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ON THE ASYMPTOTIC STABILITY OF CAPUTO FRACTIONAL DELAY SINGULAR SYSTEMS

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Abstract. In this paper, a Caputo fractional-order singular system is considered. An asymptotic stability theorem, which has sufficient conditions, is proved in relation to this system. Razumikhin method is used in the proof. Two numerical examples are given as applications of the given result. Our result is new and provides new contributions to the topic of the paper.

1. Introduction

The qualitative topic of various fractional singular and non-singular differential systems have been extensively studied in recent years and numerous interesting qualitative results have been obtained in the relevant literature (see, [1–20]).

Now, let us summarize some of the recent studies on delay systems.

Stability of nonlinear system of fractional-order volterra delay integro differential equations with Caputo fractional derivative are discussed in [2] by Graef et al. In [2], some sufficient conditions for the stability are obtained using Razumikhin method. In [7], Phat and Hien proved new exponential stability conditions for non-autonomous linear-delay systems using Razumikhin's stability theorem. In [8], Phat et al. proved the guaranteed cost problem of fractional order delay systems subject to nonlinear perturbations and parametric time-varying uncertainties with help of the Razumikhin method. Yige and Meirong [15] proved some new results for the stability of fractional order linear time delay systems using of the Razumikhin method. They gave two examples to support these results.

Motivated from the papers [8, 16] and those in the references of this paper, we consider the following linear Caputo fractional singular system with constant delay:

$$ND_t^q x(t) = Ex(t) + Mx(t-h) \qquad (1.1)$$
$$x(t) = \varphi(t), t \in [-h, 0],$$

where $x(t) \in R^n$ is the state vector function of the given system, $q \in (0,1), E, N, M \in R^{n\times n}$ are constant system matrices with suitable dimensions,the matrix $N \in R^{n\times n}$ is singular and $rankN = r \leq n, n \geq 1, h > 0$ is constant delay.

We start our study by converting the given singular system (1.1) to a neutral system with the help of useful definitions and lemmas. For this, let us give the following lemmas and definitions.

Definition 1.1 ([5]). The pair (N, E) or system (1.1) is said to be regular if $det(\lambda N - E) \neq 0$ and impulse free if $deg(det(\lambda N - E)) = rank(N), \lambda = s^q$.

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Lemma 1.2 ([1])**.** *If the pair (N,E) is regular and impulse-free, then there exist two non-singular matrices* $S, F \in R^{n\times n}$ *such that*

$$SNF = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, SEF = \begin{bmatrix} E_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

Definition 1.3 ([8])**.** The Caputo fractional-order derivative of order $q > 0$ for a function $f(t)$ is defined by

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^n(s)}{(t-s)^{q+1-n}} ds, t \geq 0, n-1 < q \leq n,$$

where n is a positive number. In particular, when $q \in (0,1)$, we get

$$D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f^{'}(s)}{(t-s)^q} ds, t \geq 0.$$

Lemma 1.4 ([8])**.** *Suppose* $x(t) \in R^n$ *be a differentiable function. Then the following inequality is satisfied:*

$$D_t^q(x^T(t) \sum x(t)) \leq 2x^T(t) \sum D_t^q x(t), \forall q \in (0,1), \forall t \geq 0,$$

where $\sum \in R^{n\times n}, \sum = \sum^T \geq 0$, *is a constant matrix.*

Lemma 1.5 (Fractional Razumikhin Theorem [8])**.** *Let* $u, v, w : R^+ \to R^+$ *are continuous and non-decreasing, and* $u(0) = v(0) = w(0) = 0, v(.)$ *is strictly increasing. If there exists a continuous function* $V(.) : R^+ \times R^n \to R^+$ *such that*

i) $u(||x||) \leq V(t, x(t)) \leq v(||x||), t \geq 0, x \in R^n$ *and*
ii) $D_t^q V(t, x(t)) \leq -w(||x||)$

provided that

$$V(t+s, x(t+s)) < kV(t, x(t)), k > 1, \forall s \in [-h, 0], t \geq 0,$$

then the zero solution of fractional order system $D_t^q x(t) = f(t, x(t))$ *is asymptotically stable.*

