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PERIODIC MILD SOLUTIONS OF INFINITE DELAY INTEGRO-DIFFERENTIAL INCLUSIONS WITH NON INSTANTANEOUS IMPULSES

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Abstract. In this paper, we investigates the existence of periodic mild solutions for a class of impulsive Integro-differential inclusions. We base our arguments on fixed point theory paired with the approach of measure of noncompactness using the resolvent operator. Finally, an illustration of our results is presented.

1. Introduction

In recent years, there has been substantial progress in functional evolution equations; see, for example, the papers [2, 3, 30]. In [1], an iterative approach is utilized to find mild solutions of evolution equations. Olszowy and Wcedrychowicz [23] examined a class of evolution equations over unbounded intervals using Tichonov's fixed point theorem. Liang *et al.* [20] investigated the existence of periodic mild solutions in Banach spaces to a class of impulsive differential equations with infinite delay. Although, in prior publications, several constraints were assumed. Various researchers have recently gotten further results by using the approach of measure of noncompactness; see [4, 5, 29], and the sources within.

In recent years, impulsive differential inclusions have gained much importance in several mathematical models of real phenomena, particularly in biological or medical fields, as well as in control theory. Recent researches and results on impulsive differential inclusions and equations can be found in the monographs [7–9, 15], and the papers [26, 28].

In [25, 27], the authors investigated several types of impulsive differential equations with non-instantaneous impulses.

Motivated by the preceding articles, in this paper, we consider the following problem:
(1.1)
$$\begin{cases} \mathfrak{p}'(\vartheta) - \mathcal{Z}\mathfrak{p}(\vartheta) - \int_0^{\vartheta} \mathfrak{N}(\vartheta-\varrho)\mathfrak{p}(\varrho)d\varrho \in \Psi(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{p}_{\vartheta});\ \vartheta \in \Theta_j,\ j = 0, \dots, \\ \mathfrak{p}(\vartheta) = \Phi_j(\vartheta, \mathfrak{p}(\vartheta_j^-));\ \vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \\ \mathfrak{p}(\vartheta) = \psi(\vartheta);\ \text{ if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases}$$

where $\Theta_0 := [0, \vartheta_1],\ \tilde{\Theta}_j := (\vartheta_j, \varrho_j],\ \Theta_j := (\varrho_j, \vartheta_{j+1}];\ j = 1, \dots, 0 = \varrho_0 < \vartheta_1 \le \varrho_1 < \vartheta_2 \le \varrho_2 < \cdots \le \varrho_{\nu-1} < \vartheta_\nu \le \varrho_\nu < \vartheta_{\nu+1} = \varsigma \le \varrho_{\nu+1} < \vartheta_{\nu+2} \le \cdots < +\infty, \Psi :$

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$\Theta_j \times \Xi \times \chi \to \mu(\Xi)$; $j = 0, \ldots,$ is $\varsigma-$periodic compact multivalued map, ϑ, $\varsigma > 0$, is the family of subsets of Ξ is denoted by $\mu(\Xi)$, χ is a phase space given in the sequel, $\Phi_j : \tilde{\Theta}_j \times \Xi \to \Xi$ are given functions, and $\varsigma-$periodic in ϑ, $\varsigma > 0, \psi : \mathbb{R}_- \to \Xi$ is a given function, and $(\Xi, \| \cdot \|)$ is a Banach space, $\mathfrak{p}'(\vartheta) := \frac{d\mathfrak{p}}{d\vartheta}$, $\mathcal{Z} : D(\mathcal{Z}) \subset \Xi \to \Xi$ generates a C_0-semigroup on the Banach space Ξ, $\mathfrak{N}(\vartheta)$ is a closed linear operator on Ξ, and $\varsigma-$periodic in ϑ, $\varsigma > 0$, with $D(\mathcal{Z}) \subset D(\mathfrak{N})$. For each continuous function $\mathfrak{p}$ and any $\vartheta \in \mathbb{R}_+$, $\mathfrak{p}_\vartheta$ is the element of χ given by $\mathfrak{p}_\vartheta(\varepsilon) = \mathfrak{p}(\vartheta + \varepsilon)$ for $\varepsilon \in \mathbb{R}_-$.

The following is an outline of the paper's structure. Section 2 presents some preliminary results. Section 3 introduces the main results, which are proved by applying Darbo fixed point theorem, Poincaré operator and the notion of measure of noncompactness in Banach spaces, while Section 4 provides an illustration.

2. Preliminaries

Let $\Theta := [0, \varsigma]$, $\varsigma > 0$, and $\Upsilon(\Xi)$ the Banach space of the bounded linear operators from Ξ into Ξ, with the norm

$$\|\mathfrak{T}\|_{\Upsilon(\Xi)} = \sup_{\|\mathfrak{p}\|=1} \|\mathfrak{T}(\mathfrak{p})\|.$$

By $L^1(\Theta, \Xi)$, we denotes the Banach space of measurable functions $\mathfrak{p} : \Theta \to \Xi$ which are Bochner integrable with the norm

$$\|\mathfrak{p}\|_{L^1} = \int_\Theta \|\mathfrak{p}(\vartheta)\| d\vartheta.$$

Let $\mathfrak{F}(\Theta) := C(\Theta, \Xi)$ be the Banach space of all continuous functions from $\Theta := [0, \varsigma]$ into Ξ with the norm

$$\|\mathfrak{p}\|_\infty = \sup_{\vartheta \in \Theta} \|\mathfrak{p}(\vartheta)\|.$$

Let $L^\infty(\Theta)$ be the Banach space of measurable functions $v : \Theta \to \mathbb{R}$ that are essentially bounded with the norm

$$\|\mathfrak{p}\|_{L^\infty} = \inf\{\varepsilon > 0 : |\mathfrak{p}(\vartheta)| \le \varepsilon,\ a.e.\ \vartheta \in \Theta\}.$$

Consider the space

$$\begin{aligned}
&\tilde{\mathfrak{F}}((-\infty, 0], \Xi) \\
&\quad = \{\mathfrak{p} : (-\infty, 0] \to \Xi :\ \mathfrak{p} \text{ is continuous and there exist } \ \tau_j \in (-\infty, 0); \\
&\qquad j = 1, \ldots, \nu,\ \text{where}\ \ \mathfrak{p}(\tau_j^-)\ \text{ and } \mathfrak{p}(\tau_j^+) \text{ exist with } \mathfrak{p}(\tau_j^-) = \mathfrak{p}(\tau_j)\big\}.
\end{aligned}$$

