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ON A MULTI-TERM FUNCTIONAL DIFFERENTIAL EQUATION WITH PARAMETERS SUBJECT TO A QUADRATIC INTEGRO-DIFFERENTIAL CONDITION

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Abstract. Here, we study a problem of a multi-term functional-differential equation with parameters subject to a quadratic integro-differential condition. The existence and the uniqueness of the solution will be proved. The Hyers-Ulam stability will be studied. The continuous dependence of the unique solution on some parameters will be studied. Special cases and examples will be given.

1. Introduction

Functional and differential equations serve as the essential building blocks for modeling complex real-world processes in a variety of disciplines, including physics, engineering, biology, and economics [4, 14, 15]. Particularly, the complex mathematical models that are crucial in many different scientific domains are produced by multi-term functional-differential equations, whose variables depend not only on their current state but also on their previous and future values [10, 16, 18, 20]. As a key tenet of stability theory, the concepts of Hyers-Ulam stability and continuous dependency emphasize how modest changes in the mathematical problem and its parameters cause only a slight change in the solution of the problem [5, 12, 13, 17, 21]. These equations get even more complex when the nonlocal integral condition is present, complicating their analysis in a way that is known as nonlocal problems which has been discussed by numerous authors (see [1, 2, 8, 9, 19, 22]).
In light of the aforementioned results, we investigate the nonlocal solvability of the multi-term functional-differential equation with parameters.

$$\frac{dx(t)}{dt} = f\left(t, \lambda_1 \frac{d}{dt}x(\gamma_1 t), \lambda_2 \frac{d}{dt}x(\gamma_2 t), ..., \lambda_n \frac{d}{dt}x(\gamma_n t)\right),\ t \in (0, T], \tag{1.1}$$

with the nonlocal quadratic integral condition

$$x(0) = x_0 + \int_0^T g\left(s, x(s)\frac{dx(s)}{ds}\right)ds, \tag{1.2}$$

where $\lambda_i > 0$ and $\gamma_i \in (0, 1]$ are parameters, $i = 1, 2, ..., k$.
Our aim in this paper is to study the existence of solution of the problem (1.1)-(1.2) under suitable conditions, then we study the uniqueness of the solution. Additionally, we study the Hyers-Ulam stability of the problem. Moreover, we investigate

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the continuous dependence of the unique solution on the initial data x_0, the functions g and f, and parameters λ_i and γ_i. Finally, we introduce some examples and special cases to illustrate our results.

2. Existence of Solutions

In this section, we prove the existence of at least one continuous solution of (1.1)-(1.2), for this aim, we assume that:

(i) $f : [0,T] \times R^n \to R$ is continuous and satisfy the lipschitz condition in $x \in R^n$ such that

$$|f(t, x_1, x_2, ..., x_n) - f(t, y_1, y_2, ..., y_n)| \leq \sum_{i=1}^{n} k_i |x_i - y_i| \text{ with constants } k_i > 0.$$

(ii) $\sup_{t\in[0,T]} |f(t, 0, 0, ..., 0)| \leq M$.

(iii) $\sum_{i=1}^{n} k_i \lambda_i < 1$.

(iv) $g : [0,T] \times R \to R$ satisfies Carathéodory condition, i.e. it is measurable in $t \in [0,T]$ for all $x \in R$ and continuous in $x \in R$ for almost all $t \in [0,T]$.

(v) There exist a function $a \in L^1[0,T]$ and constant $b > 0$ such that

$$|g(t,x)| \leq |a(t)| + b|x|.$$

(vi) $bTr_1 < 1$.

Lemma 2.1. *Let x be a solution of (1.1)-(1.2), then it can be given by the integral equation*

$$x(t) = x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^t y(s)ds, \; t \in [0,T], \tag{2.1}$$

where $y(t)$ is the solution of the functional equation

$$y(t) = f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), \lambda_2\gamma_2 y(\gamma_2 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big), \; t \in [0,T]. \tag{2.2}$$

Proof. Let x be a solution of (1.1)-(1.2) and $\frac{dx(t)}{dt} = y \in C[0,T]$, then

$$x(t) = x(0) + \int_0^t y(s)ds,$$

using (1.2), we obtain (2.1)

$$x(t) = x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^t y(s)ds \in C^1[0,T],$$

and

$$x(\gamma_i t) = x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^{\gamma_i t} y(s)ds,$$

hence

$$\frac{d}{dt}x(\gamma_i t) = \gamma_i y(\gamma_i t). \tag{2.3}$$

Using (2.3) in (1.1), we obtain (2.2)

$$y(t) = f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), \lambda_2\gamma_2 y(\gamma_2 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big), \; t \in [0,T].$$

