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q-ABEL INTEGRAL EQUATION WITH n TERMS

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Abstract. In this paper solution of Abel's integral equation with n terms is discussed by using Laplace transform. Furthermore we derive the results about the solution of q-Abel integral equation as well as q-Abel integral equation with n-terms using different techniques.

1. Introduction and basic results

Abel's integral equation is an important singular integral equation and Abel found this equation from a problem of mechanics, namely the tautochrone problem, which is considered to be the first application of fractional calculus to an engineering problem [2, 3, 12]. This equation and some variants of it found applications in heat transfer between solids and gases under non-linear boundary conditions, theory of superfluidity, percolation of water, subsolutions of a non-linear diffusion problem, propagation of shock-waves in gas fields tubes, microscopy, seismology, radio astronomy, satellite photometry of airglows, electron emission, atomic scattering, radar ranging, optical fiber evaluation, X-ray radiography, flame and plasma diagnostics [6,10]. Abel's integral equation, is the very first integral equation which is studied, and the relevant integral equation have never ceased to inspire mathematicians to investigate and to generalize them [4,7,8]. The Abel's integral equation are use in different fields of physics and experimental sciences (for example scattering theory, spectroscopy, seismology, elasticity theory, plasma physics, etc)] [5,11]. One of the recently influential works on the subject of Abel integral eqation is the monograph of Gorenflo and Vessella [13].

Abel's integral equation is connected to the first fractional integral operator which is defined by Riemann and Liouville

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\Phi(t)dt = f(x),\ x>a, \alpha>0,\ f\in L_1(a,b). \tag{1.1}$$

Let us state result about existance for solution of Abel's integral equation (1.1).

Theorem 1.1. *[12] The function $f_{1-\alpha}$ defined by*

$$f_{1-\alpha}(x) := \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha} f(t)dt$$

with $f_{1-\alpha}(a) = 0$ and absolutely continuous on $[a, b]$ if and only if the Abel's integral equation (1.1) with $0 < \alpha < 1$ has a unique solution in $L_1(a, b)$. If the

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formar conditions are fulfilled, then the unique solution Φ *is given by*

$$\Phi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}f(t)dt=\frac{d}{dx}f_{1-\alpha}(x), a.e. \tag{1.2}$$

If $f\in \mathcal{AC}\,[a,b]$, *then* $f_{1-\alpha}\in \mathcal{AC}\,[a,b]$ *and from equation (1.2) we have*

$$\Phi(x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(x-a)^{\alpha}}+\int_a^x \frac{f^{'}(s)}{(x-a)^{\alpha}}ds\right].$$

Consider the cross section of a weir notch. The cross section is symmetrical with respect to the x-axis. The flow rate through the notch per unit of time will be determined by

$$Q=C\int_0^h \sqrt{h-x}\,\Phi(x)dx, \tag{1.3}$$

where the form of the notch is determined by $y=\Phi(x)$; $x\geq 0$. From equation (1.3) determining $\Phi(x)$ so that the quantity of flow per unit of time shall be proportional to a given power of the depth of stream; i.e., $Q=kh^m, m>0$. Hence we must find $\Phi(x)$ from an integral equation of the form

$$\int_0^h \sqrt{h-x}\,\Phi(x)dx=kh^m. \tag{1.4}$$

Differentiating (1.4), we have:

$$\frac{1}{2}\int_0^h (h-x)^{-1/2}\Phi(x)dx=mkh^{m-1}.$$

More generally, we can extend above equation as follows

$$\int_0^h \frac{\Phi(x)dx}{\sqrt{h-x}}=2kmh^{m-1} \tag{1.5}$$

and a solution of equation (1.5) will be a also solution of equation (1.4). But equation (1.5) comes under the form of Abel's integral equation,

$$\int_a^x \frac{\Phi(y)dy}{(x-y)^{\alpha}}=g(x),\ (0<\alpha<1).$$

This Abel's integral equation can be converted into Riemann and Liouville fractional integral operator. Furthermore by using (1.1) one can find its continuous solution Φ as given in [6].
Let us consider the basic notions of q-calculus and q-fractional calculus from [1]: Let $q\in[0,1]$ is fixed real number, a subset A of $\mathbb{R}$ is called q-geometric if $qz\in A$ whenever $z\in A$.

