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TWO STRONG CONVERGENCE THEOREMS OF NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY MONOTONE MAPPINGS WITH ERRORS

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Abstract. In this paper, we introduce two iterative schemes for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space. Then we show that the sequence converges to a common element of two sets. Further, we consider the problem finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping. Finally, we apply the conclusions to some feasibility problems.

1. Introduction-preliminaries

Let C be a closed convex subset in a real Hilbert space H and let P_C be a metric projection of H onto C. The variational inequality problem is to find a point $u \in C$ about $\langle v-u, Au\rangle \geq 0$ for all $v \in C$. And we can define a self mapping $f : C \to C$, contraction, on C if there is a constant $k \in (0,1)$ such that $\|f(x)-f(y)\| \leq k\,\|x-y\|$ for all $x, y \in C$. A mapping T of C into H is called inverse-strongly monotone if there exists a positive real number α about $\langle x- y, Tx-Ty\rangle \ \geq \alpha\|Tx-Ty\|^2$, for all $x, y \in C$, and a mapping S of C into H is called monotone if for all $x, y \in C, \langle x-y, Tx-Ty\rangle \geq 0$. Takahashi et al. [14] proposed an algorithm: $x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C\left(x_n - \lambda_n Ax_n\right)$, where $\{\lambda_n\} \subset [a,b]$ and $\{\alpha_n\}$ is a sequence in $(0,1)$, $0 < a < b < 2\alpha$, S is a nonexpansive mapping. In that paper, he introduced an iteration process of finding a common element of the set of fixed points of a nonexpansive mapping. And he used this result, he obtained a weak convergence theorem for a pair of a nonexpansive mapping and a strictly pseudocontractive mapping. And then Chen [2] proposed the viscous form of the algorithm :

$$x_{n+1} = \alpha_n f\left(x_n\right) + (1-\alpha_n)SP_C\left(x_n - \lambda_n Ax_n\right),$$

where $\{\lambda_n\} \subset [a,b]$ and $\{\alpha_n\}$ is a sequence in $(0,1)$, $0 < a < b < 2\alpha$, f is a contraction with coefficient $k(0 < k < 1)$ and S is a nonexpansive mapping. Recently, these method has received great attention by many authors, who improved them in various ways, we refer to [1, 6–8, 15] and the references therein. In particular, Xu [17] proposed the algorithm with an error sequence such that $y_n := \left(I + c_n T\right)^{-1}\left(x_n\right) + e_n$. $x_{n+1} := \alpha_n x_0 + \left(1-\alpha_n\right) y_n$. Assume that (i) $\alpha_n \to 0$; (ii) $\sum_n \alpha_n = \infty$; (iii) $c_n \to \infty$;

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(iv) $\sum_n \|e_n\| < \infty$. This method is an appropriately modified proximal point algorithm which guarantees strong convergence and which does not substantially increase calculations. For recent results on variational inequalities and fixed points of nonexpansive mappings via fixed point methods, we refer to [3,9,10,12,16] and the references therein.

In this paper, we introduce two iterative schemes: One is to find a common element about the set of fixed points of a nonexpansive mapping. And another one is to find a common element about the set of solutions of the variational inequality for an inversestrongly monotone mapping in a Hilbert space. Then we show that this methods converge strongly to a common element of two sets which solves some variational inequality.

2. Preliminaries

We state the following well-known lemmas which will be used in our convergence analysis in the sequel.

Lemma 2.1. *The following well-known results in a real Hilbert space: for each* $x, y, z \in H$ *and* $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha + \beta + \gamma = 1$, *we have 1.* $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$. *2.* $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$.

Lemma 2.2. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *. Given* $x \in H$ *and* $z \in C$. *Then* $z = P_C x \Leftrightarrow \langle\ x - z, z - y\rangle\ \geq 0, \forall y \in C$.

Lemma 2.3. *Let* C *be a closed and convex subset in a real Hilbert space* $H, x \in H$. *Then*

(1) $\|P_C x - P_C y\|^2 \leq\ \langle P_C x - P_C y, x - y\rangle$.
(2) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2$.
(3) $\|(I - P_C)\, x - (I - P_C)\, y\|^2 \leq\ \langle (I - P_C)\, x - (I - P_C)\, y, x - y\rangle$.
(4) *In the context of the variational inequality problem:* $u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u)$ *for all* $\lambda \geq 0$.

Lemma 2.4 ([18]). *Let* $\{x_n\}$ *be a sequence in* H, *assume* $x_n \to x$, *then the inequality* $\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$ *holds for evey* $y \in H$ *with* $y \neq x$.

Lemma 2.5 ([17]). *Assume that* $\{s_k\}_{k=0}^{\infty}$ *is a sequence of nonnegative real numbers such that* $s_{k+1} \leq (1 - \lambda_k)\, s_k + \lambda_k b_k + c_k$, *where* $\{\lambda_k\}, \{b_k\}$ *and* $\{c_k\}$ *satisfy the conditions:*

(a) $\lim_{k\to\infty} \lambda_k = 0, \sum_{n=0}^{\infty} \lambda_k = \infty$;
(b) *either* $\lim_{k\to\infty} sup b_k \leq 0$ *or* $\sum_{k=0}^{\infty} |\lambda_k b_k| < \infty$;
(c) $c_k \geq 0\ (k \geq 0),\ \sum_k c_k < \infty$.