Also, we can get back to (1.1)-(1.2) by differentiating (2.1) and using (2.2) and (2.3) as follows

$$\begin{aligned}\frac{dx(t)}{dt} &= y(t),\ t\in(0,T]\\ &= f\big(t,\lambda_1\gamma_1 y(\gamma_1 t),\lambda_2\gamma_2 y(\gamma_2 t),...,\lambda_n\gamma_n y(\gamma_n t)\big)\\ &= f\bigg(t,\lambda_1\frac{d}{dt}x(\gamma_1 t),\lambda_2\frac{d}{dt}x(\gamma_2 t),...,\lambda_n\frac{d}{dt}x(\gamma_n t)\bigg),\end{aligned}$$

and the nonlocal integral condition holds when substituting $t=0$ and $y=\frac{dx(t)}{dt}$ in (2.1). □

Theorem 2.2. *Let the assumptions $(i)-(iii)$ be satisfied, then (2.2) has a unique solution $y\in C[0,T]$.*

Proof. Define the set Q_{r_1} and the operator F_1 associated with (2.2) by

$$Q_{r_1}:=\big\{y\in R:||y||_C\le r_1\big\}\subset C[0,T],\ \text{where } r_1=\frac{M}{1-\sum_{i=1}^n k_i\lambda_i},$$

and

$$F_1y(t)=f\big(t,\lambda_1\gamma_1 y(\gamma_1 t),\lambda_2\gamma_2 y(\gamma_2 t),...,\lambda_n\gamma_n y(\gamma_n t)\big),\ t\in[0,T].$$

Using assumption (i), we obtain

$$|f(t,y_1,y_2,...,y_n)|\le|f(t,0,0,...,0)|+\sum_{i=1}^n k_i|y_i|.$$

Let $y\in Q_{r_1}$, then for $t\in[0,T]$, we get

$$\begin{aligned}|F_1y(t)| &= \big|f\big(t,\lambda_1\gamma_1 y(\gamma_1 t),\lambda_2\gamma_2 y(\gamma_2 t),...,\lambda_n\gamma_n y(\gamma_n t)\big)\big|\\ &\le |f(t,0,0,...,0)|+\sum_{i=1}^n k_i\big|\lambda_i\gamma_i y(\gamma_i t)\big|\\ &\le sup_{t\in[0,T]}|f(t,0,...,0)|+\sum_{i=1}^n k_i\lambda_i\sup_{\gamma_i t\in[0,T]}|y(\gamma_i t)|\\ &\le M+||y||_C\sum_{i=1}^n k_i\lambda_i.\end{aligned}$$

Then, we have

$$||F_1y||_C\le M+r_1\sum_{i=1}^n k_i\lambda_i=r_1.$$

This proves that $F_1:Q_{r_1}\to Q_{r_1}$. Now let $u,v\in Q_{r_1}$, then

$$\begin{aligned}|F_1u(t)-F_1v(t)| &= \big|f\big(t,\lambda_1\gamma_1 u(\gamma_1 t),\lambda_2\gamma_2 u(\gamma_2 t),...,\lambda_n\gamma_n u(\gamma_n t)\big)\\ &- f\big(t,\lambda_1\gamma_1 v(\gamma_1 t),\lambda_2\gamma_2 v(\gamma_2 t),...,\lambda_n\gamma_n v(\gamma_n t)\big)\big|\\ &\le \sum_{i=1}^n k_i\big|\lambda_i\gamma_i u(\gamma_i t)-\lambda_i\gamma_i v(\gamma_i t)\big|\end{aligned}$$

$$\leq \sum_{i=1}^{n} k_i\lambda_i|u(\gamma_i t) - v(\gamma_i t)|.$$

Thus

$$||F_1u - F_1v||_C \leq \left[\sum_{i=1}^{n} k_i\lambda_i\right]||u - v||_C.$$

Since $\sum_{i=1}^{n} k_i\lambda_i < 1$, then F_1 is a contraction. Now all conditions of Banach fixed point Theorem [11] are satisfied, then F_1 has a unique fixed point $y \in Q_{r_1}$, hence (2.2) has a unique solution $y \in C[0,T]$. □

Theorem 2.3. *Let the assumptions $(i) - (vi)$ be satisfied, then (2.1) has at least one solution $x \in C[0,T]$. Consequently, (1.1)-(1.2) has at least one solution $x \in C^1[0,T]$.*

Proof. Define the set Q_{r_2} and the operator F_2 associated with (2.1) by

$$Q_{r_2} := \big\{x \in R : ||x||_C \leq r_2\big\} \subset C[0,T], \text{ where } r_2 = \frac{|x_0| + ||a||_{L^1} + r_1T}{1 - bTr_1},$$

and

$$F_2x(t) = x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^t y(s)ds, \ t \in [0,T].$$