Now, let us convert the system (1.1) to a neutral system. In the light of Lemma 1.2, there exist two regular matrices $S, F \in R^{n\times n}$ such that

$$SNF = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \overline{N}, SEF = \begin{bmatrix} E_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} = \overline{E},$$

$$SMF = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = \overline{M}, F^{-1}x(t) = \xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}.$$

Then, we can write the system (1.1) as:

$$\overline{N} D_t^q \xi(t) = \overline{E}\xi(t) + \overline{M}\xi(t-h).$$

This system can be decomposed to the following system:

$$D_t^q \xi_1(t) = E_1\xi_1(t) + M_1\xi_1(t-h) + M_2\xi_2(t-h), \tag{1.2}$$

$$0 = \xi_2(t) + M_3\xi_1(t-h) + M_4\xi_2(t-h). \tag{1.3}$$

If we take the fractional derivative of the equation (1.3), then we obtain

$$0 = D_t^q \xi_2(t) + M_3 D_t^q \xi_1(t-h) + M_4 D_t^q \xi_2(t-h). \tag{1.4}$$

Now, combining equations (1.3) and (1.4), we obtain

$$\begin{aligned} D_t^q\xi_2(t) = &-\xi_2(t) - M_3\xi_1(t-h) - M_4\xi_2(t-h) \\ &- M_3D_t^q\xi_1(t-h) - M_4D_t^q\xi_2(t-h). \end{aligned} \tag{1.5}$$

In light of equations (1.2) and (1.5), we have

$$\begin{aligned} \begin{bmatrix} D_t^q\xi_1(t) \\ D_t^q\xi_2(t) \end{bmatrix} = &\begin{bmatrix} E_1\xi_1(t) + M_1\xi_1(t-h) + M_2\xi_2(t-h) \\ -\xi_2(t) - M_3\xi_1(t-h) - M_4\xi_2(t-h) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ -M_3D_t^q\xi_1(t-h) - M_4D_t^q\xi_2(t-h) \end{bmatrix}, \end{aligned}$$

which is equivalent to the fractional-order delay neutral system given by

$$\begin{aligned} D_t^q\xi(t) - \widehat{A}D_t^q\xi(t-h) &= \widehat{E}\xi(t) + \widehat{M}\xi(t-h), \\ \xi(t) &= \varphi(t), t \in [-h, 0], \end{aligned} \tag{1.6}$$

where

$$\widehat{E} = \begin{bmatrix} E_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix}, \widehat{M} = \begin{bmatrix} M_1 & M_2 \\ -M_3 & -M_4 \end{bmatrix}, \widehat{A} = \begin{bmatrix} 0 & 0 \\ -M_3 & -M_4 \end{bmatrix}. \tag{1.7}$$

Clearly, the system (1.1) and the system (1.6) are not equivalent, but the stability property for both systems remains the same. That is, the stability of the system (1.6) guarantees the stability of system (1.1), and vice versa.

2. Main results and numerical applications

The following assumptions apply throughout this study.

A. Assumptions

- (A1) Let the pair (N, E) is impulse-free and regular and $||\widehat{A}|| < 1$.
- (A2) For given $\epsilon_i > 0, (i = 1, 2, ..., 7)$, and $Z = Z^T > 0$ such that the following matrix inequalities are satisfied:

$$\Theta = \begin{bmatrix} \Theta_{11} & Z\widehat{E}^T & I & I \\ * & -\epsilon_3 I & 0 & 0 \\ * & * & -\epsilon_2 I & 0 \\ * & * & * & -\epsilon_1 I \end{bmatrix} < 0, \tag{2.1}$$

$$\Phi_3 = -2Z + \epsilon_3 I + \epsilon_4^{-1} I + \epsilon_5^{-1} I + \epsilon_7 Z < 0, \tag{2.2}$$