Then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.6 ([2]). *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $T : C \to C$ *be a nonexpansive mapping such that Fix* $(T) \neq \emptyset$. *If a sequence* $\{x_n\}$ *in* C *is such that* $x_n \to z$ *and* $x_n - Tx_n \to 0$, *then* $z = Tz$.

Lemma 2.7 ([13]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space and let* $\{\beta_n\}$ *be a sequence of* $[0,1]$ *such that* $0 < \lim_{n\to\infty} \inf \beta_n \leq \lim_{n\to\infty} \sup \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)\, y_n + \beta_n x_n$, $\forall n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \sup\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.8 ([4]). *Let C be a nonempty closed subset of H and let $\{x_n\}_{n\in N}$ be a sequence in H which is quasi-Fejer monotone with respect to C, i.e., there exists a summable sequence $\{\varepsilon_n\}_{n\in N}$ in $[0,+\infty)$ such that $(\forall x\in C)(\forall n\in N)\,\|x_{n+1}-x\|\leq \|x_n-x\|+\varepsilon_n$. Then*

(i) *The sequence $\{x_n\}_{n\in N}$ is bounded.*
(ii) *The sequence $\{x_n\}_{n\in N}$ converges weakly to a point in C if and only if $w\,(x_n)_{n\in N}\subset C$, where $w\,(x_n)_{n\in N}$ denotes the set of weak cluster points of sequence $\{x_n\}_{n\in N}$.*

Remark 2.9 ([5]). A mapping S of C in H is called strongly monotone and S is also called γ-strongly monotone if and only if there has a positive number γ such that $\langle x-y, Sx-Sy\rangle \geq \gamma\|x-y\|^2$ for all $x,y\in C$. A mapping S is γ/k^2-inverse-strongly monotone if and only if S is γ-strongly monotone and k-Lipschitz continuous, satisfied $\|Sx-Sy\|\leq k\|x-y\|,\ \forall x,y\in C$.

Remark 2.10 ([2]). If S is an α-inverse-strongly monotone mapping of C in H, then S is $1/\alpha$-Lipschitz continuous. We also get that $\forall x,y\in C$ and $\lambda>0$,

$$\begin{aligned}&\|(I-\lambda S)x-(I-\lambda S)y\|^2\\ &\leq \|x-y\|^2+\lambda(\lambda-2\alpha)\|Sx-Sy\|^2\end{aligned}$$

So, if $\lambda\leq 2\alpha$, the $I-\lambda S$ is a nonexpansive mapping of C into H.

Remark 2.11 ([11]). A set-valued mapping $S: H\to 2^H$ is called monotone if $\forall x,y\in H, f\in Sx$ and $g\in Sy$ so that $\langle x-y, f-g\rangle\geq 0$. A monotone mapping $S: H\to 2^H$ is maximal if the graph $G(S)$ of S is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping S is maximal if and only if $\forall(x,f)\in H\times H, \langle x-y, f-g\rangle\geq 0$, $\forall(y,g)\in G(S)$ implies $f\in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let N_Cv be the normal cone to C at $v\in C$, $N_Cv=\{w\in H:\langle v-u,w\rangle\geq 0, \forall u\in C\}$, and define

$$Sv=\begin{cases} Av+N_Cv, & v\in C\\ \emptyset, & v\notin C\end{cases}$$

Thus S is maximally monotone and $0\in Sv$ if and only if $v\in VI(C,A)$; see [11].

3. Main results

In this section, according to the above remarks, we can put forward the following two theorems.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S)\cap VI(C,A)\neq\varnothing$. Suppose $x_1=x\in C, \{e_n\}$ is regard as an error sequence and $e_n\in H$ and $\sum_{n=1}^{\infty}\|e_n\|<+\infty$, and $\{x_n\}$ is given by $x_0,x_1\in C$,*

$$\begin{cases} w_n=x_n+\theta_n\left(x_n-x_{n-1}\right);\\ y_n=P_C\left(I-\lambda_nA\right)\left(w_n-e_n\right);\\ x_{n+1}=\alpha_nx_n+\left(1-\alpha_n\right)Sy_n.\end{cases}$$

For every $n = 1, 2 \ldots$*where* $\{\alpha_n\}$ *is a sequence in* $[0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *Choose* $\theta_n \in [0,1]$ *and* $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* a, b *with* $0 < a < b < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. *Then* $\{x_n\}$ *converges weakly to* $z \in F(S) \cap VI(C, A)$.