It clear that Q_{r_2} is nonempty, closed, bounded and convex subset of $C[0,T]$. Let $x \in Q_{r_2}$, then for $t \in [0,T]$, we get

$$\begin{aligned}
|F_2x(t)| &= \left|x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^t y(s)ds\right| \\
&\leq |x_0| + \int_0^T \big|g\big(s, x(s)y(s)\big)\big|ds + \int_0^t |y(s)|ds \\
&\leq |x_0| + \int_0^T \big[|a(s)| + b|x(s)y(s)|\big]ds + \int_0^t |y(s)|ds \\
&\leq |x_0| + \int_0^T |a(s)|ds + b\int_0^T |x(s)||y(s)|ds + \int_0^T |y(s)|ds \\
&\leq |x_0| + ||a||_{L^1} + bT||x||_C||y||_C + ||y||_CT.
\end{aligned}$$

Then, we have

$$||F_2x||_C \leq |x_0| + ||a||_{L^1} + bTr_1r_2 + r_1T = r_2.$$

This proves that $F_2 : Q_{r_2} \to Q_{r_2}$ and the class $\{F_2x(t)\}$ is uniformly bounded on Q_{r_2}. Let $x \in Q_{r_2}$ and $t_1, t_2 \in [0,T]$, where $t_2 > t_1$ and $|t_2 - t_1| \leq \delta$, thus

$$\begin{aligned}
|F_2x(t_2) - F_2x(t_1)| &= \bigg|x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^{t_2} y(s)ds \\
&- x_0 - \int_0^T g\big(s, x(s)y(s)\big)ds - \int_0^{t_1} y(s)ds\bigg| \\
&\leq \int_{t_1}^{t_2} |y(s)|ds \\
&\leq r_1|t_2 - t_1| \leq r_1\delta = \epsilon.
\end{aligned}$$

This shows that the class $\{F_2x(t)\}$ is equi-continuous on Q_{r_2}. Thus, by Arzela-Ascoli Theorem [3], $\{F_2x(t)\}$ is relatively compact, hence F_2 is compact operator. Let $\{x_n\} \subset Q_{r_2}$ $s.t.$ $x_n \to x$, then

$$F_2x_n(t) = x_0 + \int_0^T g\big(s, x_n(s)y(s)\big)ds + \int_0^t y(s)ds, \; n = 0, 1, 2, ... ,$$

and

$$\lim_{n\to\infty} F_2x_n(t) = x_0 + \lim_{n\to\infty} \int_0^T g\big(s, x_n(s)y(s)\big)ds + \int_0^t y(s)ds.$$

Using the Lebesgue dominated convergence Theorem [7] and assumption (iv), we have

$$\begin{aligned}
\lim_{n\to\infty} F_2x_n(t) &= x_0 + \int_0^T \lim_{n\to\infty} g\big(s, x_n(s)y(s)\big)ds + \int_0^t y(s)ds \\
&= x_0 + \int_0^T g\big(s, y(s) \lim_{n\to\infty} x_n(s)\big)ds + \int_0^t y(s)ds \\
&= x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^t y(s)ds \\
&= F_2x(t).
\end{aligned}$$

Thus, F_2 is continuous operator. Then by applying Schauder fixed point Theorem [6], F_2 has at least one fixed point $x \in Q_{r_2}$, hence (2.1) has at least one solution $x \in C[0, T]$. Consequently, by Lemma 2.1, it follows that (1.1)-(1.2) has at least one solution $x \in C^1[0, T]$. □

3. Uniqueness of solution

Here, we prove the existence of a unique solution of (1.1)-(1.2), for this aim, we assume that:

$(i)'$ $g : [0, T] \times R \to R$ is measurable in $t \in [0, T]$ and satisfy the lipschitz condition in $x \in R$ such that

$$|g(t, x) - g(t, y)| \leq b|x - y| \;\text{ with constant }\; b > 0.$$

$(ii)'$ $g(t, 0) \in L^1[0, T]$.

Theorem 3.1. *Let the assumptions* $(i) - (iii)$ *and* (vi) *of Theorem 2.3 and* $(i)'$-$(ii)'$ *be satisfied, then the solution of (1.1)-(1.2),* $x \in C^1[0, T]$*, is unique.*

Proof. Assumptions (iv)-(v) of Theorem 2.3 can be deduced from assumptions $(i)'$ and $(ii)'$ as follows, putting $y = 0$ in $(i)'$, we obtain

$$|g(t, x)| \leq |g(t, 0)| + b|x|,$$

where $a(t) = g(t, 0) \in L^1[0, T]$.
Hence, we deduce that all assumptions of Theorem 2.3 are satisfied and (2.1) has at least one solution $x \in C[0, T]$. Now let x_1, x_2 be two solutions of (2.1), then

$$|x_2(t) - x_1(t)| \;=\; \bigg| x_0 + \int_0^T g\big(s, x_2(s)y(s)\big)ds + \int_0^t y(s)ds$$

$$\begin{aligned} &- \quad x_0 - \int_0^T g\big(s, x_1(s)y(s)\big)ds - \int_0^t y(s)ds\Big| \\ &\leq \quad \int_0^T \big|g\big(s, x_2(s)y(s)\big) - g\big(s, x_1(s)y(s)\big)\big|ds \\ &\leq \quad b\int_0^T |x_2(s)y(s) - x_1(s)y(s)|ds \\ &= \quad b\int_0^T |y(s)||x_2(s) - x_1(s)|ds. \end{aligned}$$

Thus

$$||x_2 - x_1||_C \leq bTr_1||x_2 - x_1||_C.$$

Since $bTr_1 < 1$, hence $x_1 = x_2$ and the solution of (2.1), $x \in C[0, T]$, is unique. Consequently, the solution of (1.1)-(1.2), $x \in C^1[0, T]$, is unique. □

4. Hyers-Ulam stability

Here, we investigate the concept of Hyers-Ulam stability of the problem (1.1)-(1.2) associated with (2.1) and (2.2).