Definition 1.2. A function g which is defined on a q-geometric set A, $0\in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} g\left(zq^n\right)=g\left(0\right) \text{ for all } z\in A.$$

Furthermore, if A is also q^{-1}-geometric, then we say that g is q-regular at infinity if there exists a constant C such that

$$\lim_{n\to\infty} g\left(zq^{-n}\right) = C \text{ for all } z \in A.$$

Henceforth, if $A \subseteq \mathbb{R}$ is q-geometric and g is a q-regular at zero function defined on A, we define $g\left(0^{+}\right)$ and $g\left(0^{-}\right)$ by

$$g\left(0^{+}\right) := \lim_{\substack{n\to\infty,\\ x>0}} g\left(xq^{k}\right),\ g\left(0^{-}\right) := \lim_{\substack{n\to\infty,\\ x<0}} g\left(xq^{k}\right).$$

Remark 1.3. Clearly, if g is q-regular at zero, then

$$g(0) = g\left(0^{+}\right) = g\left(0^{-}\right).$$

The q-regularity at zero plays the role of continuity in the classical sense in some setting. On the other hand, continuity at zero implies q-regularity at zero, but the converse is not necessarily true.

Example 1.4. The function $u : [0,1] \to \mathbb{R}$

$$u(x) = \begin{cases} 1 & x = a_n = \frac{1}{\sqrt{n}},\ n \text{ is prime} \\ x & \text{otherwise} \end{cases}$$

is q-regular at zero for rational q, but it is not continuous at zero.

Let us now define the fractional q-integral and q-derivative:

Definition 1.5. The fractional q-integral is

$$I_{q,c}^{\alpha} g(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_c^x (qt/x;q)_{\alpha-1} g(t) d_q t,$$

and the fractional q-derivative is

$$D_{q,c}^{\alpha} g(x) = D_q^{[\alpha]} I_{q,c}^{\alpha-[\alpha]} g(x).$$

Definition 1.6. A function g defined on $[0,a]$ is called q-absolutely continuous if g is q-regular at zero,and there exist $K > 0$, such that

$$\sum_{j=0}^{\infty} \left| g(tq^j) - g(tq^{j+1}) \right| \le K, \text{ for all } t \in (qa, a]. \tag{1.6}$$

If equation (1.6) holds then it can be extended through out $(0,a]$. To see this, it suffices to investigate the case when $x \in (0,a]$ then there exists $t \in (qa,a]$ and $k \in \mathbb{N}$ such that $x = tq^k$. Then

$$\begin{aligned} \sum_{j=0}^{\infty} \left| g(xq^j) - g(xq^{j+1}) \right| &= \sum_{j=k}^{\infty} \left| g(tq^j) - g(tq^{j+1}) \right| \\ &\le \sum_{j=0}^{\infty} \left| g(tq^j) - g(tq^{j+1}) \right| < \infty. \end{aligned}$$

We shall use $\mathbb{AC}_q[0,a]$ to denote the class of q-absolutely continuous function on $[0,a]$.

Definition 1.7. q-Laplace transform of a function g can be defined as

$$ {}_q\mathcal{L}_s\left[g(x)\right] = \Phi(s) = \frac{1}{1-q}\int_0^{s^{-1}} E_q(-qsx)g(x)d_qx. $$

Theorem 1.8. *Let g be a function on $\left[0,a\right]$. Then ,the function $g \in \mathbb{AC}_q\left[0,a\right]$ if and only if there exist a constant Φ in $\mathcal{L}_q^1\left[0,a\right]$ such that*

$$ g \in \mathbb{AC}_q\left[0,a\right] \Leftrightarrow g(x) = c + \int_0^x \Phi(x)d_qu, \forall x \in \left[0,a\right]. \tag{1.7} $$

Moreover, the constant c and the function Φ are uniquely determined via $c = f(0)$ and

$$ \Phi(x) = D_qg(x), \forall x \in \left(0,a\right]. $$

Example 1.9. Let $g : [0,a] \to \mathbb{R}$,$g(x) = x^2$ is a continuous $\Rightarrow g \in \mathbb{AC}_q\left[0,a\right]$.