$$(\epsilon_2 \widehat{M}^T\widehat{M} + \epsilon_4 \widehat{M}^T\widehat{M})Z \leq \epsilon_6 I, \tag{2.3}$$

$$(\epsilon_1 \widehat{A}^T\widehat{A} + \epsilon_5 \widehat{A}^T\widehat{A})Z \leq \epsilon_7 I, \tag{2.4}$$

where

$$\Theta_{11} = \widehat{E}Z + Z\widehat{E}^T + \epsilon_6 Z.$$

Theorem 2.1. *If conditions (A1) and (A2) are satisfied, then system (1.6) is asymptotically stable and the system (1.1) is asymptotically admissible.*

Proof of Theorem 2.1.

$$V(\xi(t)) = \xi^T(t)Z^{-1}\xi(t).$$

It can be easily shown that the following inequality is satisfied:

$$\lambda_{min}(Z^{-1})||\xi(t)||^2 \leq V(\xi(t)) \leq \lambda_{max}(Z^{-1})||\xi(t)||^2.$$

Thus, the condition (i) given in Lemma 1.5 is satisfied. In the light of Lemma 1.4, we have

$$\begin{aligned} D_t^q V(\xi(t)) \leq& 2\xi^T(t)Z^{-1}D_t^q\xi(t) \\ =& \xi^T(t)[Z^{-1}\widehat{E} + \widehat{E}^T Z^{-1}]\xi(t) \\ &+ 2\xi^T(t)Z^{-1}\widehat{M}\xi(t-h) + 2\xi^T(t)Z^{-1}\widehat{A}D_t^q\xi(t-h). \end{aligned} \tag{2.5}$$

By applying the Cauchy matrix inequality, we can write the following inequalities

$$\begin{aligned} 2\xi^T(t)Z^{-1}\widehat{A}D_t^q\xi(t-h) \leq& \epsilon_1(D_t^q\xi(t-h))^T\widehat{A}^T\widehat{A}(D_t^q\xi(t-h)) \\ &+ \epsilon_1^{-1}\xi^T(t)Z^{-1}Z^{-1}\xi(t), \end{aligned} \tag{2.6}$$

$$\begin{aligned} 2\xi^T(t)Z^{-1}\widehat{M}\xi(t-h) \leq& \epsilon_2\xi^T(t-h)\widehat{M}^T\widehat{M}\xi(t-h) \\ &+ \epsilon_2^{-1}\xi^T(t)Z^{-1}Z^{-1}\xi(t). \end{aligned} \tag{2.7}$$

Furthermore, by applying the following identities

$$0 = -D_t^q\xi(t) + \widehat{E}\xi(t) + \widehat{M}\xi(t-h) + \widehat{A}D_t^q\xi(t-h),$$

we get

$$\begin{aligned} &2(D_t^q\xi(t))^T Z^{-1}[-D_t^q\xi(t) + \widehat{E}\xi(t) + \widehat{M}\xi(t-h) + \widehat{A}D_t^q\xi(t-h)] \\ &= -2(D_t^q\xi(t))^T Z^{-1}(D_t^q\xi(t)) + 2(D_t^q\xi(t))^T Z^{-1}\widehat{E}(D_t^q\xi(t)) \\ &\quad + 2(D_t^q\xi(t))^T Z^{-1}\widehat{M}\xi(t-h) + 2(D_t^q\xi(t))^T Z^{-1}\widehat{A}(D_t^q\xi(t-h)) = 0. \end{aligned} \tag{2.8}$$