Proof. Put $y_n = P_C (I - \lambda_n A)(x_n - e_n)$. Let $u \in F(S) \cap VI(C, A)$. Since $I - \lambda_n A$ is nonexpansive, from Lemma 2.3, we have

$$\begin{aligned}\|y_n - u\| &= \|P_C (I - \lambda_n A)(w_n - e_n) - P_C (I - \lambda_n A) u\| \\ &\le \|w_n - u\| + \|e_n\| \\ &\le \|x_n - u\| + \theta_n \|x_n - x_{n-1}\| + \|e_n\| .\end{aligned}$$

So, we can have

$$\begin{aligned}\|x_{n+1} - u\| &\le \alpha_n \|x_n - u\| + (1 - \alpha_n) \|Sy_n - u\| \\ &\le \alpha_n \|x_n - u\| + (1 - \alpha_n) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\| + \|e_n\|] \\ &\le \|x_n - u\| + \theta_n \|x_n - x_{n-1}\| + \|e_n\| .\end{aligned}$$

Since $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$, so we can have $\theta_n \|x_n - x_{n-1}\| \to 0$, so there exists a positive integer M_1 make sure $\theta_n\|x_n - x_{n-1}\| \le M_1$. By Lemma 2.8, we can get $\{x_n\}$ is bounded. Hence $\{w_n\}, \{y_n\}, \{Sy_n\}, \{Ax_n\}$ are also bounded. Since $I - \lambda_n A$ is nonexpansive, we also have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|(I - \lambda_{n+1} A)(w_{n+1} - e_{n+1}) - (I - \lambda_n A)(w_n - e_n)\| \\ &\le \|(I - \lambda_{n+1} A) w_{n+1} - (I - \lambda_{n+1} A) w_n\| + |\lambda_n - \lambda_{n+1}| \|Aw_n\| \\ &\quad + |\lambda_n - \lambda_{n+1}| \|Ae_n\| + \|e_{n+1} - e_n\| \\ &\le \|w_{n+1} - w_n\| + |\lambda_n - \lambda_{n+1}| \|Aw_n\| + |\lambda_n - \lambda_{n+1}| \|Ae_n\| \\ &\quad + \|e_{n+1} - e_n\| \\ &\le \|x_{n+1} - x_n\| + \theta_{n+1} \|x_{n+1} - x_n\| + \theta_n \|x_n - x_{n-1}\| \\ &\quad + |\lambda_n - \lambda_{n+1}| \|Aw_n\| + |\lambda_n - \lambda_{n+1}| \|Ae_n\| + \|e_{n+1} - e_n\| .\end{aligned}$$

Since

$$\begin{aligned}\|Sy_{n+1} - Sy_n\| &\le \|x_{n+1} - x_n\| + \theta_{n+1} \|x_{n+1} - x_n\| + \theta_n \|x_n - x_{n-1}\| \\ &\quad + |\lambda_n - \lambda_{n+1}| \|Aw_n\| + |\lambda_n - \lambda_{n+1}| \|Ae_n\| + \|e_{n+1} - e_n\| .\end{aligned}$$

We can obtain $\lim_{n\to\infty} \sup (\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|) \le 0$. By Lemma 2.7, $Sy_n - x_n \to 0$. Thus $\|x_{n+1} - x_n\| \to 0$. So we can have $\|y_{n+1} - y_n\| \to 0$. For $u \in F(S) \cap VI(C, A)$, we can obtain

$$\begin{aligned}\|x_{n+1} - u\|^2 &\le \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|Sy_n - u\|^2 \\ &\le \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|y_n - u\|^2 \\ &\le \alpha_n \|x_n - u\|^2 + \|w_n - u\|^2 + (1 - \alpha_n) \|e_n\|^2 \\ &\quad + 2(1 - \alpha_n) \|w_n - u\| \|e_n\| \\ &\quad + (1 - \alpha_n) a(b - 2\alpha) \|Aw_n - Au\|^2 \\ &\quad + (1 - \alpha_n) a(b - 2\alpha) \|Ae_n\|^2\end{aligned}$$

$$+ 2(1-\alpha_n)\, a(b-2\alpha)\, \|Aw_n - Au\|\, \|Ae_n\| .$$

Therefore, we can have

$$\begin{aligned}
&- (1-\alpha_n)\, a\,(b-2\alpha)\, \|Aw_n - Au\|^2 \\
&\le \alpha_n \|x_n - u\|^2 + \|x_n - u\|^2 + 2\theta_n \|x_n - u\|\, \|x_n - x_{n-1}\| + {\theta_n}^2 \|x_n - x_{n-1}\|^2 \\
&\quad - \|x_{n+1} - u\|^2 + (1-\alpha_n) \|e_n\|^2 + 2(1-\alpha_n) \|w_n - u\|\, \|e_n\| \\
&+ (1-\alpha_n)\, a(b-2\alpha) \|Ae_n\|^2 + 2(1-\alpha_n)\, a(b-2\alpha) \|Aw_n - Au\|\, \|Ae_n\| \\
&\le \alpha_n \|x_n - u\|^2 + (\|x_n - u\| + \|x_{n+1} - u\|) \times (\|x_n - u\| - \|x_{n+1} - u\|) \\
&\quad + 2\theta_n \|x_n - u\|\, \|x_n - x_{n-1}\| + \theta_n^2 \|x_n - x_{n-1}\|^2 + (1-\alpha_n) \|e_n\|^2 \\
&\quad + 2(1-\alpha_n) \|w_n - u\|\, \|e_n\| + (1-\alpha_n)\, a(b-2\alpha) \|Ae_n\|^2 \\
&\quad + 2(1-\alpha_n)\, a(b-2\alpha) \|Aw_n - Au\|\, \|Ae_n\| .
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, and $\sum_{n=0}^{\infty} \|e_n\| < +\infty$, we can get $\|Aw_n - Au\| \to 0$. From Lemma 2.3, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \langle (I - \lambda_n A)(w_n - e_n) - (I - \lambda_n A)\, u, y_n - u\rangle \\
&= \frac{1}{2} \Big[\|(I - \lambda_n A)(w_n - e_n) - (I - \lambda_n A)\, u\|^2 + \|y_n - u\|^2 \\
&\quad - \|(I - \lambda_n A)(w_n - e_n) - (I - \lambda_n A)\, u - (y_n - u)\|^2 \Big] \\
&\le \frac{1}{2} \Big[\|w_n - u\|^2 + 2\|w_n - u\|\, \|e_n\| + \|e_n\|^2 + \|y_n - u\|^2 - \|w_n - y_n\|^2 \\
&\quad + 2\lambda_n \langle w_n - y_n, Aw_n - Ae_n - Au\rangle - {\lambda_n}^2 \|Aw_n - Ae_n - Au\|^2 \\
&\quad + 2\|(w_n - y_n) - \lambda_n (Aw_n - Ae_n - Au)\|\, \|e_n\| - \|e_n\|^2 \Big] .
\end{aligned}$$

