,\nu)$ is defined by

$$PAP_0(\mathbb{T},\mathbb{R}^n,\nu) = \left\{\phi \in BCU(\mathbb{T},\mathbb{R}^n) : \lim_{r\to+\infty} \frac{1}{\nu(Q_r)} \int_{Q_r} \|g(t)\| \,\nu(t)\Delta t\right\} = 0.$$

All weighted pseudo-almost periodic functions which go from $\mathbb{T}$ to $\mathbb{R}^n$, will be denoted by $PAP_0(\mathbb{T},\mathbb{R}^n,\nu)$.

Denote

$$\Lambda_\infty^+ := \left\{\nu \in \Lambda_\infty : \text{ for all } s \in \Pi, \ \limsup_{|t|\to\infty} \frac{\nu(t+s)}{\nu(t)} < \infty, \ \limsup_{|t|\to\infty} \frac{\nu(Q_{t+\tau})}{\nu(Q_t)} < \infty\right\}.$$

Lemma 2.19 ([15]). *Let* $\nu \in \Lambda_\infty^+$. *Then* $(PAP(\mathbb{T},\mathbb{R}^n,\nu), \|\cdot\|_\infty)$ *is a Banach space.*

Lemma 2.20 ([15]). *Let* $\nu \in \Lambda_\infty^+$, *then the space* $PAP(\mathbb{T},\mathbb{R}^n,\nu)$ *is translation invariant.*

Lemma 2.21 ([15])**.** *Let $\nu \in \Lambda_\infty^+$. If $f, g \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$, then $f+g, fg \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$. If $f \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$, $g \in AP(\mathbb{T}, \mathbb{R}^n)$ then $fg \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$.*

Lemma 2.22 ([15])**.** *Let $\nu \in \Lambda_\infty^+$. If $f : \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition and $\phi \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$ then $\Gamma : t \mapsto f(\phi(t))$ belongs to $PAP(\mathbb{T}, \mathbb{R}^n, \nu)$*

3. Existence of weighted pseudo almost periodic solutions

For convenience, we introduce the following notations:

$$\delta_i^- = \inf_{t\in\mathbb{T}} |\delta_i(t)|, a_{ij}^- = \inf_{t\in\mathbb{T}} |a_{ij}(t)|, b_{ij}^- = \inf_{t\in\mathbb{T}} |b_{ij}(t)|, c_{ij}^- = \inf_{t\in\mathbb{T}} |c_{ij}(t)|,$$
$$a_{ij}^+ = \sup_{t\in\mathbb{T}} |a_{ij}(t)|, b_{ij}^+ = \sup_{t\in\mathbb{T}} |b_{ij}(t)|, c_{ij}^+ = \sup_{t\in\mathbb{T}} |c_{ij}(t)|,$$
$$\tau_{ij}^+ = \sup_{t\in\mathbb{T}} |\tau_{ij}(t)|, \eta_{ij}^+ = \sup_{t\in\mathbb{T}} |\eta_{ij}(t)|, \pi = \max_{1\le i,j\le n} \left\{\eta_{ij}^+, \tau_{ij}^+, \xi\right\}.$$

Throughout this paper, we assume that the following conditions hold

H_1) $\delta_i \in AP(\mathbb{T}, \mathbb{R})$ with $\delta_i(t) > 0$, $-\delta_i(t) \in \mathcal{R}^+$, $\forall t \in \mathbb{T}$ and $\min_{1\le i\le n}\left\{\inf_{t\in\mathbb{T}} \delta_i(t)\right\} > 0$, $\eta_{ij}(t), \tau_{ij}(t) \in AP(\mathbb{T}, \Pi) \cap C^1(\mathbb{T}, \Pi)$ with $\alpha := \inf\Big\{(1-\eta_{ij}^\Delta(t)),$ $(1-\tau_{ij}^\Delta(t))\Big\} > 0$, $a_{ij}, b_{ij}, c_{ij} \in PAP(\mathbb{T}, \mathbb{R}, \nu)$ and $I_i(t) \in PAP(\mathbb{T}, \mathbb{R}, \nu)$.

H_2) $f_j, g_j, h_j \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants $L_j^f, L_j^g, L_j^h, M_j^f, M_j^h$ and M_j^g such that for all $u, v \in \mathbb{R}$ and $j = 1, 2, ..., n$.

$$|f_j(u) - f_j(v)| \le L_j^f |u-v|, \ |g_j(u) - g_j(v)| \le L_j^g |u-v|,$$
$$|h_j(u) - h_j(v)| \le L_j^h |u-v|, f_j(u) \le M_j^f, \ g_j(u) \le M_j^g, \ h_j(u) \le M_j^h,$$

and $f_j(0) = g_j(0) = h_j(0)$.

H_3 $F(\tau) = \sup_{t \ \mathbb{T}} \dfrac{\nu(t+\tau)}{\nu(t)}$ is bounded and continuous on arbitrary closed subinterval of $[0, +\infty)_\mathbb{T}$

H_4) $$k = \max_{1\le i\le n} \left\{ \frac{\left[\sum_{j=1}^n a_{ij}^+ L_j^f + \sum_{j=1}^n b_{ij}^+ L_j^g + \sum_{j=1}^n c_{ij}^+ L_j^h \xi\right]}{\delta_i^-} \right\} < 1.$$

Lemma 3.1. *Let $\nu \in \Lambda_\infty^+$. Suppose that H_1 and H_3 hold. If $f \in PAP(\mathbb{T}, \mathbb{R}, \nu)$, then $f(\cdot - \tau(\cdot)) \in PAP(\mathbb{T}, \mathbb{R}, \nu)$*