By applying the Cauchy matrix inequality again, we can obtain the following relationships:

$$\begin{aligned} 2(D_t^q\xi(t))^T Z^{-1}\widehat{E}\xi(t) \leq& \epsilon_3(D_t^q\xi(t))^T Z^{-1}Z^{-1}(D_t^q\xi(t)) \\ &+ \epsilon_3^{-1}\xi^T(t)\widehat{E}^T\widehat{E}\xi(t), \end{aligned} \tag{2.9}$$

$$\begin{aligned} 2(D_t^q\xi(t))^T Z^{-1}\widehat{M}\xi(t-h) \leq& \epsilon_4\xi^T(t-h)\widehat{M}^T\widehat{M}\xi(t-h) \\ &+ \epsilon_4^{-1}(D_t^q\xi(t))^T Z^{-1}Z^{-1}(D_t^q\xi(t)), \end{aligned} \tag{2.10}$$

$$\begin{aligned} 2(D_t^q\xi(t))^T Z^{-1}\widehat{A}(D_t^q\xi(t-h)) \leq& \epsilon_5(D_t^q\xi(t-h))^T\widehat{A}^T\widehat{A}(D_t^q\xi(t-h)) \\ &+ \epsilon_5^{-1}(D_t^q\xi(t))^T Z^{-1}Z^{-1}(D_t^q\xi(t)). \end{aligned} \tag{2.11}$$

Combining the relationships (2.5)-(2.11), we get

$$\begin{aligned} D_t^q V(\xi(t)) \leq& \xi^T(t)\Omega\xi(t) + (D_t^q\xi(t))^T\Phi(D_t^q\xi(t)) \\ &+ \xi^T(t-h)[\epsilon_2\widehat{M}^T\widehat{M} + \epsilon_4\widehat{M}^T\widehat{M}]\xi(t-h) \\ &+ (D_t^q\xi(t-h))^T[\epsilon_1\widehat{A}^T\widehat{A} + \epsilon_5\widehat{A}^T\widehat{A}](D_t^q\xi(t-h)), \end{aligned}$$

where

$$\begin{aligned}\Omega =& Z^{-1}\widehat{E} + \widehat{E}^T Z^{-1} + \epsilon_1^{-1} Z^{-1} Z^{-1} + \epsilon_2^{-1} Z^{-1} Z^{-1} + \epsilon_3^{-1} \widehat{E}^T \widehat{E},\\ \Phi =& -2Z^{-1} + \epsilon_3 Z^{-1} Z^{-1} + \epsilon_4^{-1} Z^{-1} Z^{-1} + \epsilon_5^{-1} Z^{-1} Z^{-1}.\end{aligned}$$

By applying conditions (2.3) and (2.4), we can obtain the following relationships:

$$\xi^T(t-h)[\epsilon_2 \widehat{M}^T \widehat{M} + \epsilon_4 \widehat{M}^T \widehat{M}]\xi(t-h) \le \epsilon_6 \xi^T(t-h) Z^{-1} \xi(t-h),$$

$$(D_t^q \xi(t-h))^T [\epsilon_1 \widehat{A}^T \widehat{A} + \epsilon_5 \widehat{A}^T \widehat{A}](D_t^q \xi(t-h)) \le \epsilon_7 (D_t^q \xi(t-h))^T Z^{-1} (D_t^q \xi(t-h)),$$

and thus

$$\begin{aligned}D_t^q V(\xi(t)) \le& \xi^T(t)\Omega\xi(t) + (D_t^q \xi(t))^T \Phi (D_t^q \xi(t))\\ &+ \xi^T(t-h)\epsilon_6 Z^{-1}\xi(t-h) + (D_t^q \xi(t-h))^T \epsilon_7 Z^{-1} (D_t^q \xi(t-h)).\end{aligned}$$

From Lemma 1.5, for any $\epsilon > 0$ and $p = \epsilon + 1 > 1$ and $\forall s \in [-h, 0], t \ge 0$, we have

$$\xi^T(t+s) Z^{-1} \xi(t+s) < p\xi^T(t) Z^{-1} \xi(t)$$

and

$$(D_t^q \xi(t+s))^T Z^{-1} (D_t^q \xi(t+s)) < p (D_t^q \xi(t))^T Z^{-1} (D_t^q \xi(t)).$$

