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PROPERTIES OF SOLUTIONS FOR A CLASS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to present new results on three classical questions related to the asymptotic behavior of solutions to a certain kind of third order neutral delay differential equations. Results are obtained for the asymptotic stability, the boundedness and the square integrability of the solutions.

1. Introduction

Recently, there has been many investigations on the asymptotic behavior of nonlinear neutral delay differential equations. In this article, we consider the following differential equations

$$\begin{aligned} x'''(t) + \phi(t)x'''(t-r) \quad &+h(x'(t))x''(t) + g(x'(t)) + f(x(t-r)) \\ &= e(t, x, x(t-r), x'(t), x'(t-r), x''), \end{aligned} \tag{1.1}$$

for all $t \geq t_1 \geq t_0 + r$, where $r > 0$. We assume that the functions $h(x'(t)), g(x'(t)))$, $f(x)$, $\phi(t)$ and $e(\cdot) := e(t, x(t), x(t-r), x'(t), x'(t-r), x''(t))$ are continuous in their respective arguments. It is also supposed that the derivatives $f'(x), g'(y)$, and $\phi'(t)$ are continuous for all x, y with $f(0) = g(0) = 0, 0 \leq \phi(t) < 1$ and $-\alpha \leq \phi'(t) \leq 0$.

It is implicitly assumed that a solution for equation (1.1) is a continuous function $x(t) \in C^3([t_x, \infty), \mathbb{R})$ satisfying equation (1.1) on $[t_x, \infty)$.

Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and defined on some ray $[t_x, \infty)$. Moreover, we tacitly assume that (1.1) possesses such solutions.

The study of the asymptotic behavior of solutions of equations of the form (1.1) has received much less attention, which is due mainly to the technical difficulties arising in its analysis. In many references, authors dealt with questions related to this kind of equations, see for example [3, 4, 10, 15, 16, 18–21, 23, 24, 28, 29].

The object of this paper is to establish sufficient conditions for the asymptotic stability of (1.1) for the case $e(\cdot) = 0$, the boundedness and the square integrability of solutions of (1.1) for the case $e(\cdot) \neq 0$. By the construction of suitable Lyapunov function, this results are obtained. This technique permits us to eliminate some restrictions that are usually imposed on the coefficients of the studied neutral differential equations.

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2. Assumptions and main results

This section contains the major results of the paper.

2.1. **Assumptions.**

In this subsection, we make the assumptions and notations we will use in the sequel. For the sake of convenience, we insert the next notation:

$$\begin{aligned} \Delta(t) &= x'''(t) + \phi(t)x'''(t-r), \\ Y &= x'(t) + \phi(t)x'(t-r), \\ Z &= x''(t) + \phi(t)x''(t-r). \end{aligned}$$

Equation (1.1) is reformulated as the equivalent system

$$\begin{cases} x' = y, \\ y' = z, \\ Z' = \phi'(t)z(t-r) - h(y)z - \ g(y) - f(x) + \displaystyle\int_{t-r}^{t} \ f'(x(s)y(s)ds + e(\cdot). \end{cases} \tag{2.1}$$

To arrive to the desired results, suppose that the following conditions which will be used on the functions that appeared in equation (1.1) are satisfied

i) $h_0 < h(y) < h_1$,

ii) $\dfrac{f(x)}{x} \geq M > 0$ $(x \neq 0)$, and $|f'(x)| \leq \delta$ for all x,

iii) $d^2 < d_0 \leq \dfrac{g(y)}{y} \leq d_1$,

iv) $\dfrac{\delta}{2} < d < h_0$,

v) $\displaystyle\int_{t_1}^{t} \left|\phi'(s)\right| ds < \rho$,

where $d_0, d_1, d, M, \delta, h_0, h_1$ and ρ are positive constants.