Thus

$$\begin{aligned}
\|y_n - u\|^2 \le\; & \|w_n - u\|^2 + 2\|w_n - u\|\, \|e_n\| - \|w_n - y_n\|^2 \\
& + 2\lambda_n \langle w_n - y_n, Aw_n - Au\rangle - 2\lambda_n \langle w_n - y_n, Ae_n\rangle \\
& - {\lambda_n}^2 \|Aw_n - Au\|^2 + 2{\lambda_n}^2 \|Aw_n - Au\|\, \|Ae_n\| - {\lambda_n}^2 \|Ae_n\|^2 \\
& + 2\|(w_n - y_n) - \lambda_n (Aw_n - Ae_n - Au)\|\, \|e_n\| ,
\end{aligned}$$

which shows that

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&\le \alpha_n \|x_n - u\|^2 + (1-\alpha_n) \Big[\|w_n - u\|^2 + 2\|w_n - u\|\, \|e_n\| - \|w_n - y_n\|^2 \\
&\quad + 2\lambda_n \langle w_n - y_n, Aw_n - Au\rangle - 2\lambda_n \langle w_n - y_n, Ae_n\rangle \\
&\quad - {\lambda_n}^2 \|Aw_n - Au\|^2 + 2{\lambda_n}^2 \|Aw_n - Au\|\, \|Ae_n\| - {\lambda_n}^2 \|Ae_n\|^2 \\
&\quad + 2\|(w_n - y_n) - \lambda_n (Aw_n - Ae_n - Au)\|\, \|e_n\|] \\
&\le \|x_n - u\|^2 + 2\theta_n \|x_n - u\|\, \|x_n - x_{n-1}\| + {\theta_n}^2 \|x_n - x_{n-1}\|^2 \\
&\quad + 2\|w_n - u\|\, \|e_n\| - \|w_n - y_n\|^2 + 2\lambda_n \langle w_n - y_n, Aw_n - Au\rangle
\end{aligned}$$

$$- 2\lambda_n\langle w_n - y_n, Ae_n\rangle - \lambda_n^2 \|Aw_n - Au\|^2 + 2\lambda_n^2 \|Aw_n - Au\| \|Ae_n\|$$
$$- {\lambda_n}^2 \|Ae_n\|^2 + 2 \|(w_n - y_n) - \lambda_n (Aw_n - Ae_n - Au)\| \|e_n\| .$$

Since $\|x_{n+1} - x_n\| \to 0, \|Aw_n - Au\| \to 0, \|e_n\| \to 0, \|Ae_n\| \to$, we can get $\|w_n - y_n\| \to 0$. Thus $|x_n - y_n| \leq \|x_n - w_n\| + \|w_n - y_n\| \to 0$ and $\|Sx_n - x_n\| \leq \|Sx_n - Sy_n\| + \|Sy_n - x_n\| \leq \|x_n - y_n\| + \|Sy_n - x_n\| \to 0$. We assume that there is a sequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to z. Then we can have $z \in F(S) \cap VI(C, A)$. Since $x_n - y_n \to 0$, we can have $y_{n_i} \to z$. Let $Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$. This prove that operator T is maximally monotone. Let $(v, w) \in G(T)$. From $w - Av \in N_C v$ and $y_n \in C$, one concludes $\langle v - y_n, w - Av\rangle \geq 0$. By $y_n = P_C (I - \lambda_n A) (w_n - e_n)$, we have $\langle v- y_n, y_n - (I - \lambda_n A) (w_n - e_n)\rangle \geq 0$. Thus $\langle v - y_n, \frac{y_n - w_n + e_n}{\lambda_n} + Aw_n - Ae_n\rangle \geq 0$. Therefore, we can have

$$\begin{aligned} \langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\ &\geq \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Aw_{n_i}\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - w_{n_i}}{\lambda_{n_i}}\rangle \\ &\quad - \langle v - y_{n_i}, \frac{e_{n_i}}{\lambda_{n_i}} + Ae_{n_i}\rangle. \end{aligned}$$