Consider the space

$$\begin{aligned}
\mathcal{P}_c = &\big\{\mathfrak{p} : (-\infty, \varsigma] \to \Xi : \mathfrak{p}|_{\mathbb{R}_-} \in \chi,\ \mathfrak{p}|_{\tilde{\Theta}_j} = \Phi_j;\ j = 1, \ldots, \nu,\ \mathfrak{p}|_{\Theta_j};\ j = 1, \ldots, \nu \\
&\text{is continuous and there exist } \mathfrak{p}(\varrho_j^-),\ \mathfrak{p}(\varrho_j^+),\ \mathfrak{p}(\vartheta_j^-)\ \text{ and }\ \mathfrak{p}(\vartheta_j^+) \\
&\text{with }\ \mathfrak{p}(\varrho_j^+) = \Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-))\ \text{ and } \mathfrak{p}(\vartheta_j^-) = \Phi_j(\vartheta_j, \mathfrak{p}(\vartheta_j^-))\big\},
\end{aligned}$$

with the norm

$$\|\mathfrak{p}\|_{\mathcal{P}_c} = \max\{\|\mathfrak{p}\|_\infty, \|\psi\|_\chi\}.$$

Consider the space

$$\widetilde{\mathcal{P}_c} = \{\mathfrak{p} : \mathbb{R} \to \Xi : \mathfrak{p}|_{\mathbb{R}_-} \in \chi,\ \mathfrak{p}|_{\tilde{\Theta}_j} = \Phi_j;\ j = 1, \dots;\ \mathfrak{p}|_{\Theta_j};\ j = 1, \dots;$$
$$\text{is continuous and there exist } \mathfrak{p}(\varrho_j^-),\ \mathfrak{p}(\varrho_j^+),\ \mathfrak{p}(\vartheta_j^-) \text{ and } \mathfrak{p}(\vartheta_j^+)$$
$$\text{with } \mathfrak{p}(\varrho_j^+) = \Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-)) \text{ and } \mathfrak{p}(\vartheta_j^-) = \Phi_j(\vartheta_j, \mathfrak{p}(\vartheta_j^-))\}.$$

A semigroup of bounded linear operators $\mathfrak{S}(\vartheta)$ is uniformly continuous if

$$\lim_{\vartheta \to 0} \|\mathfrak{S}(\vartheta) - I\|_{\Xi} = 0,$$

where I is the identity operator in Ξ.

Note that if a semigroup $\mathfrak{S}(\vartheta)$ is of class (C_0) then it verifies the growth condition

$$\|\mathfrak{S}(\vartheta)\|_{\Upsilon(\Xi)} \leq \kappa e^{\bar{\kappa}\vartheta}, \text{ for } 0 \leq \vartheta < \infty \text{ with some constants } \kappa > 0 \text{ and } \bar{\kappa} \geq 0.$$

If, for instance $\kappa = 1$ and $\bar{\kappa} = 0$, i.e; $\|\mathfrak{S}(\vartheta)\|_{\Upsilon(\Xi)} \leq 1$, for $\vartheta \geq 0$, then the semigroup $\mathfrak{S}(\vartheta)$ is called a *contraction semigroup.*

Let $(\mathfrak{W}, \|\cdot\|)$ be a Banach space. let $\mu_{cl}(\mathfrak{W}) = \{\Lambda \in \mu(\mathfrak{W}) : \Lambda \text{ closed}\}$, $\mu_b(\mathfrak{W}) = \{\Lambda \in \mu(\mathfrak{W}) : \Lambda \text{ bounded}\}$, $\mu_{cp}(\mathfrak{W}) = \{\Lambda \in \mu(\mathfrak{W}) : \Lambda \text{ compact}\}$, $\mu_{cp,cv}(\mathfrak{W}) = \{\Lambda \in \mu(\mathfrak{W}) : \Lambda \text{ compact and convex}\}$.

Definition 2.1. A multivalued map $\mathfrak{S} : \mathfrak{W} \to \mu(\mathfrak{W})$ is convex(closed) valued if $\mathfrak{S}(\mathfrak{w})$ is convex (closed) for all $\mathfrak{w} \in \mathfrak{W}$. $\mathfrak{S}$ is bounded on bounded sets if $\mathfrak{S}(\Omega) = \cup_{\mathfrak{w}\in\Omega}\mathfrak{S}(\mathfrak{w})$ is bounded in $\mathfrak{W}$ for all $\Omega \in \mu_b(\mathfrak{W})$ (i.e. $\sup_{\mathfrak{w}\in\Omega}\{\sup\{|\lambda| : \lambda \in \mathfrak{S}(\mathfrak{w})\}\} < \infty$). $\mathfrak{S}$ is called upper semi-continuous (u.s.c.) on $\mathfrak{W}$ if for each $\mathfrak{w}_0 \in \mathfrak{W}$, the set $\mathfrak{S}(\mathfrak{w}_0)$ is a nonempty closed subset of $\mathfrak{W}$, and if for each open set $\tilde{\Omega}$ of $\mathfrak{W}$ containing $\mathfrak{S}(\mathfrak{w}_0)$, there exists an open neighborhood $\tilde{\Omega}_0$ of $\mathfrak{w}_0$ where $\mathfrak{S}(\tilde{\Omega}_0) \subseteq \tilde{\Omega}$. A map $\mathfrak{S} : \mathfrak{W} \to \mu(\mathfrak{W})$ has a fixed point if there is $\mathfrak{w} \in \mathfrak{W}$ where $\mathfrak{w} \in \mathfrak{S}(\mathfrak{w})$. We denoted by $Fix\mathfrak{S}$ the fixed point set of $\mathfrak{S}$.

Definition 2.2. The map $\mathfrak{S} : \Theta_j \to \mu_{cl}(\Xi);\ j = 0, 1, \cdots$, is measurable if for all $\mathfrak{p} \in \Xi$, the function

$$\vartheta \longmapsto d(\mathfrak{p}, \mathfrak{S}(\vartheta)) = \inf\{\|\mathfrak{p} - \bar{\mathfrak{p}}\| : \bar{\mathfrak{p}} \in \mathfrak{S}(\vartheta)\}$$

is measurable.

Definition 2.3. The map $\Psi : \Theta_j \times \Xi \times \chi \to \mu(\Xi);\ j = 0, 1, \cdots;$ is called L^1-Carathéodory if

(a) $\vartheta \longmapsto \Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})$ is measurable for each $(\mathfrak{p}, \bar{\mathfrak{p}}) \in \Xi \times \chi$;
(b) $\mathfrak{p} \longmapsto \Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})$ and $\bar{\mathfrak{p}} \longmapsto \Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})$ are u.s.c. for almost all $\vartheta \in \Theta_j$.
(c) for each $\delta > 0$, there exists $\eta_\delta \in L^1(\Theta_j, \mathbb{R}_+)$ where

$$\|\Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})\|_\mu = \sup\{\|\lambda\| : \lambda \in \Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})\} \leq \eta_\delta(\vartheta),$$

$$\text{for all } \|\mathfrak{p}\| \leq \delta,\ \|\bar{\mathfrak{p}}\|_\chi \leq \delta \text{ and for } a.e.\ \vartheta \in \Theta_j.$$

Ψ is Carathéodory if (a) and (b) are met.
For $\mathfrak{p} \in \mathfrak{F}(\Theta_j);\ j = 0, 1, \cdots$, the set of selections of Ψ is given by

$$S_{\Psi \circ \mathfrak{p}} = \{\bar{\mathfrak{p}} \in L^1(\Theta_j, \Xi) : \bar{\mathfrak{p}}(\vartheta) \in \Psi(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{p}_\vartheta)\ a.e.\ \vartheta \in \Theta_j\}.$$

Let $(\mathfrak{W}, d)$ be a metric space induced from the normed space $\mathfrak{W}$. Consider $\Omega_d : \mu(\mathfrak{W}) \times \mu(\mathfrak{W}) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$\Omega_d(\mathfrak{Q}, \bar{\mathfrak{Q}}) = \max\left\{\sup_{\iota\in\mathfrak{Q}} d(\iota, \bar{\mathfrak{Q}}), \sup_{\bar{\iota}\in\bar{\mathfrak{Q}}} d(\mathfrak{Q}, \bar{\iota})\right\},$$

where $d(\mathfrak{Q}, \bar{\iota}) = \inf_{\iota\in\mathfrak{Q}} d(\iota, \bar{\iota}),\ d(\iota, \bar{\mathfrak{Q}}) = \inf_{\bar{\iota}\in\bar{\mathfrak{Q}}} d(\iota, \bar{\iota})$.

