Applied Analysis and Optimization
Volume 8, Number 1, 2024, 39–50

Yokohama Publishers
ISSN 2189-1664 Online Journal

BOUNDEDNESS AND CONVERGENCE OF SOLUTIONS OF A CLASS OF PERTURBED NONAUTONOMOUS DIFFERENTIAL EQUATIONS

MOHAMED ALI HAMMAMI AND RABIE HAMLILI

ABSTRACT. Lyapunov's method has been applied by many researchers in the past century to investigate the stability of nonlinear systems to show that a system is stable in the sense of Lyapunov, a positive definite function of the system states, which decreases along system trajectories (Lyapunov function), should be found. Based on Lyapunov techniques, this study presents some growth conditions for the boundedness and convergence of solutions for a class of time-varying differential equations that are nonlinear under some restrictions on the term of perturbation. Some axamples are given to prove the validity of the main result.

1. Introduction

One of the major areas of differential equations theory is the asymptotic behaviors of solutions and their properties such as boundedness ans stability. Time-varying differential equations seem to be a natural way to explain the observed evolution processes of some real-world issues where the notion of boundedness of solutions is very important. It is commonly known that, for a dynamical system characterized by ordinary differential equations or nonlinear systems with perturbations, the second Lyapunov approach ([8, 19]) offers sufficient conditions to guarantee different kinds of stability and boundedness of solutions (see [2, 3, 6, 7, 11–13, 15, 21–24]). Errors in modeling a nonlinear system, aging of parameters, uncertainties, and shocks can all lead to the perturbation term. Generally speaking, we have some knowledge of the term of perturbation's upper bound. What can be said about the perturbed system's behavior if we assume that the nominal system has the propertiy of stability or boundedness of solutions for its equilibrium pointsupposed at the origin. This encourages us to investigate the issue of asymptotic behavior of solutions of perturbed systems by making the assumption that, subject to certain limitations on the scale of perturbations, the nominal related system has the same property ([1, 4, 5]). This potent class of results has been used extensively for analysis and design in system and control theory research over the years. The concept of Lipschitz stability (see [9, 10]) lies somewhere between uniformly stability on one side and the notions of asmptotic stability in variation and uniformly stability in variation on the other side. Motivated by the interesting concept of stability, the autors in (see [14, 16–18, 20]) provide some sufficient conditions for the convergence of solutions for different classes of nonlinear systems including Volterra integro-differential

2020 *Mathematics Subject Classification.* 45J05, 45D05, 26D10, 45M10.
Key words and phrases. Differential equations, Lyapunov functions, convergence of solutions.

equations. Here, by placing some limitations on the perturbation term, we investigate the problem of figuring out how the solutions of a perturbed dynamic equation behave in relation to those of an original unperturbed dynamic system.

2. Stability analysis

Consider the time-varying system described by the following time-varying differential equation:

$$\dot{x} = f(t, x) \tag{2.1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and locally Lipschitz with respect to x such that $f(t, 0) = 0$, $\forall t \geq 0$, and the associated perturbed systems:

$$\dot{x} = f(t, x) + g(t) \tag{2.2}$$

where $t \in \mathbb{R}^+$, $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ is a continuous function. The link between the stability of the origin studied as the equilibrium point of the system (2.1) and the null solution of (2.2) has been widely studied by many authors using Lyapunov techniques or non-integral inequalities. Noting that, there exist systems of the form (2.1) such that the origin is stable, but there exists bounded disturbances $g(t)$ such that (2.2) becomes unstable.

Consider the time-varying system (2.1). Unless otherwise stated, we assume throughout the paper that the function $f(.,.)$ encountered is sufficiently smooth. We often omit arguments of function to simplify notation, $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$. $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\},\ r > 0$.