Proof. By the weighted pseudo almost periodicity of f, we have

$$f(t-\tau(t)) = f_1(t-\tau(t)) + f_2(t-\tau(t)) := F_1(t) + F_2(t), \quad t \in \mathbb{T}, \tag{3.1}$$

where $f_1 \in AP(\mathbb{T}, \mathbb{R}^n)$ and $f_2 \in PAP_0(\mathbb{T}, \mathbb{R}, \nu)$. It's obvious that $F_1(\cdot) = f_1(t - \tau(t)) \in AP(\mathbb{T}, \mathbb{R})$. It remains to show that $F_2(\cdot) = f_2(\cdot - \tau(\cdot)) \in PAP_0(\mathbb{T}, \mathbb{R}^n, \nu)$. In view of H_3 we have

$$\frac{\nu(t)}{\nu(t-\tau(t))} = \frac{\nu(t-\tau(t)+\tau(t))}{\nu(t-\tau(t))} \le \sup_{\varrho\in[\tau^-,\tau^+]} F(\varrho).$$

Set $\gamma = \sup_{t\in\mathbb{T}}\left(\frac{1}{1-\tau^{\Delta}(t)}\right)\sup_{\varrho\in[\tau^-,\tau^+]} F(\varrho)$, then

$$\begin{aligned}
0 &\le \lim_{r\to\infty}\frac{1}{\nu(Q_r)}\int_{-r}^{r}\|f_2(t-\tau(t))\|\,\nu(t)\Delta t\\
&\le \lim_{r\to\infty}\frac{1}{\nu(Q_r)}\left(\int_{-r}^{r}\|f_2(t-\tau(t))\|\,\nu(t-\tau(t))\Delta t\right)\sup_{t\in\mathbb{T}}\frac{\nu(t)}{\nu(t-\tau(t))}\\
&\le \lim_{r\to\infty}\frac{1}{\nu(Q_r)}\left(\int_{-r-\tau(-r)}^{r-\tau(r)}|f(s)|\,\nu(s)\sup_{t\in\mathbb{T}}\frac{1}{1-\tau^{\Delta}(t)}\Delta s\right)\sup_{t\in\mathbb{T}}\frac{\nu(t)}{\nu(t-\tau(t))}\\
&\le \lim_{r\to\infty}\gamma\frac{1}{\nu(Q_r)}\int_{-r-\tau(-r)}^{r-\tau(r)}\|f(s)\|\,\nu(s)\Delta\\
&\le \gamma\lim_{r\to\infty}\sup\frac{\nu(Q_{r+\tau^+})}{\nu(Q_r)}\frac{1}{(Q_{r+\tau^+})}\int_{-(r+\tau^+)}^{r+\tau^+}\|f(s)\|\,\nu(s)\Delta\\
&= 0
\end{aligned}$$

which implies that $f(\cdot-\tau(\cdot))\in PAP_0(\mathbb{T},\mathbb{R}^n,\nu)$. Hence $f(\cdot-\tau(\cdot))\in PAP(\mathbb{T},\mathbb{R}^n,\nu)$ □

Lemma 3.2. *Let $\nu\in\Lambda^+_\infty$. Assume that H_1-H_4 hold and $y_j(\zeta)\in PAP(\mathbb{T},\mathbb{R}^n,\nu)$, then $\chi(t)=\int_{t-\xi}^{t}h_j(y_j(\zeta))\Delta\zeta\in PAP(\mathbb{T},\mathbb{R}^n,\nu)$.*

Proof. Since the function χ satisfies

$$\begin{aligned}
|\chi| &= \left|\int_{t-\xi}^{t}h_j(y_j(\zeta))\Delta\zeta\right|\\
&\le \xi M_j^h
\end{aligned}$$

which gives that the integral $\int_{t-\sigma}^{t}h_j(y_j(\zeta))\Delta\zeta$ is absolutely convergent and the function χ_i is bounded. We will now show the the continuity of χ. For any rd-dense point $t\in\mathbb{T}$, let $\{t_n\}\in\mathbb{T}$ a sequence such that $t_n>t$ and $\lim_{n\to\infty}t_n=t$. The continuity of the function y gives that for any $\epsilon>0$ there exists a constant $N\in\mathbb{N}$ such that for any integer $n>N$, $s\in\mathbb{T}$ with $t_n-s\in\mathbb{T}$ and $t-s\in\mathbb{T}$, we obtain

$$|y_j(t_n-s)-y_j(t-s)|<\frac{\epsilon}{L_j^h\xi}.$$

Which yields,

$$\begin{aligned}
&|\chi(t_n)-\chi(t)| \\
&= \left| \int_{t_n-\xi}^{t_n} h_j(y_j(\zeta))\Delta\zeta - \int_{t-\xi}^{t} h_j(y_j(\zeta))\Delta\zeta \right| \\
&= \left| \int_{t_n-\xi}^{t_n} h_j(y_j(\zeta))\Delta\zeta - \int_{t-\xi}^{t} h_j(y_j(\zeta))\Delta\zeta \right| \\
&\leq \left| \int_{0}^{\xi} h_j(y_j(u+t_n-\xi))\Delta u - \int_{0}^{\xi} h_j(y_j(u+t-\xi))\Delta u \right| \\
&\leq L_j^h \xi \left| y_j(u+t_n-\xi) - y_j(u+t_n-\xi) \right| \\
&\leq \epsilon.
\end{aligned}$$