Then, it is clear that

$$\begin{aligned}D_t^q V(\xi(t)) \le& \xi^T(t)\Omega\xi(t) + (D_t^q \xi(t))^T \Phi (D_t^q \xi(t))\\ &+ \epsilon_6 p \xi^T(t) Z^{-1} \xi(t) + \epsilon_7 p (D_t^q \xi(t))^T Z^{-1} (D_t^q \xi(t))\\ \le& \xi^T(t)\Omega_1\xi(t) + (D_t^q \xi(t))^T \Phi_1 (D_t^q \xi(t)),\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}\Omega_1 =& Z^{-1}\widehat{E} + \widehat{E}^T Z^{-1} + \epsilon_1^{-1} Z^{-1} Z^{-1} + \epsilon_2^{-1} Z^{-1} Z^{-1} + \epsilon_3^{-1} \widehat{E}^T \widehat{E} + \epsilon_6(\epsilon+1) Z^{-1},\\ \Phi_1 =& -2Z^{-1} + \epsilon_3 Z^{-1} Z^{-1} + \epsilon_4^{-1} Z^{-1} Z^{-1} + \epsilon_5^{-1} Z^{-1} Z^{-1} + \epsilon_7(\epsilon+1) Z^{-1}.\end{aligned}$$

Since ϵ is a positive definite arbitrary parameter and the both sides $D_t^q V(\xi(t))$ does not depend on ϵ, then taking $\epsilon \to 0$, we can write the inequality (2.12) as the following form

$$D_t^q V(\xi(t)) \le \xi^T(t)\Omega_2\xi(t) + (D_t^q \xi(t))^T \Phi_2 (D_t^q \xi(t)), \tag{2.13}$$

where

$$\begin{aligned}\Omega_2 =& Z^{-1}\widehat{E} + \widehat{E}^T Z^{-1} + \epsilon_1^{-1} Z^{-1} Z^{-1} + \epsilon_2^{-1} Z^{-1} Z^{-1} + \epsilon_3^{-1} \widehat{E}^T \widehat{E} + \epsilon_6 Z^{-1},\\ \Phi_2 =& -2Z^{-1} + \epsilon_3 Z^{-1} Z^{-1} + \epsilon_4^{-1} Z^{-1} Z^{-1} + \epsilon_5^{-1} Z^{-1} Z^{-1} + \epsilon_7 Z^{-1}.\end{aligned}$$

Now, pre- and post- multiplying Ω_2 and Φ_2 by Z, we can write the inequality (2.13) as:

$$D_t^q V(\xi(t)) \le \xi^T(t)\Omega_3\xi(t) + (D_t^q \xi(t))^T \Phi_3 (D_t^q \xi(t)), \tag{2.14}$$

where

$$\begin{aligned}\Omega_3 =& \widehat{E}Z + Z\widehat{E}^T + \epsilon_1^{-1} I + \epsilon_2^{-1} I + \epsilon_3^{-1} Z\widehat{E}^T \widehat{E} Z + \epsilon_6 Z,\\ \Phi_3 =& -2Z + \epsilon_3 I + \epsilon_4^{-1} I + \epsilon_5^{-1} I + \epsilon_7 Z.\end{aligned}$$

Here, Let us not forget that $\Omega_2 < 0$ is equivalent to $\Omega_3 < 0$. Similarly, $\Phi_2 < 0$ is equivalent to $\Phi_3 < 0$. By applying the Schur complement lemma (see [10]), we get $\Omega_3 < 0$ is equivalent to Θ, the linear matrix inequality (2.1). Hence, in the light of conditions (2.1), (2.14) and (2.2), we obtain

$$\exists\lambda > 0, D_t^q V(\xi(t)) \leq -\lambda||\xi(t)||^2, t \geq 0.$$

Therefore, the condition (ii) of Lemma 1.5 holds. Thus, system (1.6) is asymptotically stable. Then system (1.1) is asymptotically admissible as it is regular, impulse free and asymptotically stable. This results completes the proof of Theorem 2.1. □