Definition 2.1. *The solutions of system* (2.1) *are uniformly bounded if there exists* $\Lambda > 0$, *such that for all* $\Lambda_1 > 0$, *there exists a* $T = T(R_1)$, *such that for all* $t_0 \geq 0$

$$\|x_0\| \leq \Lambda_1 \Rightarrow \|x(t)\| \leq \Lambda, \quad \forall t \geq t_0 + T.$$

Let $x = 0$ be an equilibrium point for the nonlinear system (2.1). The origin is an equilibrium point, if

$$f(t, 0) = 0, \quad \forall t \geq 0.$$

Definition 2.2. $x = 0$ is said to be globally exponentially stable if there exist $k, \lambda > 0$, such that all trajectories satisfy:

$$\|x(t)\| \leq k\|x_0\| \exp(-\lambda(t - t_0)), \, \forall\, x_0 \in \mathbb{R}^n, \, \forall t \geq t_0 \geq 0.$$

Here, we have supposed that $f(t, 0) = 0$, $\forall t \geq 0$. For the stability study of the perturbed system (2.2), in the case where $g(t) \neq 0$ for a certain $t \geq 0$, we shall study the asymptotic behavior of solutions in a neighborhood of the origin, in the sense that the solutions converge to a certain small ball B_r, $r > 0$ centred at the origin. Therefore, we introduce the notion of exponential stability of B_r.

Definition 2.3. The ball B_r is globally uniformly exponentially stable with respect the system (2.2), if there exists $\lambda_1 > 0, \lambda_2 > 0$, such that

$$\|x(t, t_0, x_0)\| \leqslant \lambda_1(\|x_0\|)e^{-\lambda_2(t-t_0)} + r, \tag{2.3}$$

for all $t \geqslant t_0 \geqslant 0,\ \forall x_0 \in \mathbb{R}^n \setminus B_r$.

The factor λ_2 in the definition above will be named the convergence speed while factor λ_1 will be named the transient estimate. It is also, worth to notice that, in the above definition, if we take $r = 0$, then one deals with the standard concept of the global exponential stablity of the origin viewed as an equilibrium point. Moreover, we shall study the asymptotic behavior of a small ball centered at the origin for $0 \leqslant \| x(t) \| -r,\ \forall t \geqslant t_0 \geqslant 0$ so that the initial conditions are taken outside the ball B_r. If r is small enough, then the trajectories approach to the origin when t goes to infinity. Lyapunov´s direct method allows us to determine the stability of a system without explicitly integrating the differential equation. This method is a generalization of the idea that if there is an appropriate energy function in a system, then we can study the rate of change of the energy of the system to ascertain stability. To make this precise, we need the following properties.

For the linear case $\dot{x} = A(t)x$, the equilibrium point is globally uniformly exponentially stable if and only if the transition matrix $\Phi(t, t_0)$ associated to $A(t)$ satisfies:

$$\|\Phi(t, t_0)\| \leq k \exp -\gamma(t - t_0), \forall t \geq t_0, k > 0, \gamma > 0.$$

In this case, we assume that $A(t)$ is bounded for all $t \geq 0$. When $f(t, x)$ in (2.2) is linear it means that $f(t, x) = A(t)x$, the solution of the equation (2.2) with initial condition (t_0, x_0) is given by:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)g(s)ds.$$

Then, we have

$$\|x(t)\| \leq k \exp -\gamma(t - t_0)\|x(t_0)\| + \int_{t_0}^{t} ke^{-\gamma(t-s)}\|g(s)\|ds.$$

If we suppose that, $\|g(t)\| \leq \delta(t)$ where $\lambda : t \mapsto \delta(t)$ is a continuous nonnegative L^1 function, then

$$\|x(t)\| \leq k \exp -\gamma(t - t_0)\|x(t_0)\| + \int_{t_0}^{t} ke^{-\gamma(t-s)}\|g(s)\|ds.$$

Thus,

$$\|x(t)\| \leq k \exp -\gamma(t - t_0)\|x(t_0)\| + e^{-\gamma t} \int_{t_0}^{t} ke^{\gamma s}\delta(t)ds.$$

Note that, the fact that the function $\delta(t)$ is integrable, then one has the integral I defined by

$$I = e^{-\gamma t} \int_{t_0}^{t} ke^{\gamma s}\delta(t)ds$$

is bounded. Therefore there exits, $\eta > 0$ such that

$$\|x(t)\| \leq k \exp -\gamma(t-t_0)\|x(t_0)\| + \eta.$$

The last expression implies that, the ball B_η is globally uniformly exponentially stable with respect the system (2.2).