2.2. **Results.**

Here, we state our main results.
For the case $e(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) \equiv 0$, we state

Theorem 2.1. *In addition to the hypothesis (i)-(v), suppose there exist positive constants $\varepsilon, \eta_1, \eta_2$ and c such that the following is also satisfied*

$$-dd_0 + \delta + (1+c)^2 + c\left(\frac{d_1^2}{2} + \delta\right) + \frac{3\alpha}{2} \quad = \quad -\eta_1, \tag{2.2}$$

$$c\left(h_1 - d + \frac{1}{2}\right) + \frac{\alpha\,(1+d)}{2} + (1+c)^2 - (h_0 - d) + \frac{\alpha}{2} + \varepsilon \quad = \quad -\eta_2, \tag{2.3}$$

where

$$c = \phi(t_1) \geq \phi(t) \text{ for all } t \geq t_1.$$

Then trivial solution of (1.1) is asymptotically stable, provided

$$r < \frac{2}{\delta}\min\left\{\eta_2, \frac{\eta_1}{1+c+2d}, \frac{\varepsilon}{c}\right\}.$$

With respect to $e(t, x, y, x(t-r), y(t-r), z(t)) \neq 0$, our first result goes as follows:

Theorem 2.2. *Assume that all the conditions of Theorem 2.1 are satisfied and there exist positive constants* q_1 *and* q_2 *such that :*

I_1) $|e(t,x,y,x(t-r),y(t-r),z(t))| \leq q(t) < q_1$,

I_2) $\left|\int_0^t q(s)ds\right| < q_2$,

then, there exists a positive constant D*, such that any solution* $x(t)$ *of (1.1) satisfies*

$$|x(t)| \leq D, \ |y(t)| \leq D, \ |Z(t)| \leq D. \tag{2.4}$$

In the following Theorem, we are concerned with the square integrability of solutions to equation (1.1).

Theorem 2.3. *If conditions (i)–(v), (I_1) and (I_2) hold, then for any solution* x *of (1.1)*

$$\int_{t_0}^{\infty}\Big(x''^2(s) + x'^2(s) + x^2(s)\Big)ds < \infty.$$

3. Proofs and examples

Now, we will firstly focus our interest into proving the stated results. Next, we will give an example showing the applicability of the obtained results.

3.1. Proofs.

Proof of Theorem 2.1. The proof of this theorem depends on properties of the continuously differentiable function $W = W(t, x_t, y_t, z_t)$, defined by

$$W(t) = V \cdot \Omega(t), \tag{3.1}$$

where

$$\Omega(t) = e^{-\frac{1}{\omega}\int_{t_1}^{t}|\phi'(s)|\,ds},$$

$$V = V_1 + V_2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \tag{3.2}$$

$$V_1 = dF(x) + f(x)Y + Y^2, \tag{3.3}$$

$$V_2 = \frac{1}{2}Z^2 + dyZ + \int_0^y (\ g(u) + dh(u)u)\,du + \int_{t-r}^{t}\left(\mu_1 y^2(s) + \mu_2 z^2(s)\right)ds, \tag{3.4}$$

$$F(x) = \int_0^x f(u)du,$$

ω, λ, μ_1 and μ_2 are positive constants to be specified later in the proof.
By noting that

$$2\int_0^x f'(u)f(u)du = f^2(x).$$

and using (iv), we have

$$\begin{aligned} V_1 &= d\int^x f(u)du + (Y+\frac{1}{2}f(x))^2 \ -\frac{1}{4}f^2(x) \\ &\geq d\int_0^x f(u)du - \frac{1}{2}\int_0^x f'(u)f(u)du \\ &\geq \int_0^x \left(d-\frac{\delta}{2}\right) f(u)du \\ &\geq \left(d-\frac{\delta}{2}\right) F(x). \end{aligned}$$