Theorem 1.10. *The q-Abel integral equation*

$$ \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\Phi(t)d_qt = g(x), (0 < \alpha < 1, x \in (0,a]) \tag{1.8} $$

has a unique solution $\Phi \in \mathcal{L}_q^1\left[0,a\right]$ if and only if

$$ I_q^{1-\alpha}g(x) \in \mathbb{AC}_q\left[0,a\right] \text{ and } I_q^{1-\alpha}g(0) = 0. \tag{1.9} $$

Furthermore, the unique solution Φ is given by

$$ \Phi(x) = D_{q,x}I_q^{1-\alpha}g(x). \tag{1.10} $$

The following example confirm the validity of the theorem.

Example 1.11. Consider for $0 < \alpha < 1$ the q-Abel integral equation

$$ \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\Phi(t)d_qt = x^2, (0 < \alpha < 1, x \in (0,a]). \tag{1.11} $$

An easy computation of $I_q^{1-\alpha}g(x)$ gives

$$ \begin{aligned} I_q^{1-\alpha}g(x) &= \frac{x^{-\alpha}}{\Gamma_q(1-\alpha)}\int_0^x (qt/x;q)_{-\alpha}x^2d_qt \\ &= \frac{x^{2-\alpha}}{\Gamma_q(1-\alpha)}\int_0^x (qt/x;q)_{-\alpha}d_qt. \end{aligned} $$

Taking substitution $t/x = u$, above equation becomes

$$ \begin{aligned} I_q^{1-\alpha}g(x) &= \frac{x^{2-\alpha}}{\Gamma_q(1-\alpha)}\int_0^1 (qu;q)_{-\alpha}xd_qu \\ &= \frac{x^{3-\alpha}}{\Gamma_q(1-\alpha)}\int_0^1 (qu;q)_{-\alpha}d_qu \\ &= \frac{x^{3-\alpha}}{\Gamma_q(1-\alpha)}\int_0^1 x^0(qu;q)_{-\alpha}d_qu \\ &= \frac{x^{3-\alpha}}{\Gamma_q(1-\alpha)}B_q\left(1,1-\alpha\right) \end{aligned} $$

$$
\begin{aligned}
&= \frac{x^{3-\alpha}}{\Gamma_q(1-\alpha)} \frac{\Gamma_q(1)\,\Gamma_q(1-\alpha)}{\Gamma_q(1+1-\alpha)} \\
&= \frac{x^{3-\alpha}}{\Gamma_q(1-\alpha)} \frac{\Gamma_q(3)\,\Gamma_q(1-\alpha)}{\Gamma_q(3+1-\alpha)} \\
&= \frac{x^{3-\alpha}}{\Gamma_q(1-\alpha)} \frac{2\Gamma_q(2)\,\Gamma_q(1-\alpha)}{\Gamma_q(4-\alpha)} \\
&= \frac{x^{3-\alpha}}{1} \frac{2.1}{\Gamma_q(4-\alpha)} \\
&= \frac{2!x^{3-\alpha}}{\Gamma_q(4-\alpha)}.
\end{aligned}
$$

Then , $I_q^{1-\alpha} f(0) = 0$ and $I_q^{1-\alpha} f(x) \in \mathbb{AC}_q[0,a]$. Consequently, equation (1.11) has unique solution given by

$$
\begin{aligned}
\Phi(x) &= D_{q,x} I_q^{1-\alpha} f(x) \\
&= D_{q,x} f\left(\frac{2x^{3-\alpha}}{\Gamma_q(4-\alpha)}\right) \\
&= \frac{2}{\Gamma_q(3-\alpha)} \frac{f(x^{3-\alpha}) - f(xq)^{3-\alpha}}{x-qx} \\
&= \frac{2}{\Gamma_q(4-\alpha)} \frac{x^{3-\alpha} - x^{3-\alpha} q^{3-\alpha}}{x-qx} \\
&= \frac{2}{\Gamma_q(4-\alpha)} \frac{x^{3-\alpha}\left(1-q^{3-\alpha}\right)}{x(1-q)} \\
&= \frac{2}{\Gamma_q(3-\alpha)} \frac{x^{2-\alpha}\left(1-q^{3-\alpha}\right)}{(1-q)}
\end{aligned}
$$