In the sequel, we give some of the main definitions that we need to study the asymptotic behavior of the solutions. The notion of stability will be given in the sense of *"Stability in variation"* introduced by [4, 5].

For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$, we will denote by $x(t, t_0, x_0)$, or simply by $x(t)$, the unique solution of (2.1) at time t_0 starting from the point x_0. We have, $\forall t \geq t_0 \geq 0$, $x(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x(s, t_0, x_0))ds$. Let $f_x(t, x)$ be the matrix whose element in the ith row, jth column is the partial derivative of the ith component of f with respect to the jth component of x, $f_x = mat(\frac{\partial f_i}{\partial x_j})_{i,j=1,\dots,n}$. Let $x(t, t_0, x_0)$ be the solution of (2.1). We have, $\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0}(x(t, t_0, x_0)$, is a solution of the variational system:

$$\dot{z} = f_x(t, x(t, t_0, x_0))z \tag{2.4}$$

$\Phi(t, t_0, x_0)$ is called the fundamental matrix solution of (2.3) with respect to the solution $x(t, t_0, x_0)$ which is the identity matrix for $t = t_0$. We will define the notion of stability in terms of variational system with respect the solution $x = 0$, (see [4,5]).

Definition 2.4. The solution $x = 0$ of (2.1) is said to be globally uniformly stable in variation if there exists a positive constant M, such that

$$\|\Phi(t, t_0, x_0)\| \leqslant M, \quad \forall\, t \geqslant t_0 \geqslant 0,\ \forall\, x_0 \in \mathbb{R}^n. \tag{2.5}$$

Note that, the solution $x(t, t_0, x_0)$ of (2.1) satisfies the following equality:

$$x(t, t_0, x_0) = (\int_0^1 \Phi(t, t_0, sx_0)ds)x_0,$$

if $x = 0$ of (2.1) is globally uniformly stable in variation, then

$$\|x(t, t_0, x_0)\| \leqslant M\|x_0\|, \quad \forall\, t \geqslant t_0 \geqslant 0,\ \ \forall\, x_0 \in \mathbb{R}^n. \tag{2.6}$$

Definition 2.5. The solution $x = 0$ of (2.1) is said to be globally uniformly slowly growing in variation if for every $\varepsilon > 0$ there exists a positive constant M, possibly depending on ε, such that

$$\|\Phi(t, t_0, x_0)\| \leqslant Me^{\varepsilon(t-t_0)}, \quad \forall\, t \geqslant t_0 \geqslant 0,\ \ \forall\, x_0 \in \mathbb{R}^n. \tag{2.7}$$

Definition 2.6. The solution $x = 0$ of (2.1) is said to be globally uniformly exponentially stable in variation if there exist two positive constants λ_1 and λ_2, which are independent of the initial condition, such that

$$\|\Phi(t, t_0, x_0)\| \leqslant \lambda_1 e^{-\lambda_2(t-t_0)}, \quad \forall\, t \geqslant t_0 \geqslant 0,\ \forall\, x_0 \in \mathbb{R}^n. \tag{2.8}$$

Definition 2.7. The solution $x = 0$ of (2.1) is said to be uniformly asymptotically stable in variation if there exists a positive constant $\tilde{M}$, which is independent of the initial condition, such that for every $t_0 \geqslant 0$,

$$\int_{t_0}^{t} \|\Phi(t, s, 0)\| \leqslant \tilde{M}, \quad \forall\, t \geqslant t_0 \geqslant 0. \tag{2.9}$$

The following result etablished by Brauer ([4, 5]):
If the solution $x = 0$ of (2.1) is uniformly asymptotically stable in variation then there exist some positive constants c and $\tilde{M}$ such that,

$$\int_{t_0}^{t} \sup_{x_0 \leqslant c} \|\Phi(t, s, x_0)\| \leqslant \tilde{M},$$

for every sufficiently large t_0 and all $t \geqslant t_0 \geqslant 0$.