Condition (ii) implies

$$F(x) = \int_0^x f(u)du = \int_0^x \frac{f(u)}{u} udu \geq \frac{1}{2}Mx^2.$$

Hence

$$V_1 \geq \frac{M}{2}\left(d-\frac{\delta}{2}\right)x^2,$$

Since

$$\int_{t-r}^t \left(\mu_1 y^2(s) + \mu_2 z^2(s)\right) ds \geq 0,$$

then

$$V_2 \geq \frac{1}{2}Z^2 + dyZ + \int_0^y (\ g(u) + dh(u)u)\, du.$$

It follows from (iv) that

$$\int_0^y (\ g(u) + dh(u)u)\, du \geq \left(\frac{d_0}{2} + \frac{dh_0}{2}\right) y^2,$$

and from (iii) also, that

$$\begin{aligned} V_2 &\geq \frac{1}{4}\left(dy+Z\right)^2 + \frac{d_0}{4}\left(y+\frac{d}{d_0}Z\right)^2 + \left(\frac{1}{4}\left(d_0-d^2\right) + \frac{dh_0}{2}\right)y^2 + \frac{1}{4}\left(1-\frac{d^2}{d_0}\right)Z^2 \\ &\geq \left(\frac{1}{4}\left(d_0-d^2\right) + \frac{dh_0}{2}\right)y^2 + \frac{1}{4}\left(1-\frac{d^2}{d_0}\right)Z^2. \end{aligned}$$

Since $\frac{1}{4}\left(dy+Z\right)^2 + \frac{d_0}{4}\left(y+\frac{d}{d_0}Z\right)^2 \geq 0$, we can exhibit a positive k_0 which satisfy the following

$$V_2 \geq k_0(y^2 + Z^2),$$

where

$$k_0 = \min\left\{\frac{dh_0}{2} + \frac{1}{4}\left(d_0-d^2\right), \frac{1}{4}\left(1-\frac{d^2}{d_0}\right)\right\}.$$

Thus,

$$V \geq k_1(x^2+y^2+Z^2), \tag{3.5}$$

where

$$k_1 = \min\left\{\frac{M}{2}\left(d-\frac{\delta}{2}\right), k_0\right\}.$$

At last, from (v), there exists a positive constant K_0, which proves the positivity

$$W \geq K_0(x^2 + y^2 + Z^2). \tag{3.6}$$

From (2.1), one can remark the equalities

$$\begin{aligned} Y' &= Z + \phi'(t)y(t-r), \\ Z' - \phi'(t)z(t-r) &= \Delta(t). \end{aligned}$$

Now, the time derivative of the functional (3.1) along the system (2.1), leads to

$$W' = \Omega(t) \cdot \left(V' - \frac{1}{\omega}|\phi'(t)|V \right), \tag{3.7}$$

where

$$V'_{(2.1)} = U_1 + U_2 + U_3,$$

$$\begin{aligned} U_1 &= \left[f'(x) + \mu_1 + \lambda r\right] y^2 - dg(y)y + \left[\mu_2 + (d - h(y))\right] z^2 \\ &\quad -\mu_1 y^2(t-r) - \mu_2 z^2(t-r) + \phi(t)\left[d - h(y)\right] z(t-r)z \\ &\quad +\phi(t)f'(x)yy(t-r) + 2\phi(t)yz(t-r) + 2\phi(t)y(t-r)z \\ &\quad +2yz - \phi(t)z(t-r)g(y) + 2\phi^2(t)y(t-r)z(t-r), \end{aligned}$$

$$\begin{aligned} U_2 &= \phi'(t)\left[f(x)y(t-r) + 2yy(t-r) + 2\phi(t)y^2(t-r)\right] \\ &\quad +\phi'(t)\left[z(t-r)z + \phi(t)z^2(t-r) + dz(t-r)y\right], \end{aligned}$$

and

$$U_3 = \left[dy + z + \phi(t)z(t-r)\right] \int_{t-r}^{t} f'(x(s)y(s)ds - \lambda \int_{t-r}^{t} y^2(s)ds.$$