Let $(\chi, \|\cdot\|_\chi)$ be a seminormed linear space of functions from $\mathbb{R}_-$ into Ξ, and meeting the following essential assumptions:

$(\mathcal{C}d_{A_1})$: If $\mathfrak{p} \in \mathcal{P}_c$ and $\mathfrak{p}_0 \in \chi$, then for all $\vartheta \in \Theta$ the requirements that follows are met:
(i) $\mathfrak{p}_\vartheta \in \chi$;
(ii) $\|\mathfrak{p}_\vartheta\|_\chi \leq \mathfrak{E}(\vartheta)\sup_{\varrho\in[0,\vartheta]}\|\mathfrak{p}(\varrho)\| + \tilde{\mathfrak{E}}(\vartheta)\|\psi\|_\chi$;
(iii) $\|\mathfrak{p}(\vartheta)\| \leq \bar{\mathfrak{E}}\|\mathfrak{p}_\vartheta\|_\chi$;
where $\bar{\mathfrak{E}} \geq 0$, $\mathfrak{E} : \Theta \to \mathbb{R}_+$ is continuous; $\tilde{\mathfrak{E}} : \mathbb{R}_+ \to \mathbb{R}_+$ is locally bounded, and $\bar{\mathfrak{E}}, \mathfrak{E}, \tilde{\mathfrak{E}}$, are independent of $\mathfrak{p}(\cdot)$.

$(\mathcal{C}d_{A_2})$: For the function $\mathfrak{p}(\cdot)$ in $(\mathcal{C}d_{A_1})$, $\mathfrak{p}_\vartheta$ is a χ-valued continuous function on Θ.

$(\mathcal{C}d_{A_3})$: The space χ is complete.

Denote $\mathfrak{E}_\beta = \sup\{\mathfrak{E}(\vartheta) : \vartheta \in \Theta\}$ and $\tilde{\mathfrak{E}}_\beta = \sup\{\tilde{\mathfrak{E}}(\vartheta) : \vartheta \in \Theta\}$. (See [19] for more details).

Remark 2.4. Axiom $(\mathcal{C}d_{A_1})$(ii) is equivalent to $\|\psi(0)\| \leq \bar{\mathfrak{E}}\|\psi\|_\chi$, for all $\psi \in \chi$. As consequence, we have for all ψ, $\bar{\psi} \in \chi$ such that $\|\psi - \bar{\psi}\|_\chi = 0$, thus $\psi(0) = \bar{\psi}(0)$.

Lemma 2.5 ([21])**.** *Let $\alpha_0 > 1$ such that*

$$\left(\frac{1}{2}\right)^{\alpha_0-1}\kappa < 1,$$

where $\kappa = \sup_{(\vartheta)\in\Theta}\|R(\vartheta)\|_{\Upsilon(\Xi)}$ and there exists a function ξ on $\mathbb{R}_-$ where $\xi(0) = 1$, $\xi(-\infty) = +\infty$, ξ is decreasing on $\mathbb{R}_-$, and for $d \geq \beta_0 := \frac{\varsigma}{\alpha_0}$ one has $\sup_{\varrho\in(-\infty,0]}\frac{\xi(\varrho)}{\xi(\varrho-d)} \leq \frac{1}{2}$.

Example 2.6. We define the spaces:

$$C_\xi := \left\{\psi \in \tilde{\mathfrak{F}}(\mathbb{R}_-, \Xi) : \frac{\psi(\varepsilon)}{\xi(\varepsilon)} \text{ is bounded on } \mathbb{R}_-\right\},$$

and

$$C_\xi^0 := \left\{\psi \in C_\xi : \lim_{\varepsilon\to-\infty}\frac{\psi(\varepsilon)}{\xi(\varepsilon)} = 0\right\},$$

with the norm

$$\|\psi\| = \sup\left\{\frac{|\psi(\varepsilon)|}{\xi(\varepsilon)} : \varepsilon \leq 0\right\}.$$

Therefore, the spaces C_ξ and C_ξ^0 verify the condition $(\mathcal{C}d_{A_3})$. As well as conditions $(\mathcal{C}d_{A_1})$ and $(\mathcal{C}d_{A_2})$ if

$$\sup_{\vartheta\in\Theta}\sup_{-\infty<\varepsilon\leq-\vartheta}\frac{\psi(\vartheta+\varepsilon)}{\xi(\varepsilon)}<\infty.$$

Example 2.7. For any $\sigma\in(0,\infty)$, we define the space

$$C_\sigma := \{\psi\in\tilde{\mathfrak{F}}((-\infty,0]),\Xi) : \lim_{\varepsilon\to-\infty} e^{\sigma\varepsilon}\psi(\varepsilon) \textit{ exist in } \Xi\},$$

with the norm

$$\|\psi\| = \sup\{e^{\sigma\varepsilon}|\psi(\varepsilon)| : \varepsilon\leq 0\}.$$

The axioms $(\mathcal{C}d_{A_1})-(\mathcal{C}d_{A_3})$ are therefore satisfied in the space C_σ.

In everything that follows, we take into account the phase space

$$\chi := \Big\{\psi\in\tilde{\mathfrak{F}}((-\infty,0]),\Xi) : \sup_{\varrho\in(-\infty,0]}\frac{\|\psi(\varrho)\|}{\xi(\varrho)}<\infty\Big\}.$$

Therefore, χ verifies the assumption $(\mathcal{C}d_{A_3})$. As well as $(\mathcal{C}d_{A_1})$ and $(\mathcal{C}d_{A_2})$ if

$$\sup_{\vartheta\in\Theta}\sup_{-\infty<\varepsilon\leq-\vartheta}\frac{\psi(\vartheta+\varepsilon)}{\xi(\varepsilon)}<\infty.$$

The space χ equipped with the norm

$$\|\psi\|_\chi = \sup_{\varrho\in(-\infty,0]}\frac{\|\psi(\varrho)\|}{\xi(\varrho)},$$

is a Banach space [11].