This proves $\langle v - z, w\rangle \geq 0, i \to \infty$. Thus $z \in T^{-1}0$ and then $z \in VI(C, A)$. Let us now prove $z \in F(S)$. Since $\|x_n - Sx_n\| = \|Sx_n - x_n\| \to 0$, we can have $z \in F(S)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to z'$. Then, $z' \in F(S) \cap VI(C, A)$. We may show that $z = z'$. Assume that $z \neq z'$. From the Opial condition, we can have a contradiction. Thus, $z = z'$. This implies that $x_n \to z \in F(S) \cap VI(C, A)$. □

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction mapping with coefficient $k(0 < \mathrm{k} < 1)$. A is an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $\{e_n\}$ is regard as an error sequence and $e_n \in H$ and $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, and $\{x_n\}$ is given by $x_0, x_1 \in C$,*

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}) ; \\ y_n = P_C (I - \lambda_n A) (w_n - e_n) ; \\ x_{n+1} = \alpha_n f (x_n) + (1 - \alpha_n) Sy_n. \end{cases}$$

For every $n = 1, 2 \ldots$ where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. Choose $\theta_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap VI(C, A)$, which is the unique solution in the $F(S) \cap VI(C, A)$ to the following variational inequality $\langle (I - f)q, q - p\rangle \leq 0, p \in F(S) \cap VI(C, A)$.

Proof. Put $y_n = P_C (I - \lambda_n A) (x_n - e_n)$ and let $u \in F(S) \cap VI(C, A)$. Since $I - \lambda_n A$ is nonexpansive, from Lemma 2.3, we have

$$\|y_n - u\| \leq \|x_n - u\| + \theta_n \|x_n - x_{n-1}\| + \|e_n\| .$$

So, we can have

$$\begin{aligned}\|x_{n+1}-u\| &\leq \alpha_n \|f(x_n)-u\| + (1-\alpha_n)\|Sy_n - u\| \\ &\leq \alpha_n k\|x_n-u\| + (1-\alpha_n)\|x_n-u\| \\ &\quad + (1-\alpha_n)\theta_n\|x_n-x_{n-1}\| + (1-\alpha_n)\|e_n\| + \alpha_n\|f(u)-u\| \\ &\leq \alpha_n k\|x_n-u\| + (1-\alpha_n)\|x_n-u\| + \theta_n\|x_n-x_{n-1}\| \\ &\quad + (1-\alpha_n)\|e_n\| + \alpha_n\|f(u)-u\| \\ &\leq (1-(1-k)\alpha_n)\|x_n-u\| \\ &\quad + (1-k)\alpha_n\left[\frac{1}{1-k}\|f(u)-u\| + \frac{\|e_n\|+\theta_n\|x_n-x_{n-1}\|}{(1-k)\alpha_n}\right] \\ &\leq \max\left\{\|x_n-u\|, \frac{1}{1-k}\|f(u)-u\| + 2M_2\right\} \\ &\leq \cdots \\ &\leq \max\left\{\|x_1-u\|, \frac{1}{1-k}\|f(u)-u\| + 2M_2\right\}. \\ M_2 &= \max\left\{\sup\frac{\|e_n\|}{(1-k)\alpha_n}, \sup\frac{\theta_n\|x_n-x_{n-1}\|}{(1-k)\alpha_n}\right\}, n\in N.\end{aligned}$$

We can get $\{x_n\}$ is bounded. Since $I-\lambda_n A$ is nonexpansive, we also have

$$\begin{aligned}&\|y_{n+1}-y_n\| \\ &\leq \|(I-\lambda_{n+1}A)(w_{n+1}-e_{n+1}) - (I-\lambda_n A)(w_n-e_n)\| \\ &\leq \|(I-\lambda_{n+1}A)w_{n+1} - (I-\lambda_{n+1}A)w_n\| + |\lambda_n-\lambda_{n+1}|\|Aw_n\| \\ &\quad + |\lambda_n-\lambda_{n+1}|\|Ae_n\| + \|e_{n+1}-e_n\| \\ &\leq \|w_{n+1}-w_n\| + |\lambda_n-\lambda_{n+1}|\|Aw_n\| + |\lambda_n-\lambda_{n+1}|\|Ae_n\| + \|e_{n+1}-e_n\| \\ &\leq \|x_{n+1}-x_n\| + \theta_{n+1}\|x_{n+1}-x_n\| + \theta_n\|x_n-x_{n-1}\| + |\lambda_n-\lambda_{n+1}|\|Aw_n\| \\ &\quad + |\lambda_n-\lambda_{n+1}|\|Ae_n\| + \|e_{n+1}-e_n\|.\end{aligned}$$