Notting that, if the trivial solution $x = 0$ of (2.1) is uniformly asymptotically stable in variation, then for all $t_0 \geqslant 0$, and $x_0 \in \mathbb{R}^n$,

$$\lim_{t \to +\infty} \|\Phi(t, t_0, x_0)\| = 0.$$

The connection between the stability of the zero solution of (2.1) and the zero solutions of (2.2), with respect the nominal part because in presence of g the origin is not an equilibrium point, has been extensively investigated where analogous results are established here for the notion of uniform Lipschitz stability (see [9, 10]). Before giving further details, we give some of definitions that are related to the Lipschitz stability.

The zero solution of (2.1) is said to be uniformly Lipschitz stable if there exists $M > 0$ and $\delta > 0$ such that

$$\|x\left(t, t_0, x_0\right)\| \leqslant M\|x_0\|,$$

whenever $\|x_0\| \leqslant \delta$ and $t \geqslant t_0 \geqslant 0$.

The zero solution of (2.1)is said to be globally uniformly Lipschitz stable if there exists $M > 0$ such that $\|x\left(t, t_0, x_0\right)\| \leqslant M\|x_0\|$ for $\|x_0\| < \infty$ and $t \geqslant t_0 \geqslant 0$.

We remark here that the notion of global uniform Lipschitz stable implies the "global uniform stability in variation." For the linear system $x' = A(t), x$ the following statements are equivalent (see [9, 10]): The zero solution is globally uniformly Lipschitz stable in variation, uniformly Lipschitz stable in variation, globally uniformly Lipschitz stable, uniformly Lipschitz stable, uniformly stable.

Example 1. Consider the scalar equation:

$$\dot{x}(t) = f(t, x(t)) + g(t),$$

with $f(t, x) = -e^t x^3$ and $g(t) = e^{-t}$, $t \geq 0$. We have, for $\dot{x}(t) = f(t, x(t))$ and the initial condition (t_0, x_0),

$$x(t, t_0, x_0) = x_0(1 + 2x_0^2(e^t - e^{t_0}))^{1\frac{1}{2}}.$$

Therefore for $t \geq t_0 \geq 0$, we have

$$\|\Phi(t, t_0, x_0)\| = \|(1 + 2x_0^2(e^t - e^{t_0}))^{\frac{-3}{2}}\| \leq 1.$$

Thus, the zero solution of the nominal system is globally uniformly stable in variation. Now if we take a bounded perturbation $g(t)$, the solutions of the perturbed equation are bounded.

Lyapunov approach The Lyapunov approach helps us to determine the stability and convergence of the solutions of a system without explaining the general solution also without integrate the differential equation.
Consider a continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, V is said to be globally Lipschitzian in x (uniformly in $t \in \mathbb{R}^+$) if

$$|V(t,x) - V(t,y)| \leqslant \kappa \|x - y\|$$

for some $\kappa > 0$ and for all $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$. Corresponding to V we define the Dini derivative D^+V with respect to system (2.1) by

$$D^+V_f(t,x) = \limsup_{h \to 0^+} \frac{1}{h}(V(t+h, x+hf(t,x)) - V(t,x)),$$

called the upper Dini derivative of $V(.,.)$ along the trajectory of (2.1). Let $x(t)$ be a solution of (2.1) and denote by $V'(t,x)$ the upper right-hand derivative, i.e.,