Definition 4.1. Let the solution of (1.1)-(1.2) be exists. The problem (1.1)-(1.2) is Hyers-Ulam stable, $if\ \forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ *such that* for any solution x_s of (1.1)-(1.2) satisfying

$$\left|\frac{dx_s(t)}{dt} - f\left(t, \lambda_1 \frac{d}{dt}x_s(\gamma_1 t), \lambda_2 \frac{d}{dt}x_s(\gamma_2 t), ..., \lambda_n \frac{d}{dt}x_s(\gamma_n t)\right)\right| \leq \delta,$$

then

$$||x - x_s||_C \leq \epsilon.$$

Theorem 4.2. *Let the assumptions of Theorem 3.1 be satisfied, then the problem (1.1)-(1.2) is Hyers-Ulam stable.*

Proof. Let $\left|\frac{dx_s(t)}{dt} - f\left(t, \lambda_1 \frac{d}{dt}x_s(\gamma_1 t), \lambda_2 \frac{d}{dt}x_s(\gamma_2 t), ..., \lambda_n \frac{d}{dt}x_s(\gamma_n t)\right)\right| \leq \delta$, then

$$-\delta \leq \frac{dx_s(t)}{dt} - f\left(t, \lambda_1\gamma_1 \frac{dx_s(\gamma_1 t)}{d(\gamma_1 t)}, \lambda_2\gamma_2 \frac{dx_s(\gamma_2 t)}{d(\gamma_2 t)}, ..., \lambda_n\gamma_n \frac{dx_s(\gamma_n t)}{d(\gamma_n t)}\right) \leq \delta,$$

then

$$-\delta \leq y_s(t) - f\big(t, \lambda_1\gamma_1 y_s(\gamma_1 t), \lambda_2\gamma_2 y_s(\gamma_2 t), ..., \lambda_n\gamma_n y_s(\gamma_n t)\big) \leq \delta.$$

Now

$$\begin{aligned} &|y(t) - y_s(t)| \\ &= \big|f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big) - y_s(t)\big| \\ &= \big|f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big) - y_s(t) \\ &\quad - f\big(t, \lambda_1\gamma_1 y_s(\gamma_1 t), ..., \lambda_n\gamma_n y_s(\gamma_n t)\big) + f\big(t, \lambda_1\gamma_1 y_s(\gamma_1 t), ..., \lambda_n\gamma_n y_s(\gamma_n t)\big| \\ &\leq \big|f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big) - f\big(t, \lambda_1\gamma_1 y_s(\gamma_1 t), ..., \lambda_n\gamma_n y_s(\gamma_n t)\big)\big| \\ &\quad + \big|f\big(t, \lambda_1\gamma_1 y_s(\gamma_1 t), ..., \lambda_n\gamma_n y_s(\gamma_n t)\big) - y_s(t)\big| \end{aligned}$$

$$\leq \sum_{i=1}^{n} k_i \big|\lambda_i \gamma_i y(\gamma_i t) - \lambda_i \gamma_i y_s(\gamma_i t)\big| + \delta$$
$$\leq \sum_{i=1}^{n} k_i \lambda_i \big|y(\gamma_i t) - y_s(\gamma_i t)\big| + \delta.$$

Thus

$$||y - y_s||_C \leq \sum_{i=1}^{n} k_i \lambda_i ||y - y_s||_C + \delta,$$

and

$$||y - y_s||_C \leq \frac{\delta}{1 - \sum_{i=1}^{n} k_i \lambda_i} = \epsilon^*.$$

Now

$$\begin{aligned}
|x(t) - x_s(t)| &= \bigg|x_0 + \int_0^T g\big(s, x(s)y(s)\big)ds + \int_0^t y(s)ds \\
&\quad - x_0 - \int_0^T g\big(s, x_s(s)y_s(s)\big)ds + \int_0^t y_s(s)ds\bigg| \\
&\leq \int_0^T \big|g\big(s, x(s)y(s)\big) - g\big(s, x_s(s)y_s(s)\big)\big|ds + \int_0^T \big|y(s) - y_s(s)\big|ds \\
&\leq b\int_0^T |x(s)y(s) - x_s(s)y_s(s)|ds + \int_0^T ||y - y_s||_C ds \\
&\leq b\int_0^T |x(s)y(s) - x_s(s)y(s)|ds \\
&\quad + b\int_0^T |x_s(s)y(s) - x_s(s)y_s(s)|ds + \epsilon^* T \\
&= b\int_0^T |y(s)||x(s) - x_s(s)|ds + b\int_0^T |x_s(s)|y(s) - y_s(s)|ds + \epsilon^* T \\
&\leq bT||y||_C||x - x_s||_C + bT||x_s||_C||y - y_s||_C + \epsilon^* T.
\end{aligned}$$