Definition 2.8 ([6])**.** Let $\mathfrak{W}$ be a Banach space and $\Omega_{\mathfrak{W}}$ the bounded subsets of $\mathfrak{W}$. The Kuratowski measure of noncompactness is the map $\omega:\Omega_{\mathfrak{W}}\to[0,\infty]$ given by

$$\omega(\Omega) = \inf\{\epsilon>0 : \Omega\subseteq\cup_{i=1}^n\Omega_i \text{ and } \operatorname{diam}(\Omega_i)\leq\epsilon\},\ \text{ here } \Omega\in\Omega_\Xi,$$

where

$$\operatorname{diam}(\Omega_i) = sup\{\|\mathfrak{p}-\bar{\mathfrak{p}}\|_\Xi : \mathfrak{p},\bar{\mathfrak{p}}\in\Omega_i\}.$$

Lemma 2.9 ([18])**.** *Let $\mathfrak{P}\subset\mathfrak{F}(\Theta)$ be a bounded and equicontinuous set. Therefore:*

(i) *the function $\vartheta\to\omega(\mathfrak{P}(\vartheta))$ is continuous on Θ, and*

$$\omega_c(\mathfrak{P}) = \sup_{\vartheta\in\Theta}\omega(\mathfrak{P}(\vartheta)).$$

(ii) $\omega\left(\int_0^\varsigma\mathfrak{p}(\varrho)d\varrho : \mathfrak{p}\in\mathfrak{P}\right)\leq\int_0^\varsigma\omega(\mathfrak{P}(\varrho))d\varrho,$

where

$$\mathfrak{P}(\vartheta) = \{\mathfrak{p}(\vartheta) : \mathfrak{p}\in\mathfrak{P}\};\ \vartheta\in\Theta.$$

Lemma 2.10 ([22])**.** *Let $\{\mathfrak{p}_j\}_{j=1}^\infty\subset L^1(\Theta)$ be uniformly integrable. Thus, $\omega(\{\mathfrak{p}_j\}_{j=1}^\infty)$ is measurable and*

$$\omega\left(\left\{\int_0^\vartheta\mathfrak{p}_j(\varrho)d\varrho\right\}_{j=1}^\infty\right)\leq 2\int_0^\vartheta\omega(\{\mathfrak{p}_j(\varrho)\}_{j=1}^\infty)d\varrho.$$

Lemma 2.11 ([10])**.** *If Λ is a bounded subset of a Banach space $\mathfrak{W}$, then for each $\epsilon > 0$, there is a sequence $\{\lambda_j\}_{j=1}^{\infty} \subset \Lambda$ such that*

$$\omega(\Lambda) \leq 2\omega(\{\lambda_j\}_{j=1}^{\infty}) + \epsilon.$$

Definition 2.12. The map$\psi : [0,\infty) \to [0,\infty)$, is a dominating function ($D-$function) if it is an u.s.c. and monotonic nondecreasing function verifying $\psi(0) = 0$.

Definition 2.13. Let $\mathfrak{W}$ be a Banach space. A multivalued mapping $\mathfrak{S} : \mathfrak{W} \in \mu_{bd,cl}(\mathfrak{W})$ is called $D-$set-Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\omega_{\mathfrak{W}}(\mathfrak{S}(\Omega)) \leq \psi(\omega_{\mathfrak{W}}(\Omega))$, for all $\Omega \in \mu_{bd,cl}(\mathfrak{W})$ and $\psi(0) = 0$.

Remark 2.14. If $\psi(\beta) = j\beta;\ j > 0$, then $\mathfrak{S}$ is called a $j-$set-Lipschitz mapping. Moreover, if $j < 1$, then $\mathfrak{S}$ is called a $j-$set-contraction on Ξ. If $\psi(\beta) < \beta$, for $\beta > 0$, then $\mathfrak{S}$ is called a nonlinear $D-$set-contraction on $\mathfrak{W}$.

Definition 2.15. Let $\mathfrak{W}$ be a Banach space and $\omega_{\mathfrak{W}}$ be a measure of noncompactness on $\mathfrak{W}$. An operator $\mathfrak{S} : \mathfrak{W} \to \mathfrak{W}$ is called condensing if $\mathfrak{S}$ is continuous takes bounded sets into bounded sets, and $\omega_{\mathfrak{W}}(\mathfrak{S}(\Omega)) < \omega_{\mathfrak{W}}(\Omega)$ for every bounded set Ω of $\mathfrak{W}$ with $\omega_{\mathfrak{W}}(\Omega) > 0$.

Theorem 2.16 ([12])**.** *Let $\mathfrak{Y}$ be a nonempty, bounded, closed and convex subset of a Banach space Ξ and $\mathfrak{S} : \mathfrak{Y} \to \mu_{cl,cv}(\mathfrak{Y})$ be a closed and nonlinear $D-$ set contraction, Then $\mathfrak{S}$ has at least a fixed point.*

Theorem 2.17 ([21])**.** *Let $\mathfrak{S} : \mathfrak{W} \to \mathfrak{W}$ be a condensing operator where $\mathfrak{W}$ a Banach space. If $\mathfrak{S}(V) \subset V$ for a bounded,closed and convex set V of $\mathfrak{W}$, then $\mathfrak{S}$ admit a fixed point in V.*

3. Existence results

Definition 3.1 ([13, 17])**.** A resolvent operator for the problem

$$\begin{cases} \mathfrak{p}'(\vartheta) = \mathcal{Z}\mathfrak{p}(\vartheta) + \int_0^{\vartheta} \mathfrak{N}(\vartheta - \varrho)\mathfrak{p}(\varrho)d\varrho, \ \vartheta \in [0,\infty), \\ \mathfrak{p}(0) = \mathfrak{p}_0 \in \Xi, \end{cases} \tag{3.1}$$

is a bounded linear operator-valued function $R(\vartheta) \in \Upsilon(\Xi);\ \vartheta \geq 0$, verifying the following assumptions:

(i) $R(0) = I$ and $\|R(\vartheta)\| \leq Ne^{\nu\vartheta}$ for some constants $N > 0$, and $\nu \in \mathbb{R}$.

(ii) For each $\mathfrak{p} \in \Xi,\ R(\vartheta)\mathfrak{p}$ is strongly continuous for $\vartheta \geq 0$.

(iii) $R(\vartheta)$ is bounded for $\vartheta \geq 0$. For $\mathfrak{p} \in D(\mathcal{Z}),\ R(\cdot)\mathfrak{p} \in C(\mathbb{R}_+, D(\mathcal{Z})) \cap C^1(\mathbb{R}_+, \Xi)$ and

$$R'(\vartheta)\mathfrak{p} = \mathcal{Z}R(\vartheta)\mathfrak{p} + \int_0^{\vartheta} \mathfrak{N}(\vartheta - \varrho)R(\varrho)\mathfrak{p}d\varrho = R(\vartheta)\mathcal{Z}\mathfrak{p} + \int_0^{\vartheta} R(\vartheta - \varrho)\mathfrak{N}(\varrho)\mathfrak{p}d\varrho;\ \vartheta \in [0,\infty).$$

The hypotheses:

($\mathcal{C}d_{B_1}$) The operator $\mathcal{Z}$ is the infinitesimal generator of a uniformly continuous semigroup $(S(\vartheta))_{\vartheta \geq 0}$.