Since $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$

$$
\begin{aligned}
\Phi(x) &= \frac{2x^{2-\alpha}}{\Gamma_q(4-\alpha)} \frac{\Gamma_q(3-\alpha+1)}{\Gamma_q(3-\alpha)} \\
&= \frac{2x^{2-\alpha}}{\Gamma_q(3-\alpha)}.
\end{aligned}
$$

2. The n-terms Abel's integral equation

In this section, we discuss solution method for n-term Abel's integral equation. First of all we consider the general form of two terms Abel's integral equations. For $\alpha, \beta > 0$, such that $0 < \beta < \alpha < 1$

$$
\int_0^x \left\{ \frac{P}{(x-t)^\alpha} + \frac{Q}{(x-t)^\beta} \right\} u(t)dt = f(x), \; x > 0 \tag{2.1}
$$

we put $F(s) = \mathcal{L}\{f(x)\}$ and $U(s) = \mathcal{L}\{u(x)\}$, then:

$$\mathcal{L}\left\{\int_0^x \left(\frac{P}{(x-t)^\alpha} + \frac{Q}{(x-t)^\beta}\right) u(t)dt\right\} = \mathcal{L}\{f(x)\}$$

$$P\mathcal{L}\left\{\int_0^x (x-t)^{-\alpha} u(t)dt\right\} + Q\mathcal{L}\left\{\int_0^x (x-t)^{-\beta} u(t)dt\right\} = F(s).$$

Since $\mathcal{L}(x^\alpha) = \frac{\Gamma(\alpha+1)}{S^{\alpha+1}}$, or $\mathcal{L}((x-t)^\alpha) = \frac{\Gamma(\alpha+1)}{S^{\alpha+1}}$, also $\mathcal{L}((x-t)^{-\alpha}) = \frac{\Gamma(-\alpha+1)}{S^{-\alpha+1}}$,

$$U(s) = \frac{S^{1-\alpha}S^{1-\beta}}{P\Gamma(1-\alpha)S^{1-\beta} + Q\Gamma(1-\beta)S^{1-\alpha}} F(s).$$

Divided by $S^{1-\beta}$ on numerator and denominator:

$$U(s) = \frac{S^{1-\alpha}}{P\Gamma(1-\alpha)} \frac{1}{1 + \frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)} S^{\beta-\alpha}} F(s) \tag{2.2}$$

provided that

$$\left|\frac{P\Gamma(1-\beta)}{Q\Gamma(1-\alpha)}\right| |S|^{\beta-\alpha} < 1.$$

Convergence of the geometric series implies that

$$\frac{1}{1 + \frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)} S^{\beta-\alpha}} = \sum_{n=0}^{\infty} (-)^n \left(\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)} S^{\beta-\alpha}\right)^n.$$

Equation (2.2) implies that

$$U(s) = \frac{S^{1-\alpha}}{P\Gamma(1-\alpha)} \left(\sum_{n=0}^{\infty} (-)^n \left(\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)} S^{\beta-\alpha}\right)^n\right) F(s) \tag{2.3}$$

In more compact form:

$$\begin{aligned} U(s) &= \sum_{n=0}^{\infty} (-)^n \frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}} S^{(\beta-\alpha)n+1-\alpha} F(s) \\ &= \sum_{n=0}^{\infty} (-)^n \frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}} \frac{1}{S^\eta} F(s). \end{aligned}$$

Here $\eta = (n+1)\alpha - n\beta - 1$. Since $\mathcal{L}(x^\eta) = \frac{\Gamma(\eta+1)}{S^{\eta+1}}$. Also $\mathcal{L}(x^{\eta-1}) = \frac{\Gamma(\eta-1+1)}{S^{\eta-1+1}} = \frac{\Gamma(\eta)}{S^\eta}$, so $\frac{1}{S^\eta} = \frac{\mathcal{L}(x^{\eta-1})}{\Gamma(\eta)}$.

$$U(s) = \sum_{n=0}^{\infty} (-)^n \frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}} \frac{\mathcal{L}(x^{\eta-1})}{\Gamma(\eta)} \mathcal{L}(f(x))$$