Apply the assumption $|f(x)| < \delta\,|x|$ and the inequality $2uv \leq u^2 + v^2$, to get

$$\begin{aligned} U_1 &\leq \left[-dd_0 + 1 + \frac{c\delta}{2} + c + \delta + \frac{cd_1^2}{2} + \mu_1 + \lambda r\right] y^2 \\ &\quad + \left[-\mu_1 + c + \frac{c\delta}{2} + c^2\right] y^2(t-r) \\ &\quad + \left[\mu_2 + 1 - (h_0 - d) + c\left(\frac{h_1 - d}{2} + 1\right)\right] z^2 \\ &\quad + \left[-\mu_2 + c\left(c + \frac{3}{2} + \frac{h_1 - d}{2}\right)\right] z^2(t-r). \end{aligned}$$

In the same spirit and by the fact that $\phi'(t) \leq 0$ and $\phi(t) \leq c$, we obtain

$$\begin{aligned} U_2 &\leq \frac{\alpha}{2}z^2 + \frac{3\alpha}{2}y^2(t-r) + \frac{\alpha\,(1+d)}{2}z^2(t-r) \\ &\quad +|\phi'(t)|\left(\frac{\delta^2}{2}x^2 + \left[1 + \frac{d}{2}\right]y^2\right), \end{aligned}$$

and

$$U_3 \leq \frac{d\delta r}{2}y^2 + \frac{\delta r}{2}z^2 + \frac{c\delta r}{2}z^2(t-r) + \left[\frac{\delta}{2}(1 + c + d) - \lambda\right] \int_{t-r}^{t} y^2(s)ds.$$

So, after rearrangement

$$\begin{aligned} V' &\leq \left[-dd_0+\delta+1+c\left(1+\frac{d_1^2}{2}+\frac{\delta}{2}\right)+\mu_1+\left(\frac{d\delta}{2}+\lambda\right)r\right]y^2 \\ &+\left[\mu_2+1-(h_0-d)+c\left(1+\frac{h_1-d}{2}\right)+\frac{\alpha}{2}+\frac{\delta}{2}r\right]z^2 \\ &+\left[-\mu_1+c\left(1+\frac{\delta}{2}+c\right)+\frac{3\alpha}{2}\right]y^2(t-r) \\ &+\left[-\mu_2+c\left(c+\frac{h_1-d}{2}+\frac{3}{2}\right)+\frac{\alpha\,(1+d)}{2}+\frac{c\delta}{2}r\right]z^2(t-r) \\ &+\left[\frac{\delta}{2}\,(1+c+d)-\lambda\right]\int_{t-r}^{t}y^2(s)ds \\ &+|\phi'(t)|\left(\frac{\delta^2}{2}x^2+\left[1+\frac{d}{2}\right]y^2\right). \end{aligned}$$

Choose

$$\begin{aligned} \mu_1 &= c\left(1+\frac{\delta}{2}+c\right)+\frac{3\alpha}{2}, \\ \mu_2 &= c\left(c+\frac{h_1-d}{2}+\frac{3}{2}\right)+\frac{\alpha\,(1+d)}{2}+\varepsilon, \end{aligned}$$

and

$$\lambda=\frac{\delta}{2}\,(1+c+d)\,.$$

With this choice of constants, we get

$$\begin{aligned} V' &\leq \left[-dd_0+\delta+(1+c)^2+c\left(\frac{d_1^2}{2}+\delta\right)+\frac{3\alpha}{2}+[\delta\,(1+c+2d)]\,\frac{r}{2}\right]y^2 \\ &+\left[c\left(h_1-d+\frac{1}{2}\right)+\frac{\alpha\,(1+d)}{2}+(1+c)^2-(h_0-d)+\frac{\alpha}{2}+\varepsilon+\frac{\delta r}{2}\right]z^2 \\ &+\left[-\varepsilon+\frac{c\delta}{2}r\right]z^2(t-r) \\ &+|\phi'(t)|\left(\frac{\delta^2}{2}x^2+\left[1+\frac{d}{2}\right]y^2\right). \end{aligned}$$