That yields

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ &\leq |\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Sy_{n-1}\| \\ &\quad + (1-\alpha_n)\|Sy_n-Sy_{n-1}\| + \alpha_n k\|x_n-x_{n-1}\| \\ &\leq (1-\alpha_n)[\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\| + \theta_{n-1}\|x_{n-1}-x_{n-2}\| \\ &\quad + |\lambda_{n-1}-\lambda_n|\|Aw_{n-1}\| + |\lambda_{n-1}-\lambda_n|\|Ae_{n-1}\| + \|e_n-e_{n-1}\|] \\ &\quad + |\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Sy_{n-1}\| + \alpha_n k\|x_n-x_{n-1}\| \\ &\leq (1-(1-k)\alpha_n)\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\| + \theta_{n-1}\|x_{n-1}-x_{n-2}\| \\ &\quad + (L_1+Q_1)|\lambda_{n-1}-\lambda_n| \\ &\quad + M_3|\alpha_n-\alpha_{n-1}| + \|e_n-e_{n-1}\|,\end{aligned}$$

where

$$L_1 = \sup\{\|Aw_{n-1}\|, n\in N\}, Q_1 = \sup\{\|Ae_{n-1}\|, n\in N\},$$

and

$$M_3 = \sup\left\{\|f(x_{n-1}) - Sy_{n-1}\|, n \in N\right\}.$$

By Lemma 2.5, we can get $\|x_n - x_{n-1}\| \to 0$, and then $\|x_{n+1} - x_n\| \to 0$. From $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\|e_{n+1} - e_n\| \to 0$, we can obtain $\|y_{n+1} - y_n\| \to 0$, and then $\|y_{n-1} - y_n\| \to 0$. Observe that $\lim_{n\to\infty} \|x_n - Sy_n\| = 0$. By remark 2.10, we can obtain

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&\le \alpha_n \|f(x_n) - u\|^2 + (1-\alpha_n)\|Sy_n - u\|^2 \\
&\le \alpha_n \|f(x_n) - u\|^2 \\
&\quad + (1-\alpha_n)\left[\|w_n - e_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A(w_n - e_n) - Au\|^2\right] \\
&\le \alpha_n \|f(x_n) - u\|^2 + \|w_n - u\|^2 + (1-\alpha_n)\|e_n\|^2 + 2(1-\alpha_n)\|w_n - u\|\|e_n\| \\
&\quad + (1-\alpha_n)a(b-2\alpha)\|Aw_n - Au\|^2 + (1-\alpha_n)a(b-2\alpha)\|Ae_n\|^2 \\
&\quad + 2(1-\alpha_n)a(b-2\alpha)\|Aw_n - Au\|\|Ae_n\|.
\end{aligned}$$

After simple calculations, we can have $\|Aw_n - Au\| \to 0$. Considering $\|w_n - e_n - u\| \le \|w_n - u\| + \|e_n\|$, we can have

$$\|w_n - e_n - u\|^2 \le \|w_n - u\|^2 + 2\|w_n - u\|\|e_n\| + \|e_n\|^2.$$

From Lemma 2.3, we have

$$\begin{aligned}
&\|y_n - u\|^2 \\
&\le \langle (I - \lambda_n A)(w_n - e_n) - (I - \lambda_n A)u, y_n - u\rangle \\
&\le \frac{1}{2}\Big[\|w_n - e_n - u\|^2 + \|y_n - u\|^2 - \|(w_n - y_n) - \lambda_n(Aw_n - Ae_n - Au)\|^2 \\
&\quad +2\|(w_n - y_n) - \lambda_n(Aw_n - Ae_n - Au)\|\|e_n\| - \|e_n\|^2\Big] \\
&\le \frac{1}{2}\Big[\|w_n - u\|^2 + 2\|w_n - u\|\|e_n\| + \|e_n\|^2 + \|y_n - u\|^2 - \|w_n - y_n\|^2 \\
&\quad + 2\lambda_n\langle w_n - y_n, Aw_n - Ae_n - Au\rangle - \lambda_n^2\|Aw_n - Ae_n - Au\|^2 \\
&\quad +2\|(w_n - y_n) - \lambda_n(Aw_n - Ae_n - Au)\|\|e_n\| - \|e_n\|^2\Big].
\end{aligned}$$

So, we can obtain

$$\begin{aligned}
&\|y_n - u\|^2 \\
&\le \|w_n - u\|^2 + 2\|w_n - u\|\|e_n\| - \|w_n - y_n\|^2 \\
&\quad + 2\lambda_n\langle w_n - y_n, Aw_n - Au\rangle - 2\lambda_n\langle w_n - y_n, Ae_n\rangle \\
&\quad - {\lambda_n}^2\|Aw_n - Au\|^2 + 2{\lambda_n}^2\|Aw_n - Au\|\|Ae_n\| - {\lambda_n}^2\|Ae_n\|^2 \\
&\quad + 2\|(w_n - y_n) - \lambda_n(Aw_n - Ae_n - Au)\|\|e_n\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&\le \alpha_n \|f(x_n) - u\|^2 + (1-\alpha_n)\|Sy_n - u\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \left\|f\left(x_n\right)-u\right\|^2+\left\|x_n-u\right\|^2+2\theta_n\left\|x_n-u\right\|\left\|x_n-x_{n-1}\right\| \\
&\quad+{\theta_n}^2\left\|x_n-x_{n-1}\right\|^2+2\left\|w_n-u\right\|\left\|e_n\right\|-\left\|w_n-y_n\right\|^2 \\
&\quad+2\lambda_n\langle w_n-y_n, Aw_n-Au\rangle-2\lambda_n\langle w_n-y_n, Ae_n\rangle \\
&\quad-{\lambda_n}^2\left\|Aw_n-Au\right\|^2+2{\lambda_n}^2\left\|Aw_n-Au\right\|\left\|Ae_n\right\|-{\lambda_n}^2 \\
&\quad\left\|Ae_n\right\|^2+2\left\|\left(w_n-y_n\right)-\lambda_n\left(Aw_n-Ae_n-Au\right)\right\|\left\|e_n\right\|.
\end{aligned}
$$