Then

$$||x - x_s||_C \leq bTr_1||x - x_s||_C + (b||x_s||_C + 1)\epsilon^* T,$$

and

$$||x - x_s||_C \leq \frac{(b||x_s||_C + 1)\epsilon^* T}{1 - bTr_1} = \epsilon.$$

□

5. Continuous dependence

In this section, we study the continuous dependence of the unique solution of (1.1)-(1.2) on the initial data x_0, the functions g and f, and parameters λ_i and γ_i.

Definition 5.1. The solution $y \in C[0,T]$ of the functional equation (2.2) depends continuously on the function f and parameters λ_i and γ_i, *if* $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ *such that*

$$\max\big\{|f(t, x_1, ..., x_n) - f^*(t, x_1, ..., x_n)|,\ |\lambda_i - \lambda_i^*|,\ |\gamma_i - \gamma_i^*|\big\} \leq \delta,$$

then

$$||y - y^*||_C \leq \epsilon.$$

where y^* is the unique solution of the functional equation

$$y^*(t) = f^*\big(t, \lambda_1^*\gamma_1^* y^*(\gamma_1^* t), \lambda_2^*\gamma_2^* y^*(\gamma_2^* t), ..., \lambda_n^*\gamma_n^* y^*(\gamma_n^* t)\big),\ t \in [0, T]. \tag{5.1}$$

Theorem 5.2. *Let the assumptions of Theorem 2.2 be satisfied, then the solution $y \in C[0, T]$ of (2.2) depends continuously on the function f and parameters λ_i and γ_i.*

Proof. Let y and y^* be the two solution of (2.2) and (5.1) respectively, then we have

$$\begin{aligned}
&|y(t) - y^*(t)| \\
&= \big|f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big) - f^*\big(t, \lambda_1^*\gamma_1^* y^*(\gamma_1^* t), ..., \lambda_n^*\gamma_n^* y^*(\gamma_n^* t)\big)\big| \\
&\leq \big|f\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big) - f^*\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big)\big| \\
&\quad + \big|f^*\big(t, \lambda_1\gamma_1 y(\gamma_1 t), ..., \lambda_n\gamma_n y(\gamma_n t)\big) - f^*\big(t, \lambda_1^*\gamma_1^* y^*(\gamma_1^* t), ..., \lambda_n^*\gamma_n^* y^*(\gamma_n^* t)\big)\big| \\
&\leq \delta + \sum_{i=1}^n k_i \big|\lambda_i\gamma_i y(\gamma_i t) - \lambda_i^*\gamma_i^* y^*(\gamma_i^* t)\big| \\
&= \delta + \sum_{i=1}^n k_i \big|\lambda_i\gamma_i y(\gamma_i t) - \lambda_i\gamma_i y^*(\gamma_i t) + \lambda_i\gamma_i y^*(\gamma_i t) - \lambda_i^*\gamma_i^* y^*(\gamma_i^* t)\big| \\
&\leq \delta + \sum_{i=1}^n k_i\lambda_i \big|y(\gamma_i t) - y^*(\gamma_i t)\big| + \sum_{i=1}^n k_i \big|\lambda_i\gamma_i y^*(\gamma_i t) - \lambda_i^*\gamma_i^* y^*(\gamma_i^* t)\big| \\
&\leq \delta + ||y - y^*||_C \sum_{i=1}^n k_i\lambda_i \\
&\quad + \sum_{i=1}^n k_i \big|\lambda_i\gamma_i y^*(\gamma_i t) - \lambda_i\gamma_i y^*(\gamma_i^* t) + \lambda_i\gamma_i y^*(\gamma_i^* t) - \lambda_i^*\gamma_i^* y^*(\gamma_i^* t)\big| \\
&\leq \delta + ||y - y^*||_C \sum_{i=1}^n k_i\lambda_i + \sum_{i=1}^n k_i\lambda_i \big|y^*(\gamma_i t) - y^*(\gamma_i^* t)\big| \\
&\quad + \sum_{i=1}^n k_i |\lambda_i\gamma_i - \lambda_i^*\gamma_i^*| |y^*(\gamma_i^* t)|.
\end{aligned}$$