By using convolution theorem, we obtain:

$$\mathcal{L}(u(x)) = \mathcal{L}\left[\sum_{n=0}^{\infty} (-)^n \frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}} \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt\right]$$

Applying Laplace inverse on both sides, we obtain,

$$u(x) = \sum_{n=0}^{\infty}(-)^n \frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}} \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt$$
$$= \sum_{n=0}^{\infty} c_n D_x^{-\eta} f(x), 0 < \beta < \alpha < 1$$

where $c_n = (-)^n \frac{(B\Gamma(1-\beta))^n}{(A\Gamma(1-\alpha))^{n+1}}, D_x^{-\eta} f(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x \cdot -t)^{\eta-1} f(t)dt$, $\eta = (n+1)\alpha - n\beta - 1$.

Now we consider the three terms Abel's integral equation in the general form:

$$\int_0^x \left\{ \frac{P}{(x-t)^\alpha} + \frac{Q}{(x-t)^\beta} + \frac{R}{(x-t)^\gamma} \right\} u(t)dt = f(x), 0 < \gamma < \beta < \alpha < 1, \tag{2.4}$$

Applying Laplace transforms and putting $F(s) = \mathcal{L}\{f(x)\}$ $U(s) = \mathcal{L}\{u(x)\}$, $\mathcal{L}(x^\alpha) = \frac{\Gamma(\alpha+1)}{S^{\alpha+1}}$, and $\mathcal{L}((x-t)^\alpha) = \frac{\Gamma(\alpha+1)}{S^{\alpha+1}}$,

$$\left\{ P\frac{\Gamma(1-\alpha)}{S^{1-\alpha}} + Q\frac{\Gamma(1-\beta)}{S^{1-\beta}} + R\frac{\Gamma(1-\gamma)}{S^{1-\gamma}} \right\} U(s)$$
$$= \frac{P\Gamma(1-\alpha)S^{1-\beta}S^{1-\gamma} + Q\Gamma(1-\beta)S^{1-\alpha}S^{1-\gamma} + R\Gamma(1-\gamma)S^{1-\alpha}S^{1-\beta}}{S^{1-\alpha}S^{1-\beta}S^{1-\gamma}} U(s) = F(s).$$

It implies that

$$U(s) = \frac{S^{1-\alpha}S^{1-\beta}S^{1-\gamma}}{P\Gamma(1-\alpha)S^{1-\beta}S^{1-\gamma} + Q\Gamma(1-\beta)S^{1-\alpha}S^{1-\gamma} + R\Gamma(1-\gamma)S^{1-\alpha}S^{1-\beta}} F(s).$$

Divided by $S^{1-\beta}S^{1-\gamma}$ on numerator and denominator:

$$U(s) = \frac{S^{1-\alpha}}{P\Gamma(1-\alpha) + Q\Gamma(1-\beta)S^{1-\alpha-1+\beta} + R\Gamma(1-\gamma)S^{1-\alpha-1+\gamma}} F(s)$$
$$= \frac{S^{1-\alpha}}{P\Gamma(1-\alpha) + Q\Gamma(1-\beta)S^{\beta-\alpha} + R\Gamma(1-\gamma)S^{\gamma-\alpha}} F(s).$$

It follows that

$$U(s) = \frac{S^{1-\alpha}}{P\Gamma(1-\alpha)} \left[\frac{1}{1 + \frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} + \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha}} \right] F(s), \tag{2.5}$$

with

$$\left| \frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)} \right| |S|^{\beta-\alpha} < 1, \left| \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)} \right| |S|^{\gamma-\alpha} < 1.$$

Using the geometric series

$$\frac{1}{1 + \frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} + \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha}} = \frac{1}{1 - \left(-\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} - \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha} \right)}.$$

Let

$$r = -\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} - \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha},$$

then

$$\sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty}\left(-\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} - \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha}\right)^n$$
$$= \sum_{n=0}^{\infty}(-)^n\left(\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} - \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha}\right)^n.$$