Example 2.2. Consider the following linear Caputo fractional singular system with constant delay:

$$ND_t^q x(t) = Ex(t) + Mx(t-2),$$

where

$$0 < q < 1, x(t) = (x_1(t), x_2(t)) \in R^2,$$

$$N = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 0 & -1 \\ -4 & 0.5 \end{bmatrix}, M = \begin{bmatrix} 0.24 & 0.02 \\ 0.08 & -0.01 \end{bmatrix}.$$

In the light of Definition 1.1, it can easily be shown that the pair (N, E) or the given system (1.1) is regular and impulse-free. Thus, there exist two invertible matrices

$$S = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, F = \begin{bmatrix} 0.125 & 0 \\ 1 & 0.5 \end{bmatrix}$$

such that

$$SNF = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, SEF = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, SMF = \begin{bmatrix} 0.05 & 0 \\ 0 & -0.02 \end{bmatrix}.$$

Therefore, in the light of Lemma 1.2, we obtain

$$\widehat{E} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \widehat{M} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.02 \end{bmatrix}, \widehat{A} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix},$$

and $Z = diag(14.5, 14.6), \epsilon_1 = 0.4, \epsilon_2 = 0.2, \epsilon_3 = 15, \epsilon_4 = 0.1, \epsilon_5 = 0.8, \epsilon_6 = 0.4, \epsilon_7 = 0.3$. By applying Theorem 2.1, we find that the following LMIs hold for the given particular case:

$$\Phi_3 = \begin{bmatrix} -8.75 & 0 \\ 0 & -8.92 \end{bmatrix} < 0,$$

$$(\epsilon_2 \widehat{M}^T \widehat{M} + \epsilon_4 \widehat{M}^T \widehat{M})Z \leq \epsilon_6 I \Rightarrow \begin{bmatrix} 0.0109 & 0 \\ 0 & 0.0018 \end{bmatrix} \leq \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix},$$

$$(\epsilon_1 \widehat{A}^T \widehat{A} + \epsilon_5 \widehat{A}^T \widehat{A})Z \leq \epsilon_7 I \Rightarrow \begin{bmatrix} 0.0070 & 0 \\ 0 & 0.0158 \end{bmatrix} \leq \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} -23.2 & 0 & -14.5 & 0 & 1 & 0 & 1 & 0 \\ 0 & -23.36 & 0 & -14.6 & 0 & 1 & 0 & 1 \\ -14.5 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14.6 & 0 & -15 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -0.4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -0.2 \end{bmatrix} < 0.$$

In view of Theorem 2.1, the system given in Example 2.2 is asymptotically stable. At the end, the system given in the Example 2.2, is asymptotically admissible as it is regular, impulse free and asymptotically stable.

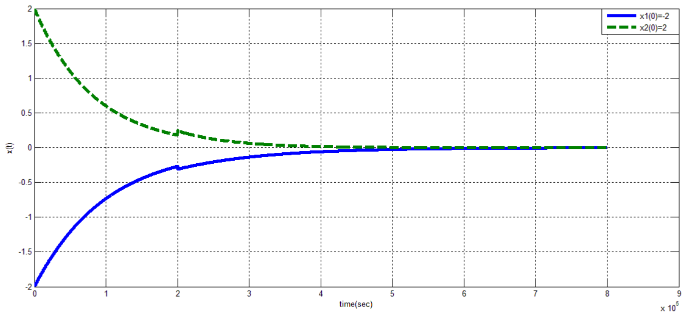

FIGURE 1. Trajectories of the system given by Example 2.2

Example 2.3. Now, let us consider the following example, which demonstrates the practical applicability of our theoretical result:

$$ND_t^q x(t) = Ex(t) + Mx(t-4),$$

where

$$0 < q < 1, x(t) = (x_1(t), x_2(t), x_3(t)) \in R^3,$$

$$N = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E = \begin{bmatrix} -2.2 & 0 & 0 \\ 0 & -1.6 & 0 \\ 0 & 0 & 2 \end{bmatrix}, M = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & -0.02 \end{bmatrix}.$$

In the light of Definition 1.1, it can easily be shown that the pair (N, E) or the given system (1.1) is regular and impulse-free. Thus, there exist two invertible matrices