Now $y^* \in C[0, T]$ and $|\gamma_i - \gamma_i^*| \leq \delta$, then $\big|y^*(\gamma_i t) - y^*(\gamma_i^* t)\big| \leq \epsilon_1$, then

$$\begin{aligned}
|y(t) - y^*(t)| &\leq \delta + ||y - y^*||_C \sum_{i=1}^n k_i\lambda_i \\
&\quad + \epsilon_1 \sum_{i=1}^n k_i\lambda_i + ||y^*||_C \sum_{i=1}^n k_i |\lambda_i\gamma_i - \lambda_i^*\gamma_i^*| \\
&\leq \delta + ||y - y^*||_C \sum_{i=1}^n k_i\lambda_i
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon_1 \sum_{i=1}^{n} k_i \lambda_i + r_1 \sum_{i=1}^{n} k_i |\lambda_i \gamma_i - \lambda_i \gamma_i^* + \lambda_i \gamma_i^* - \lambda_i^* \gamma_i^*| \\
\leq\ & \delta + ||y - y^*||_C \sum_{i=1}^{n} k_i \lambda_i \\
&+\epsilon_1 \sum_{i=1}^{n} k_i \lambda_i + r_1 \sum_{i=1}^{n} k_i \big[\lambda_i |\gamma_i - \gamma_i^*| + |\lambda_i - \lambda_i^*||\gamma_i^*|\big],
\end{aligned}
$$

and

$$||y - y^*||_C \leq \delta + ||y - y^*||_C \sum_{i=1}^{n} k_i \lambda_i + \epsilon_1 \sum_{i=1}^{n} k_i \lambda_i + r_1 \sum_{i=1}^{n} k_i (\lambda_i \delta + \delta).$$

Hence

$$||y - y^*||_C \leq \frac{\delta + \epsilon_1 \sum_{i=1}^{n} k_i \lambda_i + \delta r_1 \sum_{i=1}^{n} k_i (\lambda_i + 1)}{1 - \sum_{i=1}^{n} k_i \lambda_i} = \epsilon.$$

□

Definition 5.3. The solution $x \in C^1[0, T]$ of (1.1)-(1.2) depends continuously on the initial data x_0 and functions g and y, *if* $\forall\ \epsilon > 0$ $\exists\ \delta(\epsilon) > 0$ *such that*

$$\max\big\{|x_0 - x_0^*|,\ |g(t,x) - g^*(t,x)|,\ |y(t) - y^*(t)|\big\} \leq \delta,$$

then

$$||x - x^*||_C \leq \epsilon,$$

where x^* is unique solution of the integral equation

$$x^*(t) = x_0^* + \int_0^T g^*\big(s, x^*(s) y^*(s)\big) ds + \int_0^t y^*(s) ds,\ t \in [0, T]. \tag{5.2}$$

Theorem 5.4. *Let the assumptions of Theorem 3.1 be satisfied, then the solution $x \in C^1[0, T]$ of (1.1)-(1.2) depends continuously on the initial data x_0 and functions g and y.*

Proof. Let x and x^* be the two solution of (2.1) and (5.2) respectively, then we have

$$
\begin{aligned}
|x(t) - x^*(t)| \ =\ & \Big|x_0 + \int_0^T g\big(s, x(s) y(s)\big) ds + \int_0^t y(s) ds \\
& -x_0^* - \int_0^T g^*\big(s, x^*(s) y^*(s)\big) ds - \int_0^t y^*(s) ds\Big| \\
\leq\ & |x_0 - x_0^*| + \int_0^T \big|g\big(s, x(s) y(s)\big) - g^*\big(s, x^*(s) y^*(s)\big)\big| ds \\
& + \int_0^T \big|y(s) - y^*(s)\big| ds \\
\leq\ & \delta + \int_0^T \big|g\big(s, x(s) y(s)\big) - g^*\big(s, x(s) y(s)\big)\big| ds
\end{aligned}
$$

$$\begin{aligned}
& +\int_0^T \big|g^*\big(s, x(s)y(s)\big) - g^*\big(s, x^*(s)y^*(s)\big)\big|ds + \delta T \\
\leq\ & \delta + \delta T + b\int_0^T \big|x(s)y(s) - x^*(s)y^*(s)\big|ds + \delta T \\
\leq\ & (1+2T)\delta + b\int_0^T \big|x(s)y(s) - x^*(s)y(s)\big|ds \\
& +b\int_0^T \big|x^*(s)y(s) - x^*(s)y^*(s)\big|ds \\
=\ & (1+2T)\delta + b\int_0^T |y(s)||x(s) - x^*(s)|ds \\
& +b\int_0^T |x^*(s)||y(s) - y^*(s)|ds \\
\leq\ & (1+2T)\delta + bT||y||_C||x - x^*||_C + bT\delta||x^*||_C.
\end{aligned}$$

Thus

$$||x - x^*||_C \leq (1+2T)\delta + bTr_1||x - x^*||_C + bT\delta r_2.$$

Hence

$$||x - x^*||_C \leq \frac{(1+2T+bTr_2)\delta}{1 - bTr_1} = \epsilon.$$

□

Corollary 5.5. *Let the assumptions of Theorem 3.1 be satisfied, then the solution $x \in C^1[0,T]$ of (1.1)-(1.2) depends continuously on the function f and parameters λ_i and γ_i.*