So equation (2.5)
(2.6)

$$U(s) = \frac{S^{1-\alpha}}{P\Gamma(1-\alpha)}\left(\sum_{n=0}^{\infty}(-)^n\left(\frac{Q\Gamma(1-\beta)}{P\Gamma(1-\alpha)}S^{\beta-\alpha} + \frac{R\Gamma(1-\gamma)}{P\Gamma(1-\alpha)}S^{\gamma-\alpha}\right)^n\right)F(s).$$

More explicitly

$$U(s) = \sum_{n=0}^{\infty}(-)^n\left(\frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}}S^{(\beta-\alpha)n+1-\alpha} + \frac{R\Gamma(1-\gamma)^n}{(P\Gamma(1-\alpha))^{n+1}}S^{(\gamma-\alpha)n+1-\alpha}\right)F(s)$$
$$= \sum_{n=0}^{\infty}(-)^n\left(\frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}}\frac{1}{S^{\eta}} + \frac{R\Gamma(1-\gamma)^n}{(P\Gamma(1-\alpha))^{n+1}}\frac{1}{S^{\xi}}\right)F(s),$$

where $\eta = (n+1)\alpha - n\beta - 1,^{\xi} = (n+1)\alpha - n\gamma - 1$.

$$\mathcal{L}\left(u(x)\right) = \sum_{n=0}^{\infty}(-)^n\left(\frac{(Q\Gamma(1-\beta))^n}{(P\Gamma(1-\alpha))^{n+1}}\frac{\mathcal{L}\left(x^{\eta-1}\right)}{\Gamma(\eta)} + \frac{R\Gamma(1-\gamma)^n}{(P\Gamma(1-\alpha))^{n+1}}\frac{\mathcal{L}\left(x^{\xi-1}\right)}{\Gamma(\xi)}\right)\mathcal{L}\left(f(x)\right)$$

By using convolution theorem and inverse Laplace, we obtain

$$u(x) = C_n D_x^{-\eta} f(x) + E_n D_x^{-\xi} f(x),$$

where $C_n = (-)^n \frac{(B\Gamma(1-\beta))^n}{(A\Gamma(1-\alpha))^{n+1}}$, $E_n = (-)^n \frac{C\Gamma(1-\gamma)^n}{(A\Gamma(1-\alpha))^{n+1}}$.

Similarly the solution of the following nth terms Abel's integral equation:

$$\int_0^x \left(\sum_{i=1}^{n}\frac{P_i}{(x-t)^{\alpha i}}\right)u(t)dt = f(x), x > 0, 0 < \alpha_{i+1} < \alpha_i < 1, \forall i = 1, 2, \ldots n.$$

is

$$u(x) = \sum_{i=0}^{\infty} C_j D_x^{-\eta_i} f(x).$$

3. Two term q-Abel integral equation

Now consider the following two term q-Abel integral equation
(3.1)

$$\frac{1}{1-q}\int_0^x Px^{-\alpha}\left(qt/x;q\right)_{-\alpha}\Phi\left(t\right)d_qt + \frac{1}{1-q}\int_0^x Qx^{-\beta}\left(qt/x;q\right)_{-\beta}\Phi\left(t\right)d_qt = h\left(x\right).$$

The q-convolution implies that

$$Px^{-\alpha} *_q \Phi\left(x\right) + Qx^{-\beta} *_q \Phi\left(x\right) = h\left(x\right).$$

Applying q-Laplace on both sides, we have

$$P_q\mathcal{L}_s\left[x^{-\alpha}\right]\Phi\left(s\right) + Q_q\mathcal{L}_s\left[x^{-\beta}\right]\Phi\left(s\right) = H\left(s\right). \tag{3.2}$$

Now

$$ {}_{q}\mathcal{L}_s\left[x^{-\alpha}\right]=\frac{(1-q)^{-\alpha}\,\Gamma_q(1-\alpha)}{s^{1-\alpha}} $$

and

$$ {}_{q}\mathcal{L}_s\left[x^{-\beta}\right]=\frac{(1-q)^{-\beta}\,\Gamma_q(1-\beta)}{s^{1-\beta}}. $$