($\mathcal{C}d_{B_2}$) For all $\vartheta \geq 0$, $\mathfrak{N}(\vartheta)$ is a closed linear operator from $D(\mathcal{Z})$ to Ξ and $\mathfrak{N}(\vartheta) \in \Upsilon(\Xi)$. For any $\mathfrak{p} \in \Xi$, the map $\vartheta \mapsto \mathfrak{N}(\vartheta)\mathfrak{p}$ is bounded, differentiable and the derivative $\vartheta \mapsto \mathfrak{N}'(\vartheta)\mathfrak{p}$ is bounded and uniformly continuous on $\mathbb{R}_+$.

Theorem 3.2 ([13,17]). *Assume that ($\mathcal{C}d_{B_1}$) and ($\mathcal{C}d_{B_2}$) hold. Then there exists a unique uniformly continuous resolvent operator for the problem (3.1).*

Definition 3.3. A ς-periodic function $\mathfrak{p} \in \widetilde{\mathcal{P}_c}$ is a periodic mild solution of problem (1.1) if there exists $\bar{\xi} \in S_{\Psi \circ \mathfrak{p}}$, for a.e. $\vartheta \in \mathbb{R}_+$, such that

$$\mathfrak{p}(\vartheta) = \begin{cases} R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta - \varrho)\bar{\xi}(\varrho)d\varrho, \ \vartheta \in \Theta_0, \\ R(\vartheta - \varrho_j)\Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-)) + \int_{\varrho_j}^\vartheta R(\vartheta - \varrho)\bar{\xi}(\varrho)d\varrho, \ \vartheta \in \Theta_j, \ j = 1, \dots, \\ \Phi_j(\vartheta, \mathfrak{p}(\vartheta_j^-)), \ \vartheta \in \tilde{\Theta}_j, \ j = 1, \dots, \\ \psi(\vartheta), \quad \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

The hypotheses:

($\mathcal{C}d_{C_1}$) The multivalued map $\Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})$ is L^1-Carathéodory, and has compact and convex values and maps bounded sets into bounded sets.

($\mathcal{C}d_{C_2}$) The continuous functions $\Phi_j, \ j = 1, 2, \dots;$ map bounded sets into bounded sets.

($\mathcal{C}d_{C_3}$) For $\varsigma > 0$, $\Psi(\vartheta + \varsigma, \mathfrak{p}, \bar{\mathfrak{p}}) = \Psi(\vartheta, \mathfrak{p}, \bar{\mathfrak{p}})$, $\mathcal{Z}(\vartheta + \varsigma) = \mathcal{Z}(\vartheta)$, $\vartheta \in \Theta_j$, $j = 0, \dots, \nu$, $\mathfrak{p}, \bar{\mathfrak{p}} \in \Xi \times \chi$, $\Phi_j(\vartheta + \varsigma, z) = \Phi_j(\vartheta, z)$, $\vartheta \in \tilde{\Theta}_j$, $j = 1, \dots, \nu, z \in \Xi$, and $\psi(\varrho + \varsigma) = \psi(\varrho)$, $\varrho \in (-\infty, 0]$.

($\mathcal{C}d_{C_4}$) There exist continuous function $l_j \in L^\infty(\Theta)$, such that

$$\|\Phi_j(\vartheta, z)\|_\Xi \leq l_j(\vartheta)(1 + \|z\|), \ \textit{for a.e. } \vartheta \in \tilde{\Theta}_j, \ \textit{and each } z \in \Xi, \ j = 0, \dots, \nu.$$

($\mathcal{C}d_{C_5}$) For bounded sets $\Omega \subset \Xi$ and $\Omega_\vartheta \subset \chi$, $\vartheta \in \mathbb{R}_+$, such that

$$\Omega_\vartheta = \{\mathfrak{p}_\vartheta : \mathfrak{p}_\vartheta \in \chi\},$$

we have

$$\omega_\Xi(\Psi(\vartheta, \Omega, \Omega_\vartheta)) \leq \eta_\delta(\vartheta)\omega_\Xi(B), \ \textit{for a.e. } \vartheta \in \Theta_j, \ j = 0, \dots, \nu,$$

and

$$\omega_\Xi(\Phi_j(\vartheta, \Omega)) \leq l_j(\vartheta)\omega_\Xi(\Omega), \ \textit{for a.e. } \vartheta \in \tilde{\Theta}_j, \ j = 1, \dots, \nu,$$

where $\eta_\delta \in L^1(\Theta_j)$, and ω_Ξ is a measure of noncompactness on the Banach space Ξ.

Set

$$\eta_\delta^* := \|\eta_\delta\|_{L^1(\Theta_j)} = \int_{\Theta_j} \eta_\delta(\vartheta)d\vartheta,$$

$$l^* = \max_{j=0,\dots,\nu} \|l_j\|_{L^\infty}, \quad \kappa = \sup_{\vartheta \in \Theta} \|R(\vartheta)\|_{\Upsilon(\Xi)}.$$

Theorem 3.4. *Assume that the hypotheses ($\mathcal{C}d_{B_1}$), ($\mathcal{C}d_{B_2}$), and ($\mathcal{C}d_{C_1}$)$-$($\mathcal{C}d_{C_5}$) are met. If*

$$\delta := \max\{l^*, 4\kappa\varsigma\eta_\delta^*, \kappa(l^* + \varsigma\eta_\delta^*)\} < 1, \tag{3.2}$$

then the problem (1.1) has at least one mild solution.

Proof. Consider the problem
(3.3)
$$\begin{cases} \mathfrak{p}'(\vartheta) - \mathcal{Z}\mathfrak{p}(\vartheta) - \int_0^\vartheta \mathfrak{N}(\vartheta-\varrho)\mathfrak{p}(\varrho)d\varrho \in \Psi(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{p}_\vartheta),\ \vartheta \in \Theta_j,\ j = 0, \dots, \nu, \\ \mathfrak{p}(\vartheta) = \Phi_j(\vartheta, \mathfrak{p}(\vartheta_j^-)),\ \vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \nu, \\ \mathfrak{p}(\vartheta) = \psi(\vartheta), \quad \text{if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases}$$