$$V'(t,x(t)) = \limsup_{h \to 0^+} \frac{1}{h}(V(t+h, x(t+h)) - V(t,x)).$$

If $V(t,x)$ is continuous in t and Lipschitzian in x (uniformly in t) with the Lipschitz constant $\kappa > 0$, then $V'(t,x)$ and $D_f^+V(t,x)$ are related as follows:

$$V'(t,x(t)) \leqslant D^+V_f(t,x(t)).$$

Indeed, we have

$$\begin{aligned} V(t+h,x(t+h)) - V(t,x(t)) &= V(t+h,x(t+h)) - V(t+h,x+hf(t,x)) \\ &\quad + V(t+h,x+hf(t,x)) - V(t,x(t)). \end{aligned}$$

Since the function V is globally Lipschitzian in x (uniformly in t), then

$$\limsup_{h \to 0^+} \frac{V(t+h,x(t+h)) - V(t,x(t))}{h} \leq$$

$$\kappa\{\lim_{h \to 0^+} \|\frac{x(t+h) - x(t)}{h} - f(t,x(t))\|\} + \limsup_{h \to 0^+} \frac{1}{h}(V(t+h,x+hf(t,x)) - V(t,x)),$$

which gives

$$V'(t,x) \leqslant D_f^+V(t,x). \tag{2.10}$$

Note that, in case when the function V is differentiable, the derivative with respect to time along the trajectories of system (2.1) is given by:

$$\frac{d}{dt}V(t,x) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x).f(t,x),$$

in this case, we have

$$\frac{d}{dt}V(t,x) = V'(t,x) = D_f^+V(t,x).$$

Suppose that the jacobian matrix $[\partial f/\partial x]$ is bounded on $\mathbb{R}^n$, uniformly in t. Assume that the system (2.1) is globally exponentially stable, then there is a continuously differentiable Lyapunov function

$$V : [0,+\infty[\times \mathbb{R}^n \longrightarrow \mathbb{R}_+,$$

that satisfies the following conditions for some positive constants c_1, c_2, c_3, c_4 and for all $x \in \mathbb{R}^n$, $t \geq t_0$:

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2,$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2,$$

and

$$\|\frac{\partial V}{\partial x}(t,x)\| \leq c_4\|x\|.$$

If we suppose that, $\|g(t)\| \leq \delta(t)$ where $\lambda : t \mapsto \delta(t)$ is a continuous nonnegative bounded function, by taking the derivative along the trajectories of the perturbed system (2.2), one has

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t),$$

then

$$\dot{V}(t,x) \leq -c_3\|x\|^2 + \|\frac{\partial V}{\partial x}\|\delta(t).$$

So,

$$\dot{V}(t,x) \leq -c_3\|x\|^2 + c_4\delta(t)\|x\|.$$

Thus,

$$\dot{V}(t,x) \leq -c_3\|x\|^2 + \frac{c_4}{2}(\|x\|^2 + \delta^2(t)),$$

This implies that,

$$\dot{V}(t,x) \leq [-\frac{c_3}{c_2} + \frac{c_4}{2c_1}]V(t,x) + \frac{c_4}{2}\delta^2(t).$$

Let $\tilde{V}(t) = V(t,x(t))$, $\varsigma_1 = \frac{c_3}{c_2} - \frac{c_4}{2c_1}$ and $\varsigma_2 = \frac{c_4}{2}$, one gets

$$\dot{\tilde{V}}(t) \leq -\varsigma_1\tilde{V}(t) + \varsigma_2\delta^2(t).$$

By integration we get:

$$\tilde{V}(t) \leq \tilde{V}(0)e^{-\varsigma_1 t} + \varsigma_2 e^{-\varsigma_1 t}\int_0^t e^{\varsigma_1 s}\delta^2(s)ds.$$

Since $\delta(t)$ is a nonnegative bounded function then it is the same for its square $\delta^2(t)$. There exists a nonnegative constant $\tilde{\delta} > 0$, such that for all $t \geq 0$, $\delta^2(t) \leq \tilde{\delta}$. So,

$$\tilde{V}(t) \leq \tilde{V}(0)e^{-\varsigma_1 t} + \varsigma_2\tilde{\delta}e^{-\varsigma_1 t}\int_0^t e^{\varsigma_1 s}ds.$$