Substituting in equation (3.2), we can obatin the following form:

$$ \begin{aligned} \Phi(s) &= \frac{s^{1-\alpha}H(s)}{P(1-q)^{-\alpha}\Gamma_q(1-\alpha)+Q(1-q)^{-\beta}\Gamma_q(1-\beta)\,s^{1-\alpha}s^{\beta-1}} \\ &= \frac{s^{1-\alpha}H(s)}{P(1-q)^{-\alpha}\Gamma_q(1-\alpha)+Q(1-q)^{-\beta}\Gamma_q(1-\beta)\,s^{\beta-\alpha}} \\ &= \frac{s^{1-\alpha}(1-q)^{\alpha}}{P\Gamma_q(1-\alpha)+Q(1-q)^{\alpha-\beta}\Gamma_q(1-\beta)\,s^{\beta-\alpha}}H(s) \\ &= \frac{s^{1-\alpha}}{P\Gamma_q(1-\alpha)}\frac{(1-q)^{\alpha}}{1+\frac{Q(1-q)^{\alpha-\beta}\Gamma_q(1-\beta)}{P\Gamma_q(1-\alpha)}s^{\beta-\alpha}}H(s) \\ &= \frac{s^{1-\alpha}}{P\Gamma_q(1-\alpha)}\frac{(1-q)^{\alpha}}{1-\left(-\frac{Q(1-q)^{\alpha-\beta}\Gamma_q(1-\beta)}{P\Gamma_q(1-\alpha)}s^{\beta-\alpha}\right)}H(s). \end{aligned} $$

By using geometric series form, we obtain

$$ \begin{aligned} \Phi(s) &= \frac{s^{1-\alpha}}{P\Gamma_q(1-\alpha)}\left((1-q)^{\alpha}\sum_{n=0}^{\infty}(-1)^n\left(\frac{Q(1-q)^{\alpha-\beta}\Gamma_q(1-\beta)}{P\Gamma_q(1-\alpha)}s^{\beta-\alpha}\right)^n\right)H(s) \\ &= \frac{s^{1-\alpha}(1-q)^{\alpha}}{P\Gamma_q(1-\alpha)}\left(\sum_{n=0}^{\infty}(-1)^n\frac{(1-q)^{n(\alpha-\beta)}\left(Q\Gamma_q(1-\beta)\right)^n}{\left(P\Gamma_q(1-\alpha)\right)^n}s^{n(\beta-\alpha)}\right)H(s). \end{aligned} $$

$$ \Phi(s)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(Q\Gamma_q(1-\beta)\right)^n(1-q)^{(n+1)\alpha-n\beta}}{\left(P\Gamma_q(1-\alpha)\right)^{n+1}}s^{-(n+1)\alpha+n\beta+1)}H(s). \tag{3.3} $$

Now

$$ \begin{aligned} \frac{(1-q)^{(n+1)\alpha-n\beta}}{s^{(n+1)\alpha-n\beta-1}} &= \frac{(1-q)^{(n+1)\alpha-n\beta}}{s^{(n+1)\alpha-n\beta-2+1}} \\ &= \frac{(1-q)^{(n+1)\alpha-n\beta}(1-q)^{-2}}{(1-q)^{-2}s^{(n+1)\alpha-n\beta-2+1}} \\ &= \frac{(1-q)^{(n+1)\alpha-n\beta-2}}{(1-q)^{-2}s^{(n+1)\alpha-n\beta-2+1}} \end{aligned} $$

In q-Laplace form:

$$ \frac{(1-q)^{(n+1)\alpha-n\beta}}{s^{(n+1)\alpha-n\beta-1}}=\frac{1}{(1-q)^{-2}}\,{}_{q}\mathcal{L}_s\left[\frac{x^{(n+1)\alpha-n\beta-2}}{\Gamma_q((n+1)\alpha-n\beta-1)}\right]. $$