We conclude similarly that the function χ is ld-continuous. Consequently, χ_i is continuous on $\mathbb{T}$. By Lemma 2.22, we have $h_j(y_j(\zeta)) \in PAP(\mathbb{T},\mathbb{R}^n,\nu)$. Moreover, let

$$h_j(y_i(\zeta)) = h_{1j}(y_j(\zeta)) + h_{2j}(y_j(\zeta)) = H_1(\zeta) + H_2(\zeta)$$

where $H_{1j} \in AP(\mathbb{T},\mathbb{R}^n)$, $H_{2j} \in PAP_0(\mathbb{T},\mathbb{R}^n,\nu)$. Hence,

$$\begin{aligned}
\chi(t) &= \int_{t-\xi}^{t} H_{1j}(\zeta) + H_{2j}(\zeta)\Delta\zeta \\
&= \int_{t-\xi}^{t} H_{1j}(\zeta)\Delta\zeta + \int_{t-\xi}^{t} H_{2j}(\zeta)\Delta\zeta \\
&= \chi^1(t) + \chi^2(t).
\end{aligned}$$

The almost periodicity of H_{1j} implies that $\sup_{t\in\mathbb{T}} |H_{1j}(t+\delta) - H_{1j}(t)| < \dfrac{\epsilon}{\xi}$ for all $\epsilon > 0$

$$\begin{aligned}
\left|\chi^1(t+\delta) - \chi^1(\delta)\right| &= \left| \int_{t+\delta-\xi}^{t+\delta} H_{1j}(\zeta) - \int_{t-\xi}^{t} H_{1j}(\zeta)\Delta\zeta \right| \\
&\leq \left| \int_{t-\xi}^{t} \sum_{j=1}^{n} c_{ij}^{+} H_{1j}(\zeta+\delta)) - \int_{t-\xi}^{t} H_{1j}(\zeta)\Delta\zeta \right| \\
&\leq \int_{t-\xi}^{t} \left| H_{1j}(\zeta+\delta) - H_{1j}(\zeta) \right| \Delta\zeta
\end{aligned}$$

$$\leq \epsilon,$$

it follow that, $\chi^1(t) \in AP(\mathbb{T}, \mathbb{R}^n)$. On the other hand, since $H_{2j}(\cdot) \in PAP_0(\mathbb{T}, \mathbb{R}^n, \nu)$, we obtain from Lemma (2.20) that $H_{2j}(\cdot - s) \in PAP_0(\mathbb{T}, \mathbb{R}^n, \nu)$ for all $s \in \mathbb{R}$. Hence

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{-r}^{r} \left|\chi^2(t)\right| \nu(t)\Delta t &\leq \lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{-r}^{r} \nu(t)\Delta t \int_{t-\xi}^{t} \left|H_{2j}(\zeta)\right| \Delta\zeta \\
&\leq \lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{-r}^{r} \nu(t)\Delta t \int_{t-\xi}^{t} \left|H_{2j}(\zeta)\right| \Delta\zeta \\
&\leq \lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{-r}^{r} \nu(t)\Delta t \int_{-\xi}^{0} \left|H_{2j}(t-s)\right| \Delta s \\
&\leq \lim_{r\to\infty} \frac{1}{\nu(Q_r)} \int_{-\xi}^{0} \int_{-r}^{r} \left|H_{2j}(t-s)\right| \nu(t)\Delta t\Delta s \\
&= 0,
\end{aligned}$$

which gives that $\chi^2 \in PAP_0(\mathbb{T}, \mathbb{R}^n, \nu)$. Thus $\chi \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$. □

Lemma 3.3. *Let $\nu \in \Lambda^+_\infty$, suppose that assumptions H_1-H_4 hold. Define the non-linear operator Π as follows for each $\varphi \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$*

$$(\Pi\varphi)_i(t) = \int_{-\infty}^{t} e_{-\delta_i}(t, \sigma(s))\Gamma_i(s)\Delta s, \quad i = 1, 2, ..., n \tag{3.2}$$

where

$$\begin{aligned}
\Gamma_i(s) = \sum_{j=1}^{n} a_{ij}(s) f_j\left(\varphi_j(s - \eta_{ij}(s))\right) + \sum_{j=1}^{n} b_{ij}(s) g_j\left(\varphi_j(s - \tau_{ij}(s))\right) \\
+ \sum_{j=1}^{n} c_{ij}(t) \int_{s-\sigma}^{s} h_j(\varphi_j(\zeta))\Delta\zeta + J_i(s),
\end{aligned}$$

Then Π maps $PAP(\mathbb{T}, \mathbb{R}^n, \nu)$ into itself.

Proof. For any given $\varphi \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$, we consider the following equation

$$\begin{aligned}
y_i^{\Delta}(t) = -\delta_i(t) y_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(y_j(t - \eta_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t)(y_j(t - \tau_{ij}(t))) \\
+ \sum_{j=1}^{n} c_{ij}(t) \int_{t-\xi}^{t} h(y_j(\zeta))\Delta\zeta + J_i(t), \ i = 1, 2, ..n,
\end{aligned} \tag{3.3}$$

and its associated homogeneous equation

$$y_i^{\Delta} = -\delta_i(t)y_i(t) \ i = 1, 2, .., n. \tag{3.4}$$

From H_4 and Lemma 2.17 we deduce that (3.4) admits an exponential dichotomy. In addition, we can assume by Lemmas (2.22) and (3.1) that the functions $s \mapsto f_j(\varphi_j(s - \eta_{ij}(s)))$ and $s \mapsto g_j(\phi_j(s - \tau_{ij}(s)))$ belong to $PAP(\mathbb{T}, \mathbb{R}^n, \nu)$. Note that we have actually proved in lemma (3.2) that the function $s \mapsto \int\limits_{s-\xi}^{s} h_j(y_j(\zeta))\Delta\zeta \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$. Hence, the function $\Gamma_i \in PAP(\mathbb{T}, \mathbb{R}^n, \nu)$. Thus, by Theorem 4.2 in [15] obtain that

$$\int\limits_{-\infty}^{t} e_{-\delta_i}(t, \sigma(s))\Gamma_i(s)\Delta s, \quad i = 1, 2, ..., n \tag{3.5}$$

is a weighted pseudo almost periodic solution of (1.1). □

Theorem 3.4. *Suppose that assumptions H_1-H_4 hold. Then the delayed chaotic neural networks* (1.1) *has a unique pseudo almost periodic solution in the region*

$$E = \left\{ \phi \in PAP(\mathbb{T}, \mathbb{R}^n, \nu), \ \|\varphi - \varphi_0\| \le \frac{k \|J_i\|_\infty}{\delta_i^-(1-k)} \right\}, \tag{3.6}$$

where $\varphi_0(t) = \int\limits_{-\infty}^{s} e_{-\delta_i}(t, \sigma(s))J_i(s)\Delta$.