Combining (2.2) and (2.3) with the previous formula, we obtain

$$\begin{aligned} V' &\leq \left[-\eta_1+[\delta\,(1+c+2d)]\,\frac{r}{2}\right]y^2+\left[-\eta_2+\delta\frac{r}{2}\right]z^2 \\ &\quad+\left[-\varepsilon+c\delta\frac{r}{2}\right]z^2(t-r)+|\phi'(t)|\left(\frac{\delta^2}{2}x^2+\left[1+\frac{d}{2}\right]y^2\right). \end{aligned}$$

If

$$r<\frac{2}{\delta}\min\left\{\eta_2,\frac{\eta_1}{1+c+2d},\frac{\varepsilon}{c}\right\},$$

then

$$V'_{(2.1)} \leq -K_1 \left(y^2(t) + z^2(t)\right) + \left|\phi'(t)\right| \left(1 + \frac{d}{2}\right) y^2(t) + \frac{|\phi'(t)|}{2}\delta^2 x^2(t),$$

where

$$K_1 = \min\left\{\eta_1 - \frac{\delta\,(1 + c + 2d)}{2} r, \eta_2 - \frac{\delta}{2} r\right\}.$$

From (3.7), we observe

$$\begin{aligned} W'_{(2.1)} &= \left(V' - \frac{|\phi'(t)|}{\omega} V\right)\Omega(t) \\ &\leq \left(-K_1\left[y^2(t) + z^2(t)\right] + \left|\phi'(t)\right|\left(1 + \frac{d}{2}\right) y^2(t)\right)\Omega(t) \\ &\quad + \left(\frac{|\phi'(t)|}{2}\delta^2 x^2(t) - \frac{k_1\left|\phi'(t)\right|}{\omega}(x^2 + y^2 + Z^2)\right)\Omega(t) \\ &\leq \left(-K_1\left[y^2(t) + z^2(t)\right] + K_2\left|\phi'(t)\right|(x^2 + y^2) - \frac{k_1\left|\phi'(t)\right|}{\omega}(x^2 + y^2 + Z^2)\right)\Omega(t), \end{aligned}$$

where $K_2 = \max\left\{1 + \frac{d}{2}, \frac{\delta^2}{2}\right\}$.

By the use of (v), we obtain

$$e^{-\frac{\rho}{\omega}} < \Omega(t) < 1.$$

Choosing $\omega = \frac{k_1}{K_2}$, we conclude that

$$W'_{(2.1)} \leq -K_1 e^{-\frac{\rho}{\omega}}\left[y^2(t) + z^2(t)\right]. \tag{3.8}$$

From (3.8), $W_3(\|X\|) = K_1 e^{-\frac{\rho}{\omega}}\left[y^2(t) + z^2(t)\right]$ is positive definite function. The above discussion guarantees that the trivial solution of equation (1.1) is asymptotically stable and completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. For the case $e(t, x, y, x(t-r), y(t-r), z(t)) = e(\cdot) \neq 0$, equation (1.1) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = z, \\ Z' = \phi'(t)z(t-r) - h(y)z - \ g(y) - f(x) + e(\cdot) + \displaystyle\int_{t-r}^{t} \ f'(x(s)y(s)ds. \end{cases} \tag{3.9}$$

On differentiating (3.1) along the system (3.9), we obtain

$$W'_{(3.9)} = W'_{(2.1)} + (Ze(\cdot) + dye(\cdot))\, e^{-\frac{1}{\omega}\int_{t_1}^{t}\left|\phi'(s)\right| ds}.$$