So equation (3.3) becomes

$$
\begin{aligned}
&{}_q\mathcal{L}_s\left[\Phi\left(x\right)\right]\\
&=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}\left(1-q\right)^{-2}}\,{}_q\mathcal{L}_s\left[\frac{x^{(n+1)\alpha-n\beta-2}}{\Gamma_q\left((n+1)\alpha-n\beta-1\right)}\right]_q\mathcal{L}_s\left[h\left(x\right)\right]\\
&=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}\left(1-q\right)^{-2}}\,{}_q\mathcal{L}_s\left[\frac{x^{(n+1)\alpha-n\beta-2}}{\Gamma_q\left((n+1)\alpha-n\beta-1\right)}h\left(x\right)\right]\\
&={}_q\mathcal{L}_s\left[\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}\left(1-q\right)^{-2}}\frac{x^{(n+1)\alpha-n\beta-2}}{\Gamma_q\left((n+1)\alpha-n\beta-1\right)}h\left(x\right)\right],
\end{aligned}
$$

where $\eta=(n+1)\alpha-n\beta-1$.

$$
{}_q\mathcal{L}_s\left[\Phi\left(x\right)\right]={}_q\mathcal{L}_s\left[\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}\left(1-q\right)^{-2}}\frac{x^{(\eta-1)}}{\Gamma_q\left(\eta\right)}h\left(x\right)\right].
$$

Applying inverse Laplace on both sides, we obtain

$$
\begin{aligned}
\Phi\left(x\right)&=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n\left(1-q\right)^2}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}\Gamma_q\left(\eta\right)}\frac{1}{1-q}\int_0^x x^{\eta-1}\left(qt/x;q\right)_{\eta-1}h(t)d_qt\\
&=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n\left(1-q\right)}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}}\frac{x^{\eta-1}}{\Gamma_q\left(\eta\right)}\int_0^x\left(qt/x;q\right)_{\eta-1}h(t)d_qt\\
&=\left(1-q\right)\sum_{n=0}^{\infty}\left(-1\right)^n\frac{\left(Q\Gamma_q\left(1-\beta\right)\right)^n}{\left(P\Gamma_q\left(1-\alpha\right)\right)^{n+1}}I_{q,0}^{\eta}h(x).
\end{aligned}
$$

In compact form:

$$
\Phi\left(x\right)=\left(1-q\right)\sum_{n=0}^{\infty}c_nI_{q,0}^{\eta}h(x).
$$

where $c_n=\left(-1\right)^n\frac{\left(B\Gamma_q(1-\beta)\right)^n}{\left(A\Gamma_q(1-\alpha)\right)^{n+1}}$,

4. n-Terms q-Abel integral equation

Now let us derive the solution of the following n-term q-Abel integral equation:

$$
\begin{aligned}
\frac{1}{1-q}\int_0^x P_1x^{-\alpha_1}\left(qt/x;q\right)_{-\alpha_1}\Phi\left(t\right)d_qt+\cdots\\
+\frac{1}{1-q}\int_0^x P_nx^{-\alpha_n}\left(qt/x;q\right)_{-\alpha_n}\Phi\left(t\right)d_qt=h\left(x\right).
\end{aligned}
\tag{4.1}
$$

In q-convolution form:

$$
P_1x^{-\alpha_1}*_q\Phi\left(x\right)+\cdots+P_nx^{-\alpha_n}*_q\Phi\left(x\right)=h\left(x\right).
$$

Applying Laplace transform on both sides, we obtain

$$
P_{1q}\mathcal{L}_s\left[x^{-\alpha_1}\right]\Phi\left(s\right)+\cdots+P_{nq}\mathcal{L}_s\left[x^{-\alpha_n}\right]\Phi\left(s\right)=H\left(s\right).
\tag{4.2}
$$

Since

$$_q\mathcal{L}_s\left[x^{-\alpha_1}\right]=\frac{(1-q)^{-\alpha_1}\,\Gamma_q\left(1-\alpha_1\right)}{s^{1-\alpha_1}}$$

and

$$_q\mathcal{L}_s\left[x^{-\alpha_n}\right]=\frac{(1-q)^{-\alpha_n}\,\Gamma_q\left(1-\alpha_n\right)}{s^{1-\alpha_n}}.$$

After some basic steps we can obtain the following solution:

$$\Phi\left(x\right)=(1-q)\sum_{n=0}^{\infty}C_nI_{q,0}^{\eta_n}h\left(x\right).$$

Where $C_n=(-1)^n\frac{(P_2\Gamma_q(1-\alpha_2)\cdots P_n\Gamma_q(1-\alpha_n))^n}{(P_1\Gamma_q(1-\alpha_1))^{n+1}}$.

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