Proof. Obviously, E is a closed subset of $PAP(\mathbb{T}, \mathbb{R}, \nu)$ and

$$\|\varphi_0(t)\| = \left\| \int_{-\infty}^{t} e_{-\delta_i}(t, \sigma(s))J_i(s)\Delta(s) \right\| \le \frac{\|J_i\|_\infty}{\delta_i^-}.$$

Hence,for any $\varphi \in E$, we get

$$\|\varphi(t)\| = \|\varphi(t) - \varphi_0(t)\| + \|\varphi_0(t)\| \le \frac{k \|J_i\|_\infty}{\delta_i^-(1-d)} + \frac{\|J\|_\infty}{\delta_i^-} \le \frac{\|J_i\|_\infty}{\delta_i^-(1-d)}.$$

It is clear that the operator Π is a self-mapping from E to E . In fact, for any $\varphi \in E$, we have

$$\begin{aligned}
&\|(\Pi\varphi)(t) - \varphi_0(t)\| \\
&= \Bigg\| \int\limits_{-\infty}^{t} e_{-\delta_i}(t, \sigma(s)) \Bigg[\sum_{j=1}^{n} a_{ij}(s) f_j(\varphi_j(s - \eta_{ij}(s))) + \sum_{j=1}^{n} b_{ij}(s) g_j(\varphi_j(s - \tau_{ij}(s))) \\
&+ \sum_{j=1}^{n} c_{ij}(t) \int\limits_{s-\xi}^{s} h_j(\varphi_j(\zeta))\Delta\zeta \Bigg] \Delta s \Bigg\| \\
&\le \int\limits_{-\infty}^{t} e_{-\delta_i}(t, \sigma(s)) \Bigg[\sum_{j=1}^{n} a_{ij}^+ \|f_j(\varphi_j(s - \eta_{ij}(s))) - f(0)\| + \sum_{j=1}^{n} b_{ij}^+ \|g_j(\varphi_j(s - \tau_{ij}(s))) - g(0)\|
\end{aligned}$$

$$+\sum_{j=1}^{n} c_{ij}^{+} \int_{s-\xi}^{s} \|h_j(\varphi_j(\zeta)) - h(0)\| \Delta\zeta \Bigg] \Delta s$$

$$\leq \frac{1}{\delta_i^-} \Bigg[\sum_{j=1}^{n} a_{ij}^{+} \|f_j(\varphi_j(s-\eta_{ij}(s))) - f(0)\| + \sum_{j=1}^{n} b_{ij}^{+} \|g_j(\varphi_j(s-\tau_{ij}(s))) - g(0)\|$$

$$+\sum_{j=1}^{n} c_{ij}^{+} \int_{s-\xi}^{s} \|h_j(\varphi_j(\zeta)) - h(0)\| \Delta\zeta \Bigg]$$

$$\leq \frac{\left[\sum_{j=1}^{n} a_{ij}^{+} L_j^f + \sum_{j=1}^{n} a_{ij}^{+} L_j^g + \sum_{j=1}^{n} c_{ij}^{+} L_j^h \xi\right]}{\delta_i} \|\varphi(t)\|$$

$$\leq \frac{k \|J_i\|_\infty}{\delta_i^- (1-k)}.$$

Which yields that $(\Pi\varphi) \in \mathbb{E}$. Our next aim is to prove that Π is a contraction mapping of $\mathbb{E}$. Under H_2, for any $\varphi\psi \in \mathbb{E}$ we have

$$\|(\Pi\varphi)(t) - (\Pi\psi)(t)\|$$

$$\leq \int_{-\infty}^{t} e_{-\delta_i}(t,\sigma(s)) \Bigg[\sum_{j=1}^{n} a_{ij}^{+} \|f_j(\varphi_j(s-\eta_{ij}(s))) - f_j(\psi_j(s-\eta_{ij}(s)))\|$$

$$+\sum_{j=1}^{n} b_{ij}^{+} \|g_j(\varphi_j(s-\tau_{ij}(s))) - g_j(\psi_j(s-\tau_{ij}(s)))\|$$

$$+\sum_{j=1}^{n} c_{ij}^{+} \int_{s-\xi}^{s} \|h_j(\varphi_j(\zeta)) - h(0)\| \Delta\zeta \Bigg] \Delta s$$

$$\leq \left\{ \int_{-\infty}^{t} e_{-\delta_i}(t,\sigma(s)) \left[\sum_{j=1}^{n} a_{ij}^{+} L_j^f + \sum_{j=1}^{n} a_{ij}^{+} L_j^g + \sum_{j=1}^{n} c_{ij}^{+} L_j^h \xi \right] \Delta s \right\} \|\phi - \psi\|_\infty$$

$$\leq \frac{\left[\sum_{j=1}^{n} a_{ij}^{+} L_j^f + \sum_{j=1}^{n} a_{ij}^{+} L_j^g + \sum_{j=1}^{n} c_{ij}^{+} L_j^h \xi\right]}{\delta_i^-} \|\phi - \psi\|_\infty$$

$$\leq k \|\phi - \psi\|_\infty.$$

Since $k < 1$, we obtain that Π is a contraction mapping. Hence, system (1.1) has a unique weighted pseudo almost periodic solution on $\mathbb{E}$. □

4. Exponential stability of the weighted pseudo-almost periodic solution

In this section, we will study the exponential stability of weighted pseudo almost periodic solution on time scale of system (1.1).