Part 1. We start by demonstrating that (3.3) has a mild solution $\mathfrak{p} \in \mathcal{P}_c$, Transform the problem (3.3) into a fixed point problem. Consider the multivalued map $\mathfrak{T}_1 : \mathcal{P}_c \to \mu(\mathcal{P}_c)$ defined by
(3.4)
$$\mathfrak{T}_1(\mathfrak{p}) = \left\{ \xi \in \mathcal{P}_c : \xi(\vartheta) = \begin{cases} R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta-\varrho)\bar{\xi}(\varrho)d\varrho;\ \vartheta \in \Theta_0, \\ R(\vartheta-\varrho_j)\Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-)) \\ + \int_{\varrho_j}^\vartheta R(\vartheta-\varrho)\bar{\xi}(\varrho)d\varrho,\ \vartheta \in \Theta_j,\ j = 1, \dots, \nu, \\ \Phi_j(\vartheta, \mathfrak{p}(\vartheta_j^-)),\ \vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \nu, \\ \psi(\vartheta), \quad \text{if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases} \right\}$$

where $\bar{\xi} \in S_{\Psi \circ \mathfrak{p}}$.
Let $\xi \in \mathfrak{T}_1(\mathfrak{p})$. Thus, there exists $\bar{\xi} \in S_{\Psi \circ \mathfrak{p}}$ where

$$\xi(\vartheta) = \begin{cases} R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta-\varrho)\bar{\xi}(\varrho)d\varrho,\ \vartheta \in \Theta_0, \\ R(\vartheta-\varrho_j)\Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-)) + \int_{\varrho_j}^\vartheta R(\vartheta-\varrho)\bar{\xi}(\varrho)d\varrho,\ \vartheta \in \Theta_j,\ j = 1, \dots, \nu, \\ \Phi_j(\vartheta, \mathfrak{p}(\vartheta_j^-)),\ \vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \nu, \\ \psi(\vartheta), \quad \text{if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases}$$

Let $\gamma > 0$ be, such that

$$\gamma \geq \max\left\{ \|\psi\|_\chi, \frac{l^*}{1-l^*}, \kappa(\|\psi(0)\| + \eta_\delta^*), \frac{\kappa(l^* + \eta_\delta^*)}{1-\kappa l^*} \right\},$$

and consider the ball $\Omega_\gamma := \Omega(0,\gamma) = \{\ell \in \mathcal{P}_c : \|\ell\|_{\mathcal{P}_c} \leq \gamma\}$.
For any $\mathfrak{p} \in \Omega_\gamma$ and each $\vartheta \in \Theta_0$, we have

$$\begin{aligned} \|\xi(\vartheta)\| &\leq \kappa\|\psi(0)\| + \kappa\int_0^\vartheta \|\bar{\xi}(\varrho)\|d\varrho \\ &\leq \kappa\|\psi(0)\| + \kappa\eta_\delta^* \\ &\leq \gamma. \end{aligned}$$

Next, for each $\vartheta \in \Theta_j,\ j = 1, \dots, \nu$, we have

$$\begin{aligned} \|\xi(\vartheta)\| &\leq \kappa l^*(1+\gamma) + \kappa\int_{\varrho_j}^\vartheta \|\bar{\xi}(\varrho)\|d\varrho \\ &\leq \kappa l^*(1+\gamma) + \kappa\eta_\delta^* \\ &\leq \gamma. \end{aligned}$$

Also, for each $\vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \nu$, we have

$$\|\xi(\vartheta)\| \leq l^*(1+\gamma) \leq \gamma,$$

and for each $\vartheta \in \mathbb{R}_-$, we have

$$\|\xi(\vartheta)\| = \|\psi\|_{\mathcal{X}} \leq \gamma.$$

Hence,

$$\|\mathfrak{T}_1(\mathfrak{p})\|_{\mathcal{P}C} \leq \gamma.$$

As consequence, the operator $\mathfrak{T}_1$ transforms the ball $\Omega_\gamma := \{\mathfrak{p} \in \mathcal{P}_c : \|\mathfrak{p}\|_{\mathcal{P}_c} \leq \gamma\}$ into $\mu(\Omega_\gamma)$. Now, we will demonstrate that the operator $\mathfrak{T}_1 : \Omega_\gamma \to \mu(\Omega_\gamma)$ verifies all the requirements of Theorem 2.16.

Step 1. $\mathfrak{T}_1(\mathfrak{p}) \in \mu_{cl}(\mathcal{P}_c)$ *for each* $\mathfrak{p} \in \Omega_\gamma$.
Let $(\mathfrak{p}_n)_{n\geq 0} \in \mathfrak{T}_1(\mathfrak{p})$ such that $\mathfrak{p}_n \longrightarrow \tilde{\mathfrak{p}}$ in $\mathcal{P}_c$. Then, $\tilde{\mathfrak{p}} \in \mathcal{P}_c$ and there exists $\bar{\xi}_n \in S_{\Psi\circ\mathfrak{p}}$ such that

$$\mathfrak{p}_n(\vartheta) = \begin{cases} R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta-\varrho)\ \bar{\xi}_n(\varrho)d\varrho, & \text{if } \vartheta \in \Theta_0, \\ R(\vartheta-\varrho_j)\Phi_j(\varrho_j, \mathfrak{p}_n(\varrho_j^-)) \\ + \int_{\varrho_j}^\vartheta R(\vartheta-\varrho)\ \bar{\xi}_n(\varrho)d\varrho, & \text{if } \vartheta \in \Theta_j,\ j = 1, \dots, \nu \\ \Phi_j(\vartheta, \mathfrak{p}_n(\vartheta_j^-)), \text{ if } \vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \nu \\ \psi(\vartheta), & \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

Given that Ψ contains compact values and by $(\mathcal{C}d_{C_1})$, we can pass to a subsequence if required get that $\bar{\xi}_n$ converges to $\bar{\xi}$ in $L^1(\Theta, \Xi)$, and therefore $\bar{\xi} \in S_{\Psi\circ\mathfrak{p}}$. Then

$$\mathfrak{p}_n(\vartheta) \to \mathfrak{p}(\vartheta) = \begin{cases} R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta-\varrho)\ \bar{\xi}(\varrho)d\varrho, & \text{if } \vartheta \in \Theta_0, \\ R(\vartheta-\varrho_j)\Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-)) \\ + \int_{\varrho_j}^\vartheta R(\vartheta-\varrho)\ \bar{\xi}(\varrho)d\varrho, & \text{if } \vartheta \in \Theta_j,\ j = 1, \dots, \nu \\ \Phi_j(\vartheta, \mathfrak{p}(\vartheta_j^-)), \text{ if } \vartheta \in \tilde{\Theta}_j,\ j = 1, \dots, \nu \\ \psi(\vartheta), & \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

So, $\tilde{\mathfrak{p}} \in \mathfrak{T}_1(\mathfrak{p})$.