With the usage of condition (I_1), and (3.8) we have

$$W'_{(3.9)} \leq q(t)|Z| + dq(t)|y|,$$

Now, the inequality $|u| \leq\ u^2+1$, lead

$$\begin{aligned} W'_{(3.9)} &\leq K_3 q(t)\left[y^2+Z^2+2\right] \\ &\leq K_3 q(t)\left[x^2+y^2+Z^2+2\right], \end{aligned} \tag{3.10}$$

where $K_3=\max\{1,d\}$.
In view of (3.6), the above estimates imply

$$W'_{(3.9)} \leq \frac{K_3}{K_0} q(t) W + K_4 q(t), \tag{3.11}$$

with $K_4 = 2K_3$. Integrating both sides of (3.11) from t_1 to t, we easily obtain

$$W(t) - W(t_1) \leq K_4 \int_{t_1}^{t} q(s)ds + \frac{K_3}{K_0}\int_{t_1}^{t} W(s)q(s)ds.$$

Thus

$$W(t) \leq q_3 + \frac{K_3}{K_0}\int_{t_1}^{t} W(s)q(s)ds,$$

where

$$q_3 = W(t_1) + K_4 q_2. \tag{3.12}$$

By using Gronwall inequality, it follows

$$W(t) \leq q_3 \exp\left(\frac{K_3}{K_0}\int_{t_1}^{t} q(s)ds\right) \leq q_4, \tag{3.13}$$

where $q_4 = q_3 \exp\left(\frac{K_3}{K_0} q_2\right)$. This result implies that there exists a constant D such that

$$|x(t)| \leq D,\ |y(t)| \leq D, |Z(t)| \leq D.$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Define the functionnal

$$H(t) = W(t) + \xi \int_{t_1}^{t} (z^2(s) + y^2(s))ds. \tag{3.14}$$

where $\xi > 0$ is a constant to be specified later.
By differentiating $H(t)$ and using (3.11), we obtain

$$H'(t) \leq \left[\xi - K_1 e^{-\dfrac{\rho}{\omega}}\right](z^2(s)+y^2(s)) + \frac{K_3}{K_0} q(t) W + K_4 q(t).$$

If we Choose $\xi - K_1 e^{-\dfrac{\rho}{\omega}} < 0$, then from (3.13), we get

$$H'(t) \leq K_5 q(t), \tag{3.15}$$

where $K_5 = \dfrac{K_3}{K_0} q_4 + K_4$.
Integrating (3.15) from t_1 to t and using condition (I_2) of Theorem 2.2, we obtain

$$H(t) - H(t_1) = \int_{t_1}^{t} H'(s)ds \leq K_5 q_2.$$

Using (3.14) and equality $H(t_1) = W(t_1)$, we get

$$H(t) \leq K_5 q_2 + q_3 - K_4 q_2.$$

We can conclude by (3.14) that

$$\int_{t_1}^{t} (z^2(s) + y^2(s))ds < \frac{K_5 q_2 + q_3 - K_4 q_2}{\xi},$$

which imply the existence of positive constants σ_1 and σ_2 such that

$$\int_{t_1}^{t} y^2(s)ds \leq \sigma_1 \text{ and } \int_{t_1}^{t} z^2(s)ds \leq \sigma_2.$$

Hence

$$\int_{t_1}^{t} x'^2(s)ds \leq \sigma_1, \int_{t_1}^{t} x''^2(s)ds \leq \sigma_2. \tag{3.16}$$

Finally, we show that $\int_{t_1}^{+\infty} x^2(s)ds < \infty$. If we multiply both sides of (1.1) by $x(t-r)$, we obtain

(3.17)

$$\left[x'''(t) + \phi(t)x'''(t-r)\right] x(t-r) + h(x'(t))x''(t)x(t-r) + g(x'(t))x(t-r)$$
$$+ f(x(t-r))x(t-r) = e(t, x, x(t-r), x'(t), x'(t-r), x'')x(t-r).$$