Definition 4.1. The pseudo almost periodic solution y of system (1.1) with initial value φ is said to be globally exponentially stable if there exist a positive constant λ with $\ominus\lambda \in \mathcal{R}^+$ and $M > 0$ such that every solution x of system (1.1) with initial value φ^* satisfies

$$\|x(t) - y(t)\| \le M e_{\ominus\lambda} \|\psi\|_0 , \quad \forall\ t \in (0, +\infty)_{\mathbb{T}},$$

where

$$\|\psi\|_0 = \sup_{t\in[-\pi,0]} \max_{1\le i\le n} \{|\varphi_i^*(t) - \varphi_i(t)|\} , \quad t_0 \in \max \{[-\pi, 0]\} .$$

Theorem 4.2. *Assume that H_1-H_4 hold. Then system* (1.1) *has a unique weighted pseudo almost periodic solution, which is globally exponential stable.*

Proof. On account of the above theorem the system (1.1) has a weighted pseudo almost periodic solution $y = (y_1, y_2, ..y_n)^T$ with the initial value $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$. Suppose that $x = (x_1, x_2, ...x_n)$ is an arbitrary solution of (1.1) with initial value $\varphi^* = (\varphi_1^*, \varphi_2^*, ..., \varphi_n^*)$. Then, by system (1.1) we have

(4.1)

$$\begin{aligned} z_i^{\Delta}(t) = -\delta_i z_i(t) &+ \sum_{j=1}^{n} a_{ij}(t) \left[f_j(z_j(t - \eta_{ij}(t)) + y_j(t - \eta_{ij}(t))) - f y_j(t - \eta_{ij}(t))\right] \\ &+ \sum_{j=1}^{n} b_{ij}(t) \left[g(z_j(t - \tau_{ij}(t)) + y_j(t - \tau_{ij}(t))) - g(y_j(t - \tau_{ij}(t))))\right] \\ &+ \sum_{j=1}^{n} c_{ij}(t) \int_{t-\xi}^{t} \left[h(z_j(\zeta) + y_j(\zeta)) - h(y_j(\zeta))\right] \Delta\zeta, \end{aligned}$$

where $z_i(t) = x_i(t) - y_i(t), \quad i = 1, 2, ..n$, the initial condition of (4.1) is

$$\psi_i(s) = \varphi_i^*(s) - \varphi_i(s), \quad s \in [-\pi, 0]_{\mathbb{T}} , i = 1, 2, ..n.$$

Let

$$\begin{aligned} \phi_i(\beta) = \delta_i^- - \beta - \exp(\beta \sup_{s\in\mathbb{T}} \mu(s)) \Bigg(&\sum_{j=1}^{n} a_{ij}^+ L_j^f \exp(\beta\eta_{ij}^+) + \sum_{j=1}^{n} b_{ij}^+ L_j^g \exp(\beta\tau_{ij}^+) \\ &+ \sum_{j=1}^{n} \xi c_{ij}^+ L_j^h \exp(\beta\xi) \Bigg) \end{aligned}$$

where $i = 1, 2, ..n$ and $\beta \in [0, \infty[$. By H_4, we get

$$\phi_i(0) = \delta_i^- - \left(\sum_{j=1}^{n} a_{ij}^+ L_j^f + \sum_{j=1}^{n} b_{ij}^+ L_j^g + \sum_{j=1}^{n} \sigma_{ij}^+ c_{ij}^+ L_j^h \right) > 0, \quad i = 1, 2, ...n. \tag{4.2}$$

Since ϕ_i, $i = 1, 2, ..n$ are continuous on $[0, +\infty)$ and $\phi_i(\beta) \longrightarrow -\infty$ as $\beta \longrightarrow +\infty$. Hence, there exist $\varepsilon_i > 0$ such that $\phi_i(\varepsilon_i) = 0$ and $\phi_i(\beta) > 0$ for $\beta \in (0, \varepsilon)$. Let $d = \min\limits_{1\leq i\leq n} \varepsilon_i$, we have $\phi_i(d) \geq 0$, $i = 1, 2, ..n$. Then, we can take a positive constant $0 < \lambda < \min\left\{d, \min\limits_{1\leq i\leq n} \delta_i^-\right\}$ such that $\phi(\lambda) > 0$, $i = 1, 2..n$ which implies that for $i = 1, 2, .., n$.

$$\frac{\exp(\lambda \sup\limits_{s\in\mathbb{T}} \mu(s))}{\delta_i^- - \lambda}\left[\sum_{j=1}^{n} a_{ij}^+ L_j^f \exp\left(\lambda\eta_{ij}^+\right) + \sum_{j=1}^{n} b_{ij}^+ L_j^g \exp(\lambda\tau_{ij}^+) + \sum_{j=1}^{n} \xi c_{ij}^+ L_j^h \exp(\lambda\xi)\right] < 1, \tag{4.3}$$