Thus,

$$\tilde{V}(t) \leq \tilde{V}(0)e^{-\varsigma_1 t} + \frac{\varsigma_2}{\varsigma_1}\tilde{\delta}.$$

Using the fact that, $c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$, it follows that,

$$\|x(t)\| \leq (\frac{c_2}{c_1})^{\frac{1}{2}}\|x(0)\|e^{-\frac{1}{2}\varsigma_1 t} + (\frac{\varsigma_2}{c_1\varsigma_1}\tilde{\delta})^{\frac{1}{2}}.$$

The last expression implies that, the ball $B_{(\frac{\varsigma_2}{c_1\varsigma_1}\tilde{\delta})^{\frac{1}{2}}}$ is globally uniformly exponentially stable with respect the system (2.2), 2.1. **Convergence and boundedness of the solutions.** Let consider the perturbed system (2.2). If the zero solution is uniformly Lipschitz stable for (2.1) with

$$\|\Phi(t,s,x)\| \leqslant \iota\|z\|,\ \forall\, x \in \mathbb{R}^n,\ \forall t \geq t_0 \geq 0,\ \iota > 0,$$

and the perturbation term in (2.2) satisfies

$$\|g(t)\| \leqslant \xi(t),\ \forall\, t \geqslant t_0 \geqslant 0, \tag{2.11}$$

where $\xi(.) \in C[\mathbb{R}^+, \mathbb{R}^+]$ and integrable function, then the zero solution of the perturbed system is uniformly Lipschitz stable.
Note that, $\Phi\left(t, t_0, x\left(t, t_0, x_0\right)\right)$ is the fundamental matrix of (2.4), and the zero solution of (2.2) is uniformly Lipschitz stable, where $\int_0^\infty \xi(s)ds < \infty$.
Indeed, using the nonlinear variation of constants formula, the solutions $y(t)$ of the perturbed system with the same initial values $y_0 = x_0$ are related by

$$y\left(t, t_0, x_0\right) = x\left(t, t_0, x_0\right) + \int_{t_0}^{t} \Phi\left(t, s, y\left(s, t_0, x_0\right)\right) g(s)ds$$

Hence

$$\|y\left(t, t_0, x_0\right)\| \leqslant \|x\left(t, t_0, x_0\right)\| + \int_{t_0}^{t} \|\Phi(t,s,y)\|\|g(s)\|ds.$$

Since the zero solution of (2.1) is uniformly Lipschitz stable, there exist $\alpha > 0$ and $\delta > 0$ such that $|x_0| \leqslant \delta$ implies $\|x\left(t, t_0, x_0\right)\| \leqslant \alpha\|x_0\|$. Hence, using the assumption on ξ, we obtain

$$\|y\left(t, t_0, x_0\right)\| \leqslant \alpha\|x_0\| + \iota \int_{t_0}^{t} \|y\|\xi(s)ds.$$

Applying Gronwall's inequality, one obtains

$$\|y\left(t, t_0, x_0\right)\| \leqslant \alpha exp(\iota \int_{t_0}^{t} \xi(s)ds).$$

Since the function $\xi(s)$ is integrable, then there exists a nonnegative constant $\Delta > 0$ such that $\|y\left(t, t_0, x_0\right)\| \leqslant \alpha\Delta$. This implies that, the zero solution of the perturbed system is uniformly Lipschitz stable.

Remark that, for the linear system of the form $\dot{x} = A(t)x$, where $A(t)$ is continuous and bounded matrix, the perturbed system $\dot{x} = A(t)x + g(t)$ is uniformly Lipschitz stable if

$$\|g(t)\| \leqslant \xi(t) \text{ and } \int_0^\infty \xi(t)dt < \infty.$$

In the sequel, we give a result on asymptotic behavior and growth properties of the solutions of (2.2) under some restrictives conditions on the perturbation term based on the following well known comparison Lemma.