Integrating (3.17) from t_1 to t, gives

$$\int_{t_1}^{t} f(x(s-r))x(s-r)ds = L_1(t) + L_2(t) + L_3(t) + L_4(t), \tag{3.18}$$

where

$$\begin{aligned} L_1(t) &= -\int_{t_1}^{t} \left[x'''(s) + \phi(s)x'''(s-r)\right] x(s-r)ds, \\ L_2(t) &= -\int_{t_1}^{t} h(x'(s))x''(s)x(s-r)ds, \\ L_3(t) &= -\int_{t_1}^{t} g(x'(s))x(s-r)ds, \end{aligned}$$

and

$$L_4(t) = \int_{t_1}^{t} e(s, x, x(s-r), x'(t), x'(s-r), x'')x(s-r)ds.$$

We have

$$L_1(t) = -\int_{t_1}^{t} x'''(s)x(s-r)ds - \int_{t_1}^{t} \phi(s)x'''(s-r)x(s-r)ds.$$

Integrating by parts and using (3.16), we obtain

$$\begin{aligned} L_1(t) &= -\left[x(t-r)x''(t) - x(t_1-r)x''(t_1)\right] \\ &\quad - \left[\phi(t)x(t-r)x''(t-r) - \phi(t_1)x(t_1-r)x''(t_1-r)\right] \\ &\quad + \int_{t_1}^{t} x''(s)x'(s-r)ds + \int_{t_1}^{t} \phi'(s)x(s-r)x''(s-r)ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t} \phi(s)x'(s-r)x''(s-r)ds \\
\leq\ & |x(t_1-r)x''(t_1)| + D^2 + |cx(t_1-r)x''(t_1-r)| \\
&+\frac{1}{2}\int_{t_1}^{t}\left[x''^2(s)+x'^2(s-r)\right]ds \\
&+\alpha\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}}\left(\int_{t_1}^{t} x''^2(s-r)ds\right)^{\frac{1}{2}} \\
&+\frac{c}{2}\int_{t_1}^{t}\left[x''^2(s-r)+x'^2(s-r)\right]ds.
\end{aligned}
$$

Consequently

$$
L_1(t) \leq l_1 + \alpha\sqrt{\sigma_2}\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}},
$$

where

$$
l_1 = L + D^2 + \frac{1+c}{2}\left[\sigma_2+\sigma_1\right] \text{ and } L = |x(t_1-r)|\left(\left|x''(t_1)\right| + \left|cx''(t_1-r)\right|\right).
$$

In the same way, after using (i)-(iii), (I_1), (I_2) and (3.16), one arrives at

$$
\begin{aligned}
L_2(t) &\leq \int_{t_1}^{t}\left|h(x'(s))x''(s)x(s-r)\right|ds \\
&\leq \left(\int_{t_1}^{t}\left[h\left(x'(s)\right)x''(s)\right]^2 ds\right)^{\frac{1}{2}}\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}} \\
&\leq l_2\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}}, \\
L_3(t) &\leq \int_{t_1}^{t}\left|g(x'(s))x(s-r)\right|ds \\
&\leq \left(\int_{t_1}^{t}\left[g(x'(s))\right]^2 ds\right)^{\frac{1}{2}}\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}} \\
&\leq \left(d_1^2\int_{t_1}^{t} x'^2(s)ds\right)^{\frac{1}{2}}\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}} \\
&\leq l_3\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}}, \\
L_4(t) &\leq \int_{t_1}^{t}\left|e(s,x,y,x(s-r),x'(s-r),x''(s))x(s-r)\right|ds \\
&\leq D\int_{t_1}^{t}|q(s)|\,ds \leq l_4,
\end{aligned}
$$

where

$$
l_2 = \sqrt{h_1^2\sigma_2}, \quad l_3 = \sqrt{d_1^2\sigma_1}, \quad \text{and} \quad l_4 = Dq_2.
$$