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, F = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.125 \end{bmatrix}$$

such that

$$SNF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, SEF = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}, SMF = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & -0.01 \end{bmatrix}.$$

Therefore, in the light of Lemma 1.2, we can obtain

$$\widehat{E} = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \widehat{M} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}, \widehat{A} = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.04 \end{bmatrix},$$

and $Z = diag(14.6, 14.7, 14.8), \epsilon_1 = 0.4, \epsilon_2 = 0.2, \epsilon_3 = 15, \epsilon_4 = 0.1, \epsilon_5 = 0.8, \epsilon_6 = 0.4, \epsilon_7 = 0.3$. By applying Theorem 2.1, we found that the linear matrix inequality conditions (2.1)-(2.4) are satisfied as

$$\Phi_3 = \begin{bmatrix} -8.92 & 0 & 0 \\ 0 & -9.09 & 0 \\ 0 & 0 & -9.26 \end{bmatrix} < 0,$$

$$(\epsilon_2 \widehat{M}^T \widehat{M} + \epsilon_4 \widehat{M}^T \widehat{M}) Z \leq \epsilon_6 I \Rightarrow \begin{bmatrix} 0.0110 & 0 & 0 \\ 0 & 0.0018 & 0 \\ 0 & 0 & 0.0004 \end{bmatrix} \leq \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix},$$

$$(\epsilon_1 \widehat{A}^T \widehat{A} + \epsilon_5 \widehat{A}^T \widehat{A}) Z \leq \epsilon_7 I \Rightarrow \begin{bmatrix} 0.0070 & 0 & 0 \\ 0 & 0.0159 & 0 \\ 0 & 0 & 0.0284 \end{bmatrix} \leq \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} \Theta_{11} & 0 & 0 & \Theta_{14} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \Theta_{22} & 0 & 0 & \Theta_{25} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \Theta_{33} & 0 & 0 & \Theta_{36} & 0 & 0 & 1 & 0 & 0 & 1 \\ \Theta_{41} & 0 & 0 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Theta_{52} & 0 & 0 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Theta_{63} & 0 & 0 & -15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \end{bmatrix}$$

$$< 0,$$

where

$$\Theta_{11} = -26.28, \Theta_{14} = \Theta_{41} = -16.06, \Theta_{22} = -17.64, \Theta_{25} = \Theta_{52} = -11.76,$$
$$\Theta_{33} = -23.68, \Theta_{36} = \Theta_{63} = -14.8.$$

In view of Theorem 2.1, the system given in Example 2.3 is asymptotically stable. At the end, the system given in the Example 2.3, is asymptotically admissible since it is regular, impulse free and asymptotically stable.

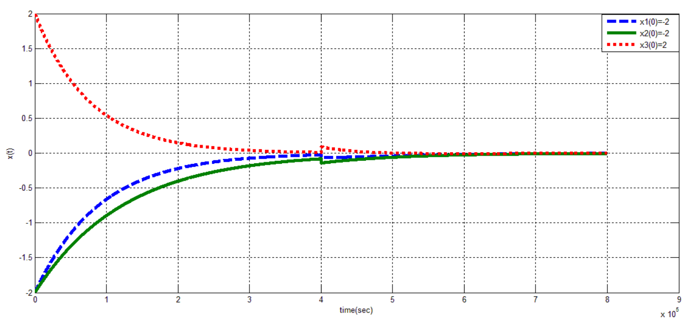

FIGURE 2. Trajectories of the system given by Example 2.3

3. Conclusion

In this paper, we have investigated the problem of asymptotic admissibility of a Caputo fractional-order singular sistem with constant delay using Razumikhin approach. We have transformed the considered system to a non-singular delayed fractional-order neutral system. In light of fractional Razumikhin stability theorem and matrix inequality technique, we have proved the new sufficient criteria for asymptotic stability of transformed singular system. Thus, we have proved that the considered singular system is asymptotically admissible since it is regular, impulse-free and asymptotically stable. Finally, we have presented some numerical examples to show applications of our result.

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