Since $\alpha_n \to 0, \|x_{n+1}-x_n\| \to 0, \|Aw_n-Au\| \to 0, \|e_n\| \to 0, \|Ae_n\| \to 0$, we can get $\|w_n-y_n\| \to 0$. Observe $\|x_n-w_n\| = \theta_n\|x_n-x_{n-1}\| \to 0$, $\|x_n-y_n\| \leq \|x_n-w_n\|+\|w_n-y_n\| \to 0$, $\|Sx_n-x_n\| \leq \|x_n-y_n\|+\|Sy_n-x_n\| \to 0$. Next we show that $\lim_{n\to\infty}\sup\langle f(u)-u, Sy_n-u\rangle \leq 0$, where $u \in F(S)\cap VI(C,A)$. To show it, choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$
\lim_{n\to\infty}\sup\langle f(u)-u, Sy_n-u\rangle = \lim_{i\to\infty}\sup\langle f(u)-u, Sy_{n_i}-u\rangle.
$$

As $\{x_n\}$ is bounded, we have that a sequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to z. Then we can have $z \in F(S)\cap VI(C,A)$. Since $x_n-y_n \to 0$, we can have $y_{n_i} \to z$. We first show that $z \in VI(C,A)$. Let $Tv = \begin{cases} Av+N_Cv, v\in C, \\ \emptyset, \quad v\notin C. \end{cases}$. Then T is maximal monotone.Let $(v,w) \in G(T)$. Since $w-Av \in N_Cv$ and $y_n \in C$, we have $\langle v-y_n, w-Av\rangle \geq 0$. By $y_n = P_C\left(I-\lambda_nA\right)\left(w_n-e_n\right)$, we have $\langle v-y_n, y_n-\left(I-\lambda_nA\right)\left(w_n-e_n\right)\rangle \geq 0$ and hence $\langle v-y_n, \frac{y_n-w_n+e_n}{\lambda_n}+Aw_n-Ae_n\rangle \geq 0$. Therefore, we can obtain $\langle v-z, w\rangle \geq 0, i\to\infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C,A)$. Let us show that $z \in F(S)$. Since $\|x_n-Sx_n\| = \|Sx_n-x_n\| \to 0$, so based on Lemma 2.6, we can have $z \in F(S)$. Thus

$$
\lim_{n\to\infty}\sup\langle f(u)-u, Sy_n-u\rangle = \langle f(u)-u, z-u\rangle \leq 0.
$$

Next we can have

$$
\begin{aligned}
&\left\|x_{n+1}-u\right\|^2 \\
&\leq \left(1-2\alpha_n+\alpha_n^2\right)\left\|x_n-u\right\|^2+2\theta_n\left\|x_n-x_{n-1}\right\|\left\|x_n-u\right\| \\
&\quad+\theta_n^2\left\|x_n-x_{n-1}\right\|^2+\left\|e_n\right\|^2+2\left\|w_n-u\right\|\left\|e_n\right\|+{\alpha_n}^2\left\|f\left(x_n\right)-u\right\|^2 \\
&\quad+2\alpha_n\left(1-\alpha_n\right)\left\|f\left(x_n\right)-f(u)\right\|\left\|Sy_n-u\right\| \\
&\quad+2\alpha_n\left(1-\alpha_n\right)\langle f(u)-u, Sy_n-u\rangle \\
&\leq \left[1-2\alpha_n+{\alpha_n}^2+2k\alpha_n\left(1-\alpha_n\right)\right]\left\|x_n-u\right\|^2+2\theta_n\left\|x_n-x_{n-1}\right\|\left\|x_n-u\right\| \\
&\quad+\theta_n^2\left\|x_n-x_{n-1}\right\|^2+\left\|e_n\right\|^2+2\left\|w_n-u\right\|\left\|e_n\right\|+{\alpha_n}^2\left\|f\left(x_n\right)-u\right\|^2 \\
&\quad+2\alpha_n\left(1-\alpha_n\right)k\theta_n\left\|x_n-u\right\|\left\|x_n-x_{n-1}\right\| \\
&\quad+2\alpha_n\left(1-\alpha_n\right)k\left\|x_n-u\right\|\left\|e_n\right\|+2\alpha_n\left(1-\alpha_n\right)\langle f(u)-u, Sy_n-u\rangle \\
&\leq \left(1-\overline{\alpha_n}\right)\left\|x_n-u\right\|^2+\overline{\alpha_n}\overline{\beta_n}+\overline{\gamma_n},
\end{aligned}
$$

where $M_4 = \sup\{\|x_n - u\|\}$, $M_5 = \sup\{\|w_n - u\|\}$, $\overline{\alpha_n} = \alpha_n[2 - \alpha_n - 2k(1 - \alpha_n)]$,

$$\overline{\beta_n} = +\frac{\alpha_n \|f(x_n) - u\|^2}{2 - \alpha_n - 2k(1 - \alpha_n)} + \frac{2(1 - \alpha_n) k\theta_n M_4 \|x_n - x_{n-1}\|}{2 - \alpha_n - 2k(1 - \alpha_n)} + \frac{2\alpha_n (1 - \alpha_n) kM_4 \|e_n\|}{2 - \alpha_n - 2k(1 - \alpha_n)} + \frac{2(1 - \alpha_n)\langle f(u) - u, Sy_n - u\rangle}{2 - alpha_n - 2k(1 - \alpha_n)}.$$

and

$$\overline{\gamma_n} = \|e_n\|^2 + 2M_5\|e_n\| + 2\theta_n M_4 \|x_n - x_{n-1}\| + {\theta_n}^2 \|x_n - x_{n-1}\|^2.$$