In the other hand from condition (ii), we have

$$\int_{t_1}^{t} x(s-r)f(x(s-r))ds \geq M\int_{t_1}^{t} x^2(s-r)ds.$$

Hence, by (3.18), we obtain

$$\begin{aligned} M\int_{t_1}^{t} x^2(s-r)ds &\leq l_1+\sqrt{\alpha^2\sigma_2}\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}}+l_2\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}} \\ &\quad +l_3\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}}+l_4. \end{aligned} \tag{3.19}$$

If

$$\int_{t_1}^{t} x^2(s-r)ds \to \infty \quad as \quad t\to\infty,$$

then dividing both sides of (3.19) by $\left(\int_{t_1}^{t} x^2(s-r)ds\right)^{\frac{1}{2}}$, we immediately obtain a contradiction. Hence, we deduce that $\int_{t_1}^{t} x^2(s-r)ds<\infty$, then $\int_{t_1}^{+\infty} x^2(s)ds<\infty$. This fact completes the proof of Theorem 2.3. □

3.2. **Example.** As a particular case of (1.1), consider the following third order neutral differential equation

$$\begin{aligned} &\left(x'''(t)+\frac{1}{10}e^{-\frac{10}{3}t}x'''(t-r)\right)+\left(\frac{11}{2}+\frac{1}{2}\sin x'(t)\right)x''(t) \\ &\quad +\frac{91}{20}x'(t)+\frac{1}{20}x'(t)\cos x'(t)+\left[\frac{19}{10}x(t-r)+\frac{x(t-r)}{\sqrt{10}+|x(t-r)|}\right] \\ &\quad =\frac{1}{1+t^2+|x|+|y|+|z|}. \end{aligned}$$

The conditions over the functions appearing in the example are as below

$$5=h_0\leq h(y)=\frac{11}{2}+\frac{1}{2}\sin y\leq h_1=6,$$

$$4=d^2<d_0=4.5\leq\frac{g(y)}{y}=\frac{91}{20}+\frac{1}{20}\cos y\leq d_1=4.6,$$

$$0<\phi(t)=\frac{1}{10}e^{-\frac{10}{3}t}\leq\frac{1}{10}=c,$$

$$\phi'(t)=-\frac{1}{3}e^{-\frac{10}{3}t}<0 \text{ and } \left|\phi'(t)\right|=\left|-\frac{1}{3}e^{-\frac{10}{3}t}\right|\leq\frac{1}{3}=\alpha,$$

and

$$1.25=\frac{\delta}{2}<d=2<h_0=5.$$

From the formula of the function

$$f(x)=\frac{19}{10}x+\frac{x}{\sqrt{10}+\mid x\mid}.$$

it is clear that, $f(0) = 0$, and since $0 < \dfrac{1}{\sqrt{10} + \mid x \mid} < 1$ for all x, we have that

$$\frac{f(x)}{x} \geq \frac{19}{10} = M,$$

for all $x \neq 0$. Moreover

$$|f'(x)| = \left| \frac{19}{10} + \frac{\sqrt{10}}{(\sqrt{10} + \mid x \mid)^2} \right| \leq 2.5 = \delta.$$

We also have

$$-dd_0 + \delta + 1 + c\left(2 + c + \frac{d_1^2}{2} + \delta\right) + \frac{3\alpha}{2} = -3.48 = -\eta_1,$$

$$c\left(c + h_1 - d + \frac{5}{2}\right) + \frac{\alpha\,(1+d)}{2} + 1 - (h_0 - d) + \frac{\alpha}{2} + \varepsilon = -0.57 = -\eta_2, \text{ for } \varepsilon = \frac{1}{10}.$$

The function

$$e(t, x, y, z) = \frac{1}{1 + t^2 + |x| + |y| + |z|} \leq \frac{1}{1 + t^2} = q(t),$$

and

$$\int_0^{+\infty} |q(t)| dt < \infty,$$

for all t, x, y, z. All assumptions of Theorem 2.3 hold true, thus, the conclusions also follow.

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