6. Special cases and examples

Corollary 6.1.
1. *Let the assumptions $(iv) - (v)$ of Theorem 2.3 be satisfied. If $\sum_{i=1}^n \lambda_i^2 < 1$, then the problem*

$$\frac{dx(t)}{dt} = \sum_{i=1}^n \lambda_i \frac{d}{dt}x(\gamma_i t),\ t \in (0,T], \tag{6.1}$$

   with the nonlocal quadratic integral condition (1.2) has at least one solution $x \in C^1[0,T]$.
2. *Let the assumptions $(i)' - (ii)'$ of Theorem 3.1 be satisfied. If $\sum_{i=1}^n \lambda_i^2 < 1$, then the solution of problem (6.1) with (1.2), $x \in C^1[0,T]$, is unique.*
3. *If $\sum_{i=1}^n \lambda_i^2 < 1$, then the solution of the initial value problem (6.1) with $x(0) = x_0$, $x \in C^1[0,T]$, is unique.*

Proof. Set $f\Big(t, \lambda_1 \frac{d}{dt}x(\gamma_1 t), ..., \lambda_n \frac{d}{dt}x(\gamma_n t)\Big) = \sum_{i=1}^n \lambda_i \frac{d}{dt}x(\gamma_i t)$, using Lemma 2.1, we can obtain the functional equation

$$y(t) = \sum_{i=1}^n \lambda_i \gamma_i y(\gamma_i t),\ t \in [0,T],$$

then $|f(t,x) - f(t,y)| \leq \sum_{i=1}^{n} \lambda_i |x-y|,\ x,y \in R^n$, and $f(t,0) = 0$, then $k_i = \lambda_i$, $M = 0$, and $r_1 = 0$. Hence all assumptions of Theorem 2.3 and 3.1 are satisfied. $\square$

Corollary 6.2. *Assume that $A : [0,T] \to R$ is continuous and $\sum_{i=1}^{n} \lambda_i^2 < 1$, then*

1. *Let the assumptions $(iv) - (vi)$ of Theorem 2.3 be satisfied, then the problem*

$$\frac{dx(t)}{dt} = A(t) + \sum_{i=1}^{n} \lambda_i \frac{d}{dt} x(\gamma_i t),\ t \in (0,T], \tag{6.2}$$

with the nonlocal condition (1.2) has at least one solution $x \in C^1[0,T]$.

2. *Let the assumptions (vi) and $(i)' - (ii)'$ of Theorem 3.1 be satisfied, then the solution of (6.2) with (1.2), $x \in C^1[0,T]$, is unique.*

Corollary 6.3. *Let the assumptions $(i) - (ii)$ of Theorem 3.1 be satisfied, if $k_1 \lambda_1 < 1$, then the initial value problem*

$$\frac{dx(t)}{dt} = f\Big(t, \lambda_1 \frac{d}{dt} x(\gamma_1 t)\Big),\ t \in (0,T], \tag{6.3}$$

$$x(0) = x_0, \tag{6.4}$$

has a unique solution $x \in C^1[0,T]$.

Corollary 6.4. *Let the assumptions $(iii), (vi)$ and $(i)' - (ii)'$ of Theorem 3.1 be satisfied, if $f_i : [0,T] \times R \to R$ is continuous such that*

$$|f_i(t,x) - f_i(t,y)| \leq k_i |x-y|,$$

where $sup_{t \in [0,T]} |f_i(t,0)| \leq M$, then the problem

$$\frac{dx(t)}{dt} = \sum_{i=1}^{n} f_i\Big(t, \lambda_i \frac{d}{dt} x\big(\gamma_i t\big)\Big), t \in (0,T], \tag{6.5}$$

with the nonlocal condition (1.2) has a unique solution $x \in C^1[0,T]$.

Example 6.5. Consider the following nonlocal problem

$$\frac{dx(t)}{dt} = \frac{e^{-t}}{t+7} + \frac{1}{2} sin\big(\frac{1}{4}t\big) \frac{d}{dt} x\big(\frac{1}{4}t\big) + \frac{1}{3} sin\big(\frac{1}{2}t\big) \frac{d}{dt} x\big(\frac{1}{5}t\big),\ t \in (0,7], \tag{6.6}$$

with the nonlocal quadratic integral condition

$$x(0) = 1 + \int_0^7 \Big(\big(\frac{s}{3}\big)^2 + \frac{1}{6} x(s) \frac{dx(s)}{ds} \Big) ds. \tag{6.7}$$

Using Lemma 2.1, x is the solution of the integral equation

$$x(t) = 1 + \int_0^7 \Big(\big(\frac{s}{3}\big)^2 + \frac{1}{6} x(s) y(s) \Big) ds + \int_0^t y(s) ds,\ t \in [0,7], \tag{6.8}$$

where $y(t)$ is given by the functional equation

$$y(t) = \frac{e^{-t}}{t+7} + \frac{1}{8} sin\big(\frac{1}{4}t\big) y\big(\frac{1}{4}t\big) + \frac{1}{15} sin\big(\frac{1}{2}t\big) y\big(\frac{1}{5}t\big),\ t \in [0,7]. \tag{6.9}$$