Lemma 2.8. *Consider a scalar differential equation:*

$$\dot{u}(t) = \Im(t,u), \quad t \geqslant 0,$$

where $\Im(t,u)$ *is a continuous function in* (t,u). *Let* $u(t)$ *be the maximal solution of this differential equation with* $u(t_0) = u_0$. *If a continuous function* $v(t)$ *with* $v(t_0) = u_0$ *satisfies*

$$v'(t) \leqslant \Im(t, u(t)), \quad \forall t \geqslant t_0 \geqslant 0,$$

then

$$v(t) \leqslant v(t_0) + \int_{t_0}^{t} \Im(s, u(s))ds, \quad \forall t \geqslant t_0 \geqslant 0.$$

This lemma can provide an estimation on $V(t,x(t))$ from some bounds on $D'V(t,x)$. Let $x(t) = x(t,t_0,x_0)$ be a solution of (2.1) existing for $t \geqslant t_0 \geqslant 0$. Let $V(t,x)$ is continuous in t and Lipschitzian in x (uniformly in t) which satisfies the inequality:

$$D'V(t,x) \leqslant \Im(t, V(t,x))$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then, for

$$V(t_0,x_0) \leq u_0,$$

we have

$$V(t,x(t)) \leqslant u(t), \quad for \quad t \geqslant t_0 \geqslant 0.$$

We will consider more general case when we do not know that $g(t) \neq 0$ for a certain $t \geq 0$. The origin may not be an equilibrium point of the perturbed system (2.2). We can non longer study the stability of the origin as an equilibrium point, nor should we expect the solution of the perturbed system to approach the origin as t goes to infinity. The best we can hope that for a small perturbation term the solution approach to the a small set which contains the origin. We first give the following result which gives an estimation on the solutions of perturbed system when we suppose that the nominal system is globally uniformly stable in variation. Suppose that the nominal system (2.1) has a uniformly exponentially stable in variation equilibrium point at the origin with $V(t,x)$ as a Lyapunov function candidate. Such Lyapunov function should verify the following assumptions:

i) $V(t,x)$ is defined and continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ such that $V(t,0) = 0$, $\forall t \geqslant 0$,

ii) $\|x\| \leqslant V(t,x) \leqslant \kappa\|x\|$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, $\kappa > 0$,

iii) $|V(t,x) - V(t,y)| \leqslant \kappa\|x-y\|$ for all $(t,x),(t,y) \in \mathbb{R}^+ \times \mathbb{R}^n$, $\kappa > 0$.

In [4, 5, 9] and [10], converse theorems are established.

Theorem 2.9. *Suppose that the perturbation term in* (2.2) *satisfies* (2.11). *Then,*

a) *If the trivial solution of* (2.1) *be globally uniformly stable in variation, then all solutions are bounded for all* $t \geq 0$.

b) *If the trivial solution of* (2.1) *be globally uniformly stable in variation and*

$$D_f^+ V(t,x) \leq -\zeta V(t,x), \ \zeta > 0,$$

it means that (2.1) *is globally uniformly exponentially stable in variation, then all solutions are bounded and converge to a small ball centered at the origin.*

Proof. a) Since the trivial solution of (2.1) is globally uniformly stable in variation, then there exists a function $V(t,x)$ that verify *i*), *ii*), *iii*) and the inequality $D_f^+ V(t,x) \leq 0$. Now, we consider the upper right-hand derivative of $V(t,x)$ with respect the perturbed system (2.2), we have

$$D_{(2.2)}^+ V(t,x) \leqslant D_f^+ V(t,x) + \kappa\|g(t)\|.$$

With the help of properties of $V(t,x)$, the above estimation implies that:

$$D^+_{(2.2)}V(t,x) \leqslant \xi(t).$$

Since the function $\xi(.)$ is an integrable function, then $x(t)$ is bounded. Thus, there exist some positive constants α and Δ such that, the solution $x(t,t_0,x_0)$ of (2.2) satisfies:

$$\|x(t,t_0,x_0)\| \leqslant \alpha\|x_0\| + \Delta, \quad \forall\, t \geqslant t_0 \geqslant 0,\ x_0 \in \mathbb{R}^n.$$

b) The condition imposed on the derivative of the Lyapunov function implies that the trivial solution of (2.1) is globally exponentially stable in variation. Let consider the upper right-hand derivative of $V(t,x)$ with respect the perturbed system (2.2), then

$$D^+_{(2.2)}V(t,x) \leqslant D^+_{(2.1)}V(t,x) + K\|g(t)\|.$$

With the help of properties of $V(t,x)$, this implies that

$$D^+_{(2.2)}V(t,x) \leqslant -\zeta V(t,x) + \kappa\|g(t)\|,$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $\zeta > 0$.
Thus,

$$D^+_{(2.2)}V(t,x) \leqslant -\zeta V(t,x) + \kappa\xi(t).$$

We will apply the comparison Lemma (2.8) where

$$g(t,u) = -\zeta u + \kappa\xi(t).$$

Let $x(t) = x(t,t_0,x_0)$ be a solution of (2.2) such that $V(t_0,x_0) \leqslant u_0$, $u_0 \geqslant 0$. Let consider the differential equation:

$$\dot{u} = -\zeta u + \kappa\xi(t), \quad u(t_0) = u_0,$$

then

$$u(t) \leq u(0)e^{-\zeta t} + \kappa e^{-\zeta t}\int_{t_0}^{t} e^{\zeta s}\xi(s)ds.$$

Note that, the fact that the function $\xi(t)$ is integrable, then one has the integral I defined by $I = e^{-\zeta t}\int_{t_0}^{t} ke^{\zeta s}\xi(t)ds$ is bounded. Therefore there exit, $\tilde{\gamma} > 0$, $\tilde{\eta} > 0$ such that

$$\|x(t)\| \leq k\exp -\tilde{\gamma}(t-t_0)\|x(t_0)\| + \tilde{\eta}.$$

The last expression implies that, all the solutions are bounded and the ball $B_{\tilde{\eta}}$ is globally uniformly exponentially stable with respect the system (2.2).

□

Example 2. For the scalar equation:

$$\dot{x}(t) = f(t,x(t)) + g(t),$$

with $f(t,x) = -2x$ and $g(t) = e^{-t}$, $t \geq 0$. We have, for initial condition (t_0,x_0),

$$x(t,t_0,x_0) = x_0 e^{-2(t-t_0)} + e^{-t} - e^{t_0}.$$

Therefore for $t \geq t_0 \geq 0$, we have in a first consequence:

a)

$$\|x(t,t_0,x_0)\| \leq \|x_0\| + e^{-t} + e^{-t_0}.$$

In a second consequence:

b)

$$\|x(t, t_0, x_0)\| \leq \|x_0\|e^{-2(t-t_0)} + 2.$$

It follows that, the ball B_2 is globally uniformly exponentially stable with respect the perturbed system.

Conclusion In this paper, some new sufficient conditions for the boundedness and convergence of solutions of a class of time-varying differential equations are studied. The notion of stability in variation is introduced for the nominal system. It is shown that the solutions of the perturbed system are exponential stable with respect a small ball by using Lyapunov functions that are not necessarily differentiable. Some axamples are given showing the validity of the main result.

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Manuscript received September 30 2023
revised October 27 2023

M. A. Hammami
University of Sfax, Faculty of Sciences of Sfax, Department of Mathematics, Tunisia
E-mail address: MohamedAli.Hammami@fss.rnu.tn

R. Hamlili
University of Sfax, Faculty of Sciences of Sfax, Department of Mathematics, Tunisia
E-mail address: hamlilirabie13@gmail.com