By Lemma 2.5, we can get $\|x_n - u\| \to 0$, so that $x_n \to u$. Hence the proof is complete. □

4. Applications

In this section we prove four theorems in a Hilbert space by using Theorem 3.1 and Theorem 3.2. A mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 < k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for every $x, y \in C$. If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then A is $(1 - k)/2$-inverse-strongly monotone. Actually, we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \le \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \| x - y \|^2 + \| Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \ge 1 - k/2\|Ax - Ay\|^2$$

Using Theorem 3.1 and Theorem 3.2, we prove strong convergence theorems for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H, let S be a nonexpansive mapping of C into itself and let T be a k-strictly pseudocontractive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Suppose $x_1 = x \in C$, $\{e_n\}$ is regard as an error sequence and $e_n \in H$ and $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, and $\{x_n\}$ is given by $x_0 \in C$,*

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}); \\ y_n = P_C (I - \lambda_n A)(w_n - e_n); \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Sy_n. \end{cases}$$

For every $n = 1, 2 \ldots.$ where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. Choose $\theta_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$.

Proof. Put $A = I - T$, then A is $1 - \alpha/2$-inverse strongly monotone. We have $F(T) = VI(C, A)$ and

$$P_C (I - \lambda_n A) (w_n - e_n) = (1 - \lambda_n) (w_n - e_n) + \lambda_n T (w_n - e_n).$$

So by Theorem 3.1, we can have the desired result. □

Theorem 4.2. *Let H be a real Hilbert space H. Let A be an α inverse-strongly monotone mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \varnothing$. Suppose $x_1 = x \in C, \{e_n\}$ is regard as an error sequence and $e_n \in H$ and $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, and $\{x_n\}$ is given by $x_0 \in C$,*

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}); \\ y_n = P_C (I - \lambda_n A) (w_n - e_n); \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n. \end{cases}$$

For every $n = 1, 2 \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. Choose $\theta_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then $\{x_n\}$ converges weakly to $F(S) \cap A^{-1}0$.

Proof. We have $A^{-1}0 = VI(H, A)$, so putting $P_H = I$, by theorem 3.1 we can get the desired. □

Theorem 4.3. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction mapping with coefficient $k \in (0, 1), S$ be a nonexpansive mapping of* C *into itself and let T be a strictly pseudocontractive mapping of C into itself with α, such that $F(S) \cap F(T) \neq \emptyset$. Suppose $\{e_n\}$ is regard as an error sequence and $e_n \in H$ and $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, and $\{x_n\}$ is given by $x_0, x_1 \in C$,*

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}); \\ y_n = P_C (I - \lambda_n A) (w_n - e_n); \\ x_{n+1} = \alpha_n f (x_n) + (1 - \alpha_n) S y_n. \end{cases}$$

For every $n = 1, 2 \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. Choose $\theta_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, which is the unique solution in the $F(S) \cap$ VI(C, A) *to the following variational inequality $\langle (I - f)q, q - p\rangle \leq 0, p \in F(S) \cap F(T)$.*

Proof. Put $A = I - T$, then A is $\frac{1-\alpha}{2}$-inverse strongly monotone. We have $F(T) = VI(C, A)$ and

$$P_C (I - \lambda_n A) (w_n - e_n) = (1 - \lambda_n) (w_n - e_n) + \lambda_n T (w_n - e_n).$$

So by Theorem 3.1, we can have the desired result. □

Theorem 4.4. *Let H be a real Hilbert space H. Let $f : C \to C$ be a contraction mapping with coefficient $k \in (0, 1), S$ be a nonexpansive mapping of H into itself and let A be a α-inverse strongly monotone mapping of H into itself, such that*

$F(S)\cap\ A^{-1}0 \neq \emptyset$. Suppose $\{e_n\}$ is regard as an error sequence and $e_n \in H$ and $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, and $\{x_n\}$ is given by $x_0, x_1 \in C$,

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}); \\ y_n = P_C (I - \lambda_n A)(w_n - e_n); \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n. \end{cases}$$

For every $n = 1, 2 \dots.$ where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. Choose $\theta_n \in [0,1]$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$ If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, which is the unique solution in the $F(S) \cap A^{-1}0$ to the following variational inequality $\langle (I-f)q, q-p\rangle \leq 0, p \in F(S) \cap A^{-1}0$.

Proof. We can obtain $VI(H,A) = A^{-1}0$. Putting $P_H = I$, by Theorem 3.1, we can get the results. □

5. Conclusions

In this paper, we propose two different algorithms with eroor equence, and prove the sequences converge to a common element of two sets under some proper conditions. Then, we introduce some theorems under some mild conditions are still convergent in applictions.

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