Multiplying (4.1) by $e_{-\delta_i}(t, \sigma(s))$ and integrating over $[t_0, t]$ for $i = 1, 2, ..., n$ we obtain

$$\begin{aligned} z_i(t) = z_i(t_0)e_{-\delta_i}(t, t_0) + \int_{t_0}^{t} e_{-\delta_i}(t, \sigma(s)) \\ \left(\sum_{j=1}^{n} a_{ij}(s)\left[f_j(z_j(s - \eta_{ij}(s)) + y_j(s - \eta_{ij}(s))) - fy_j(s - \eta_{ij}(s))\right]\right. \\ + \sum_{j=1}^{n} b_{ij}(s)\left[g(z_j(s - \tau_{ij}(s)) + y_j(s - \tau_{ij}(s))) - g(y_j(s - \tau_{ij}(s))))\right] \\ \left. + \sum_{j=1}^{n} \int_{s-\xi}^{s} c_{ij}(\varsigma)\left[h(z_j(\varsigma) + y_j(\varsigma)) - h(y_j(\varsigma))\right] \Delta\varsigma \right) \Delta s. \end{aligned} \tag{4.4}$$

Denote

$$M = \max_{1\leq i\leq n}\left\{\frac{\delta_i^-}{\left[\sum\limits_{j=1}^{n} a_{ij}^+ L_j^f + \sum\limits_{j=1}^{n} b_{ij}^+ L_j^g + \sum\limits_{j=1}^{n} \xi c_{ij}^+ L_j^h\right]}\right\},$$

by H_4 we can assume that $M > 1$. Hence,

$$\frac{1}{M} - \frac{\exp(\lambda \sup\limits_{s\in\mathbb{T}} \mu(s))}{\delta_i^- - \lambda} \left[\sum_{j=1}^{n} a_{ij}^+ L_j^f \exp\left(\lambda\eta_{ij}^+\right) + \sum_{j=1}^{n} b_{ij}^+ L_j^g \exp(\lambda\tau_{ij}^+) + \sum_{j=1}^{n} \xi c_{ij}^+ L_j^h \exp(\lambda\xi)\right] \leq 0.$$

It is obvious that

$$|z_i(t)| = |\psi_i(t)| \le \|\psi\|_0 \, Me_{\ominus\lambda}(t,t_0) \|\psi\|_0 \,, \quad \forall\, t \in [-\pi, 0]_{\mathbb{T}} \,, \quad i = 1, 2, ..n,$$

where $\lambda \in \mathcal{R}^+$ is the same as in (4.3), then

$$\|x(t) - y(t)\| = \max_{1 \le i \le n} \{|z_i(t)|\} \le Me_{\ominus\lambda}(t,t_0) \|\psi\|_0 \,, \quad \forall\, t \in [-\pi, 0]_{\mathbb{T}} \,.$$

We claim that

$$\|x(t) - y(t)\| \le Me_{\ominus\lambda}(t,t_0) \|\psi\|_0 \,, \quad \forall\, t \in (0, +\infty)_{\mathbb{T}} \,. \tag{4.5}$$

To prove that (4.5) holds, we first show that, for any $P > 1$ the following inequality holds:

$$\|x(t) - y(t)\| \le PMe_{\ominus\lambda}(t,t_0) \|\psi\|_0 \,, \quad \forall\, t \in (0, +\infty)_{\mathbb{T}} \,. \tag{4.6}$$

If (4.6) is not true, then there must be some $t_1 \in (0, +\infty)_{\mathbb{T}}$ and $C > 0$ and some k such that

$$\|x(t_1) - y(t_1)\| = |x_k(t_1) - y_k(t_1)| = CPMe_{\ominus\lambda}(t_1,t_0) \|\psi\|_0 \,, \tag{4.7}$$

and

$$\|x(t) - y(t)\| \le CPMe_{\ominus\lambda}(t,t_0) \|\psi\|_0 \,, \quad \forall\, t \in [-\pi, t_1]_{\mathbb{T}} \,. \tag{4.8}$$

By (4.4)-(4.8)

$$\begin{aligned}
&|z_i(t_1)| = \|\psi\|_0 \, e_{-\delta_i}(t_1,t_0) \\
&+ \int_{t_0}^{t_1} CPMe_{-\delta_i}(t_1,\sigma(s)) \Bigg(\sum_{j=1}^{n} a_{ij}^+ L_j^f e_{\ominus\lambda}(s - \eta_{ij}(s), t_0) + \sum_{j=1}^{n} b_{ij}^+ L_j^g e_{\ominus\lambda}(s - \tau_{ij}(s), t_0) \\
&+ \sum_{j=1}^{n} c_{ij}^+ L_j^h \int_{s-\xi}^{s} e_{\ominus\lambda}(\zeta, t_0) \Delta\zeta \Bigg) \Delta s \\
&\le CPMe_{\ominus\lambda}(t_1,t_0) \|\psi\|_0 \Bigg\{ \frac{1}{CPM} e_{-\delta_i}(t_1,t_0) e_{\ominus\lambda}(t_0,t_1) + \int_{t_0}^{t_1} e_{-\delta_i}(t_1,\sigma(s)) e_{\lambda}(t_1,\sigma(s)) \\
&\times \Bigg(\sum_{j=1}^{n} a_{ij}^+ L_j^f e_{\ominus\lambda}(s - \eta_{ij}(s), \sigma(s)) \\
&+ \sum_{j=1}^{n} b_{ij}^+ L_j^g e_{\ominus\lambda}(s - \tau_{ij}(s), \sigma(s)) + \sum_{j=1}^{n} c_{ij}^+ L_j^h \int_{s-\xi}^{s} e_{\ominus\lambda}(\zeta, \sigma(s)) \Delta\zeta \Bigg) \Bigg\} \\
&\le CPMe_{\ominus\lambda}(t_1,t_0) \|\psi\|_0 \Bigg\{ \frac{1}{M} e_{-\delta_i \oplus \lambda}(t_1,t_0) + \Bigg(\sum_{j=1}^{n} a_{ij}^+ L_j^f \exp(\lambda(\eta_{ij}^+ + \sup_{s\in\mathbb{T}} \mu(s))) \\
&+ \sum_{j=1}^{n} b_{ij}^+ L_j^g \exp(\lambda(\tau_{ij}^+ + \sup_{s\in\mathbb{T}} \mu(s)))
\end{aligned}$$