Set

$$f(t,\lambda_1\gamma_1 y(\gamma_1 t),\lambda_2\gamma_2 y(\gamma_2 t)) = \frac{e^{-t}}{t+7} + \frac{1}{8}sin\big(\frac{1}{4}t\big)y\big(\frac{1}{4}t\big) + \frac{1}{15}sin\big(\frac{1}{2}t\big)y\big(\frac{1}{5}t\big),$$

and

$$g(t,x(t)y(t)) = \big(\frac{t}{3}\big)^2 + \frac{1}{6}x(t)y(t).$$

We have $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{1}{5}$, $x_0 = 1$, $T = 7$.
Hence, $f|(t,0,0)| \leq \frac{1}{7} = M$, $g(t,0) = \big(\frac{t}{3}\big)^2 \in L^1[0,7]$,
and

$$\big|f(t,x_1,x_2) - f(t,y_1,y_2)\big| \leq \frac{1}{8}|x_1 - y_1| + \frac{1}{15}|x_2 - y_2|,$$

$$\big|g(t,u) - g(t,\bar{u})\big| \leq \frac{1}{6}|u - \bar{u}|,$$

then, $k_1 = \frac{1}{8}$, $k_2 = \frac{1}{15}$, $b = \frac{1}{6}$, $\sum_{i=1}^{2} k_i\lambda_i \approx 0.0847222 < 1$,
$r_1 \approx 0.1560806$, and $bTr_1 \approx 0.1820940 < 1$. Obviously, all assumptions of Theorem 3.1 are satisfied, then the solution of (6.6)-(6.7), $x \in C^1[0,7]$, is unique.

Example 6.6. Consider the following nonlocal problem

$$\frac{dx(t)}{dt} = \frac{1}{2}\frac{d}{dt}x(\frac{1}{3}t) + \frac{1}{4}\frac{d}{dt}x(\frac{1}{5}t) + \frac{1}{6}\frac{d}{dt}x(\frac{1}{7}t),\ t \in (0,2], \tag{6.10}$$

with the nonlocal quadratic integral condition

$$x(0) = 1 + \int_0^2 \left((\frac{s}{7})^3 + \frac{2}{9}x(s)\frac{dx(s)}{ds} \right) ds. \tag{6.11}$$

Set

$$g(t,x(t)y(t)) = \big(\frac{t}{7}\big)^3 + \frac{2}{9}x(t)y(t).$$

We have $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $\lambda_3 = \frac{1}{6}$,and $g(t,0) = \big(\frac{t}{7}\big)^3 \in L^1[0,2]$. Hence

$$\big|g(t,u) - g(t,\bar{u})\big| \leq \frac{2}{9}|u - \bar{u}|,$$

then $b = \frac{2}{9}$, $\sum_{i=1}^{3} \lambda_i^2 \approx 0.3402778 < 1$. Obviously, all assumptions of Corollary 6.1.2 are satisfied, then the solution of (6.10)-(6.11), $x \in C^1[0,2]$, is unique.

Example 6.7. The initial value problem problem (6.10) with $x(0) = 1$, has a unique solution $x \in C^1[0,2]$, by Corollary 6.1.3.

Example 6.8. Consider the nonlocal problem

$$\frac{dx(t)}{dt} = \frac{1}{t+5} + \frac{1}{2}\frac{d}{dt}x(\frac{1}{3}t) + \frac{1}{4}\frac{d}{dt}x(\frac{1}{5}t) + \frac{1}{6}\frac{d}{dt}x(\frac{1}{7}t),\ t \in (0,2], \tag{6.12}$$

with the nonlocal quadratic integral condition (6.11) of Example 6.6.
Here, we have from Example 6.6 that $T = 2$, $\sum_{i=1}^{3} \lambda_i^2 \approx 0.3402778 < 1$, and $b = \frac{2}{9}$. Set $A(t) = \frac{1}{t+5} \in C[0,2]$, then $|f(t,0)| \leq \frac{1}{5} = M$, $r_1 \approx 0.3031579$, and $bTr_1 \approx 0.1347368 < 1$. Obviously, all assumptions of Corollary 6.2.2 are satisfied, then the solution of (6.12) with (6.11), $x \in C^1[0,2]$, is unique.

7. Conclusion

This study proves the existence and uniqueness of solution $x \in C^1[0,T]$ of the problem (1.1)-(1.2) emphasizing its Hyers-Ulam stability. The study carefully examines the behavior of the unique solution of the problem, revealing that it is continuously dependent on a set of parameters. The study not only adds to theoretical understanding but also offers practical insights via analysis of particular cases and examples. These results will influence future study in the area of functional-differential equations and have relevance for a variety of scientific and engineering applications.

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