Step 2. $\mathfrak{T}_1 : \Omega_\gamma \to \mu_{cl,cv}(\Omega_\gamma)$ *is* $D-$*set-contraction.*
Using Lemmas 2.11 and 2.10, for any $\Omega \subset \Omega_\gamma$ and any $\epsilon > 0$, there exists a sequence $\{\mathfrak{p}_j\}_{j=0}^\infty \subset \Omega$, where for every $\vartheta \in \Theta_0$, we obtain

$$\begin{aligned} \omega_\Xi((\mathfrak{T}_1\Omega)(\vartheta)) &= \omega_\Xi\left(\left\{R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta-\varrho)\ \bar{\xi}(\varrho)d\varrho;\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}},\ \mathfrak{p} \in \Omega\right\}\right) \\ &\leq 2\omega_\Xi\left(\left\{\int_0^\vartheta R(\vartheta-\varrho)\bar{\xi}(\varrho)d\varrho;\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}_j}\right\}_{j=1}^\infty\right) + \epsilon \\ &\leq 4\int_0^\vartheta \omega_\Xi\left(\|R(\vartheta-\varrho)\|_{\Upsilon(\Xi)}\{\bar{\xi}(\varrho);\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}_j}\}_{j=1}^\infty\right) d\varrho + \epsilon \\ &\leq 4\kappa\int_0^\vartheta \omega_\Xi\left(\{\bar{\xi}(\varrho);\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}_j}\}_{j=1}^\infty\right) d\varrho + \epsilon \\ &\leq 4\kappa\int_0^\vartheta \eta_\delta(\varrho)\omega\left(\{\mathfrak{p}_j(\varrho)\}_{j=1}^\infty\right) d\varrho + \epsilon \end{aligned}$$

$$\begin{aligned}
&\leq 4\kappa\eta^*_\delta \int_0^\vartheta \omega_\Xi\left(\{\mathfrak{p}_j(\varrho)\}_{j=1}^\infty\right) d\varrho + \epsilon \\
&\leq 4\kappa\varsigma\eta^*_\delta \omega_{\mathcal{P}_c}(\Omega) + \epsilon,
\end{aligned}$$

and, for all $\vartheta \in \Theta_j,\ j = 1, \ldots, \nu$, we get

$$\begin{aligned}
\omega_\Xi((\mathfrak{T}_1\Omega)(\vartheta)) &= \omega_\Xi\Bigg(\Big\{R(\vartheta-\varrho_j)\Phi_j(\varrho_j, \mathfrak{p}(\varrho_j^-)) \\
&+ \int_{\varrho_j}^\vartheta R(\vartheta-\varrho)\ \bar{\xi}(\varrho)d\varrho;\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}},\ \mathfrak{p} \in \Omega\Big\}\Bigg) \\
&\leq 2\omega_\Xi\left(\left\{\int_0^\vartheta R(\vartheta-\varrho)\bar{\xi}(\varrho)d\varrho;\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}_j}\right\}_{j=1}^\infty\right) + \epsilon \\
&\leq 4\kappa\varsigma\eta^*_\delta \omega_{\mathcal{P}_c}(\Omega) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then for all $\vartheta \in \Theta_j,\ j = 0, \ldots, \nu$, we get

$$\omega_\Xi((\mathfrak{T}_1\Omega)(\vartheta)) \leq \zeta_1(\omega_{\mathcal{P}_c}(\Omega)),$$

where $\zeta_1 : \mathbb{R}_+ \to \mathbb{R}_+,\ \zeta_1(\mathfrak{w}) = 4\kappa\varsigma\eta^*_\delta\mathfrak{w}$.
Also, for all $\vartheta \in \tilde{\Theta}_j,\ j = 1, \ldots, \nu$, we get

$$\begin{aligned}
\omega_\Xi((\mathfrak{T}_1\Omega)(\vartheta)) &\leq l_j(\vartheta)\omega_{\mathcal{P}_c}(\Omega), \\
&\leq l^*\omega_{\mathcal{P}_c}(\Omega) \\
&\leq \zeta_2(\omega_{\mathcal{P}_c}(\Omega)),
\end{aligned}$$

where $\zeta_2 : \mathbb{R}_+ \to \mathbb{R}_+,\ \zeta_2(\mathfrak{w}) = l^*\mathfrak{w}$.
Hence, for all $\vartheta \in (-\infty, \varsigma]$ we obtain

$$\omega_\Xi((\mathfrak{T}_1\Omega)(\vartheta)) \leq \zeta(\omega_{\mathcal{P}_c}(\Omega)),$$

where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\begin{cases} \zeta(\vartheta) = \zeta_1(\vartheta), & \text{if } \vartheta \in \Theta_j;\ j = 0, 1, \ldots, \\ \zeta(\vartheta) = \zeta_2(\vartheta), & \text{if } \vartheta \in \tilde{\Theta}_j;\ j = 1, 2, \ldots. \end{cases}$$

As a result, we may deduce that $\mathfrak{T}_1$ admit a fixed point in $\mathfrak{p} \in \Omega_\gamma$.

Part 2. *Periodic mild solutions.*
A common method for obtaining $\varsigma-$ periodic solutions is to define the Poincaré operator $\mathfrak{T}_2 : \chi \to \chi$ by

$$\mathfrak{T}_2(\psi) = \mathfrak{p}_\varsigma(\psi) :\ \text{where } (\mathfrak{T}_2\psi)(\varrho) = \mathfrak{p}_\varsigma(\varrho, \psi) = \mathfrak{p}(\varsigma + \varrho, \psi),\ \varrho \in \mathbb{R}_-,$$

which translates a starting function ψ along the single mild solution $\mathfrak{p}(\psi)$ to our problem (1.1) by $\varsigma-$ units. We prove that $\mathfrak{T}_2$ is condensing in χ. Thus, the provided assumptions indicate that Theorem 2.17 may be used to obtain fixed points for the Poincaré operator, resulting in periodic solutions.

Step 1. *The fixed points of $\mathfrak{T}_2$ provide a periodic mild solutions of (1.1).*
Let $\psi \in \chi$ where $\mathfrak{T}_2(\psi) = \psi$. Then for the solution $\mathfrak{p}(\cdot) = \mathfrak{p}(\cdot, \psi)$ with $\mathfrak{p}_0(\cdot, \psi) = \psi$, we can define $\bar{\mathfrak{p}}(\vartheta) = \mathfrak{p}(\vartheta + \varsigma)$. And, for $\vartheta > 0$, We can make advantage of the properties of $R(\vartheta)$, and the knowledge that $\bar{\xi}$ and Φ_j are ς-periodic functions in ϑ,

to get that $\bar{\mathfrak{p}}$ is also a solution with $\bar{\mathfrak{p}}_0(\cdot,\psi) = \mathfrak{p}_\varsigma(\psi) = \mathfrak{p}(\cdot,\psi)$. Indeed, let $\xi \in \mathfrak{T}_1(\mathfrak{p})$. Then there exists $\bar{\xi} \in S_{\Psi\circ\mathfrak{p}}$ such that

$$\bar{\mathfrak{p}}(\vartheta) = \begin{cases} R(\vartheta)\psi(0) + \int_0^\vartheta R(\vartheta-\varrho)\ \bar{\xi}(\varrho)d\varrho, & \text{if } \vartheta \in \Theta_0,\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}} \\ R(\vartheta-\varrho_j)\Phi_j(\varrho_j, \bar{\mathfrak{p}}(\varrho_j^-)) & \\ + \int_{\varrho_j}^\vartheta R(\vartheta-\varrho)\ \bar{\xi}(\varrho)d\varrho, & \text{if } \vartheta \in \Theta_j,\ j=1,\dots,\nu,\ \bar{\xi} \in S_{\Psi\circ\mathfrak{p}} \\ \Phi_j(\vartheta, \bar{\mathfrak{p}}(\vartheta_j^-)), & \text{if } \vartheta \in \tilde{\Theta}_j,\ j=1,\dots,\nu \\ \psi(\vartheta), & \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

Then the uniqueness of $R(\vartheta)$ implies that $\bar{\mathfrak{p}}(\vartheta) = \mathfrak{p}(\vartheta)$, so that $\mathfrak{p}(\vartheta) = \mathfrak{p}(\vartheta+\varsigma)$ is a ς-periodic solution.