$$+\sum_{j=1}^{n}c_{ij}^{+}\xi L_j^h\exp(\lambda(\xi+\sup_{s\in\mathbb{T}}\mu(s)))\Bigg)\int_{t_0}^{t_1}e_{-\delta_i\oplus\lambda}(t_1,\sigma(s))\Bigg\}$$

$$\leq CPMe_{\ominus\lambda}(t_1,t_0)\left\|\psi\right\|_0\left\{\frac{1}{M}e_{-\delta_i\oplus\lambda}(t_1,t_0)+\left(\sum_{j=1}^{n}a_{ij}^{+}L_j^f\exp(\lambda(\eta_{ij}^{+}+\sup_{s\in\mathbb{T}}\mu(s)))\right.\right.$$

$$+\sum_{j=1}^{n}b_{ij}^{+}L_j^g\exp(\lambda(\tau_{ij}^{+}+\sup_{s\in\mathbb{T}}\mu(s)))$$

$$\left.\left.+\sum_{j=1}^{n}c_{ij}^{+}\xi L_j^h\exp(\lambda(\xi+\sup_{s\in\mathbb{T}}\mu(s)))\right)\frac{1-e_{-\delta_i\oplus\lambda}(t_1,\sigma(s))}{\delta_i^{-}-\lambda}\right\}$$

$$\leq CPMe_{\ominus\lambda}(t_1,t_0)\left\|\psi\right\|_0\left\{\frac{1}{M}-\frac{1}{\delta_i^{-}-\lambda}\left[\sum_{j=1}^{n}a_{ij}^{+}L_j^f\exp(\lambda(\eta_{ij}^{+}+\sup_{s\in\mathbb{T}}\mu(s)))\right.\right.$$

$$+\sum_{j=1}^{n}b_{ij}^{+}L_j^g\exp(\lambda(\tau_{ij}^{+}+\sup_{s\in\mathbb{T}}\mu(s)))$$

$$\left.+\sum_{j=1}^{n}c_{ij}^{+}\xi L_j^h\exp(\lambda(\xi+\sup_{s\in\mathbb{T}}\mu(s)))\right]e_{-\delta_i\oplus\lambda}(t_1,\sigma(s))$$

$$+\frac{1}{\delta_i^{-}-\lambda}\left(\sum_{j=1}^{n}a_{ij}^{+}L_j^f\exp(\lambda(\eta_{ij}^{+}+\sup_{s\in\mathbb{T}}\mu(s)))+\sum_{j=1}^{n}b_{ij}^{+}L_j^g\exp(\lambda(\tau_{ij}^{+}+\sup_{s\in\mathbb{T}}\mu(s)))\right.$$

$$\left.\left.+\sum_{j=1}^{n}c_{ij}^{+}\xi L_j^h\exp(\lambda(\xi+\sup_{s\in\mathbb{T}}\mu(s)))\right)\right\}$$

$$\leq CPMe_{\ominus\lambda}(t_1,t_0)\left\|\psi\right\|_0,$$

which contradicts (4.7) and so (4.6) holds. Letting $P\to 1$, then (4.5) holds. Therefore, the weighted pseudo-almost periodic solution of system (1.1) is globally exponentially stable. □

5. Examples and simulations

Consider the following neural network:

$$y_i^{\Delta}=-\delta_i(t)y_i(t)+\sum_{j=1}^{2}a_{ij}(t)f_j(y_j(t-\eta_{ij}(t)))+\sum_{j=1}^{2}b_{ij}(t)g_j(y_j(t-\tau_{ij}(t)))$$
$$+\sum_{j=1}^{2}c_{ij}(t)\int_{t-3}^{t}h_j(y_j(\zeta))\Delta\zeta+J_i(t)\tag{5.1}$$

5.1. **Example.** Take $\mathbb{T} = \mathbb{R}$

$$\delta_1(t) = 0.5 + 0.3\cos(\sqrt{2}t), \quad \delta_2(t) = 0.9 + 0.6\sin(\sqrt{2}t),$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.07\cos(t) & 0.08\cos(t) \\ 0.05\cos(\sqrt{2}t) & 0.05\cos(\sqrt{5}t) \end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.05\sin(\sqrt{4}t) & 0.06\cos(\sqrt{2}t) \\ 0.07\sin(\sqrt{5}t) & 0.04\sin(\sqrt{5}t) \end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0.01\cos(\sqrt{7}t) & 0.01\sin(\sqrt{4}t) \\ 0.011\cos(\sqrt{3}t) & 0.03\cos(t) \end{pmatrix}$$

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} \sin(\sqrt{2}t) + \sin(t) + \mathrm{e}^{-|t|} \\ \cos(t) \end{pmatrix}$$

$$f_1(x) = g_1(x) = h_1(x) = 0.3\sin(x), \quad f_2(x) = g_2(x) = h_2(x) = 0.75\sin(x)$$

By calculation, we have $\delta_1^- = 0.2$, $\delta_2^- = 0.3$, $L_1^f = L_1^g = L_1^f = 0.3$, $L_2^f = L_2^g = L_2^f = 0.75$
$a_{11}^+ = 0.07, a_{12}^+ = 0.08, a_{21}^+ = 0.05, a_{22}^+ = 0.05, b_{11}^+ = 0.05, b_{12}^+ = 0.06, b_{21}^+ = 0.07, b_{22}^+ = 0.04$, $c_{11}^+ = 0.01, c_{12}^+ = 0.01, c_{21}^+ = 0.011, \quad c_{22}^+ = 0.03$, $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \eta_{11} = \eta_{12} = \eta_{21} = \eta_{22} = 1$, and

$$k = \max_{1\le i\le n} \left\{ \frac{\left[\sum_{j=1}^{n} a_{ij}^+ L_j^f + \sum_{j=1}^{n} b_{ij}^+ L_j^g + \sum_{j=1}^{n} c_{ij}^+ L_j^h \xi \right]}{\delta_i^-} \right\} = \max\{0.87375, 0.45\} < 1.$$

$\mathbb{T} = \mathbb{Z}$

$$\delta_1(k) = 0.6 + 0.2\sin(\sqrt{2}k), \quad \delta_2(k) = 0.8 + 0.5\cos(\sqrt{2}k),$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.08\sin(k) & 0.07\sin(k) \\ 0.06\cos(\sqrt{2}k) & 0.04\cos(\sqrt{5}k) \end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.06\sin(\sqrt{4}k) & 0.05\cos(\sqrt{2}k) \\ 0.08\cos(\sqrt{5}k) & 0.03\sin(\sqrt{5}k) \end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0.02\cos(\sqrt{7}k) & 0.03\sin(\sqrt{4}k) \\ 0.013\cos(\sqrt{3}k) & 0.01\cos(k) \end{pmatrix}$$

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{2}k) + \cos(k) + \mathrm{e}^{-|k|} \\ \sin(k) \end{pmatrix}$$

$$f_1(k) = g_1(x) = h_1(k) = 0.25\sin(k), \quad f_2(k) = g_2(k) = h_2(k) = 0.6\sin(k)$$

$\delta_1^- = 0.4$, $\delta_2^- = 0.3$, $L_1^f = L_1^g = L_1^f = 0.25$, $L_1^f = L_1^g = L_1^f = 0.6$
$a_{11}^+ = 0.08$, $a_{12}^+ = 0.07$, $a_{21}^+ = 0.06$, $a_{22}^+ = 0.04$, $b_{11}^+ = 0.06$, $b_{12}^+ = 0.05$, $b_{21}^+ = 0.08$, $b_{22}^+ = 0.03$, $c_{11}^+ = 0.02$, $c_{12}^+ = 0.03$, $c_{21}^+ = 0.013$, $c_{22}^+ = 0.01$, $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \eta_{11} = \eta_{12} = \eta_{21} = \eta_{22} = 1$. and

$$k = \max_{1 \le i \le n} \left\{ \frac{\left[\sum_{j=1}^{n} a_{ij}^+ L_j^f + \sum_{j=1}^{n} b_{ij}^+ L_j^g + \sum_{j=1}^{n} c_{ij}^+ L_j^h \xi \right]}{\delta_i^-} \right\} = \max\{0.44, 03, 4916\} < 1.$$

Therefore, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, all the conditions of Theorems (3.4) and (4.2) are satisfied. Consequently, system (5.1) has a weighted pseudo almost periodic solution, which is globally Exponentially stable (see Figs. 1-6).

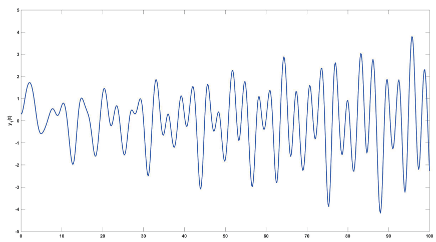

FIGURE 1. $\mathbb{T} = \mathbb{R}$: Curve of $y_1(t)$

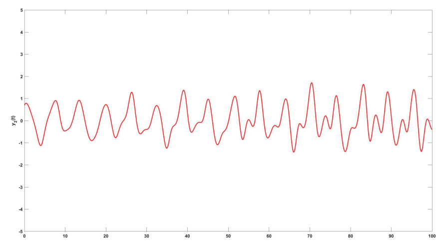

FIGURE 2. $\mathbb{T} = \mathbb{R}$: Curve of $y_2(t)$

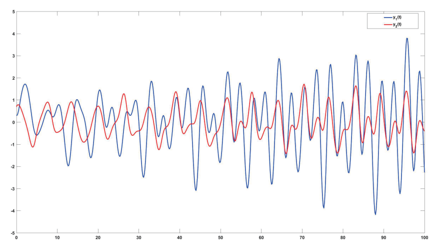

FIGURE 3. $\mathbb{T} = \mathbb{R}$: Curve of $y_1(t)$ and $y_2(t)$

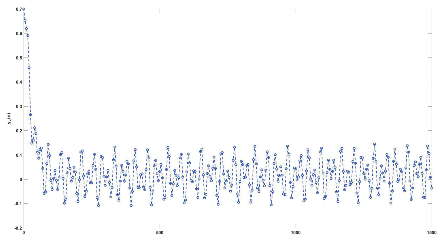

FIGURE 4. $\mathbb{T} = \mathbb{Z}$: Curve of $y_1(n)$

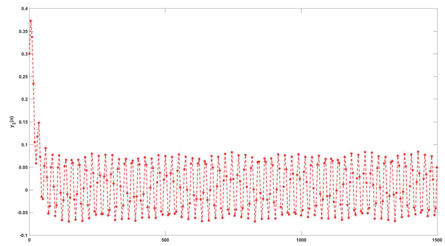

FIGURE 5. $\mathbb{T} = \mathbb{Z}$: Curve of $y_2(n)$

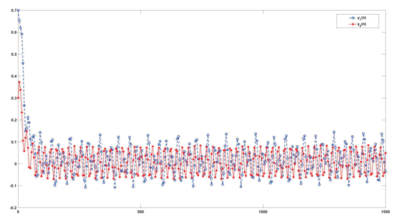

FIGURE 6. $\mathbb{T} = \mathbb{Z}$: Curve of $y_1(t)$ and $y_2(n)$

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