Step 2. $\mathfrak{T}_2$ *is condensing.*
Let $\Omega \subset \chi$ be bounded with $\omega_\chi(\Omega) > 0$. If $\mathfrak{p}_0$ is the unique solution with $\mathfrak{p}_0(\psi) = \psi$, we define $J\varrho(\Omega) = \{\mathfrak{p}_\varrho(\psi) : \psi \in \Omega\}$ and $J_{[\beta_1,\beta_2]}(\Omega) = \{\mathfrak{p}_{[\beta_1,\beta_2]}(\psi) : \psi \in \Omega\}$. Same as the proof of Theorem 4.1 in [21], we get

$$\omega_\chi(\mathfrak{T}_2(\Omega)) \le \left(\frac{1}{2}\right)^{\alpha_0 - 1} \kappa\omega_\chi(\Omega) < \omega_\chi(\Omega).$$

Therefore, Theorem 2.17 implies that $\mathfrak{T}_2$ admit a fixed point.

4. An example

Let us investigate the following problem of impulsive integro-differential inclusions:

$$(4.1)\quad \begin{cases} \frac{\partial\bar{\mathfrak{p}}}{\partial\vartheta}(\vartheta,\theta) - \frac{\partial^2\bar{\mathfrak{p}}}{\partial\theta^2}(\vartheta,\theta) \\ -\int_0^\vartheta b(\vartheta-\varrho)\frac{\partial^2}{\partial\theta^2}\bar{\mathfrak{p}}(\varrho,\theta)d\varrho \in \Psi(\vartheta, \bar{\mathfrak{p}}(\vartheta,\theta), \bar{\mathfrak{p}}_\vartheta(\cdot,\theta)), \\ \qquad \theta \in \Theta := [0,\pi],\ \vartheta \in \Theta_j,\ j = 0,\cdots, \\ \bar{\mathfrak{p}}(\vartheta,\theta) = \Phi_j(\vartheta,\theta), \qquad \theta \in \Theta,\ \vartheta \in \tilde{\Theta}_j,\ j=1,\cdots, \\ \bar{\mathfrak{p}}(\vartheta,0) = \bar{\mathfrak{p}}(\vartheta,\pi) = 0, \qquad \vartheta \in \mathbb{R}_+. \\ \bar{\mathfrak{p}}(0,\theta) = \mathcal{G}(\theta), \qquad \theta \in \Theta, \\ \bar{\mathfrak{p}}(\vartheta,\theta) = \psi(\vartheta,\theta); \qquad \vartheta \in \mathbb{R}_-,\ \theta \in \Theta, \end{cases}$$

where $\Psi : \mathbb{R}_+ \times \mathbb{R} \times C_\xi \to \mathbb{R}$, $\mathcal{G} : \Theta \to \mathbb{R}$ and $\psi : \mathbb{R}_- \times \Theta \to \mathbb{R}$ are continuous functions such that $\mathcal{G}(\theta) = \psi(0,\theta)$, $\theta \in \Theta$, and C_ξ is the phase space given in Example 2.6.

Take Let

$$\Xi := L^2(\Theta) = \left\{\mathfrak{p} : \Theta \to \mathbb{R} : \int_0^\pi |\mathfrak{p}(\theta)|^2 dx < \infty\right\}$$

be the Hilbert space with the scalar product $<\mathfrak{p},\bar{\mathfrak{p}}>=\int_0^\pi \mathfrak{p}(\theta)\bar{\mathfrak{p}}(\theta)dx$. It is known that Ξ is a Banach space with the norm

$$\|\mathfrak{p}\|_2=\left(\int_0^\pi |\mathfrak{p}(\theta)|^2dx\right)^{\frac{1}{2}},$$

and define $\mathcal{Z}: D(\mathcal{Z})\subset\Xi\to\Xi$ by $\mathcal{Z}\lambda=\lambda''$ with domain

$$D(\mathcal{Z})=\{\lambda\in\Xi,\lambda,\lambda' \quad \text{are absolutely continuous, } \lambda''\in\Xi,\lambda(0)=\lambda(\pi)=0\}.$$

Then

$$\mathcal{Z}\lambda=\sum_{n=1}^{\infty}n^2(\lambda,\lambda_n)\lambda_n,\ \lambda\in D(\mathcal{Z})$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $\lambda_n(\varrho)=\sqrt{\frac{2}{\pi}}\,sin\,ns,\ n=1,2,\ldots$ is the orthogonal set of eigenvectors in $\mathcal{Z}$. It is well known (see [24]) that $\mathcal{Z}$ is the infinitesimal generator of an analytic semigroup $\mathfrak{S}(\vartheta)$, $\vartheta\geq 0$ in Ξ and is defined by

$$\mathfrak{S}(\vartheta)\lambda=\sum_{n=1}^{\infty}exp(-n^2\vartheta)(\lambda,\lambda_n)\lambda_n,\ \lambda\in\Xi.$$

Since the analytic semigroup $\mathfrak{S}(\vartheta)$ is compact, there exists a constant $\kappa\geq 1$ such that

$$\|\mathfrak{S}(\vartheta)\|_{\Upsilon(\Xi)}\leq\kappa.$$

For $\theta\in\Theta$, we have

$$\begin{aligned}
\mathfrak{p}(\vartheta)(\theta)&=\bar{\mathfrak{p}}(\vartheta,\theta);\quad \vartheta\in\mathbb{R}_+,\\
\Psi(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}_\vartheta)&=\Psi(\vartheta,\bar{\mathfrak{p}}(\vartheta,\theta),\bar{\mathfrak{p}}_\vartheta(\cdot,\theta));\quad \vartheta\in\mathbb{R}_+,\\
\mathfrak{N}(\vartheta)&=b(\vartheta)\mathcal{Z}\\
\mathfrak{p}_0(\theta)&=\mathcal{G}(\theta);\quad \theta\in\Theta,\\
\mathfrak{p}(\vartheta)(\theta)&=\psi(\vartheta,\theta);\ \theta\in\Theta,\vartheta\in\mathbb{R}_-.
\end{aligned}$$

Therefore, by the definitions of $\bar{\xi}$, $\mathfrak{p}_0$ and $\mathcal{Z}$, the system (4.1) can be represented by (1.1). Moreover, more relevant assumptions on Ψ guarantee that $(\mathcal{C}d_{B_1}),(\mathcal{C}d_{B_2})$ and $(\mathcal{C}d_{C_1})-(\mathcal{C}d_{C_5})$ are met. As consequence, By Theorem 3.4, we can deduce that problem (4.1) has at least one mild solution on $\mathbb{R}$.

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