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PARALLEL MANN IMPLICIT SUBGRADIENT EXTRAGRADIENT ALGORITHMS FOR MONOTONE BILEVEL EQUILIBRIUM PROBLEMS

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Abstract. In a real Hilbert space, let the GSVI and CFPP represent a general system of variational inequalities and a common fixed-point problem of a countable family of uniformly Lipschitzian pseudocontraction mappings and an asymptotically nonexpansive mapping, respectively. In this paper, we introduce and analyze two parallel Mann implicit subgradient extragradient algorithms for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. Some strong convergence results for the proposed algorithms are established under the suitable assumptions, and also applied for finding a common solution of the GSVI, VIP and FPP, where the VIP and FPP stand for a variational inequality problem and a fixed-point problem, respectively.

1. Introduction

Suppose that $(\mathcal{H}, \|\cdot\|)$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, that C is a nonempty closed convex subset of $\mathcal{H}$, and that P_C is the metric projection from $\mathcal{H}$ onto C. Let $T: C \to C$ be a nonlinear mapping. We denote by $\mathrm{Fix}(T)$ the fixed-point set of T and by $\mathcal{R}$ the set of all real numbers, respectively. The mapping $T: C \to C$ is known as an asymptotically nonexpansive mapping if there exists a sequence $\{\theta_k\} \subset [0,\infty)$ such that $\lim_{k\to\infty} \theta_k = 0$ and $\|T^k u - T^k v\| \le (1+\theta_k)\|u-v\|\ \forall u,v \in C, k \ge 1$. In particular, if $\theta_k = 0\ \forall k \ge 1$, then T is said to be nonexpansive. Let A be a self-mapping on $\mathcal{H}$. The classical variational inequality problem (VIP) is to find $u^* \in C$ s.t. $\langle Au^*, v-u^*\rangle \ge 0\ \forall v \in C$. We denote by $\mathrm{VI}(C,A)$ the solution set of the VIP. It is well known that, the extragradient method suggested first by Korpelevich [36] in 1976 has become one of the most effective methods for solving the VIP till now. The weak convergence of this method to a solution of the VIP was proven in [36] if $\mathrm{VI}(C,A) \ne \emptyset$. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many scholars, who improved it via various approaches; see e.g., [1, 2, 6, 7, 10, 12–16, 18–21, 23, 25, 29, 31, 33, 34, 40, 42–47] and references therein, to name but a few. In 2011, Censor et al. [29] modified Korpelevich's extragradient method and first introduced the subgradient extragradient method, in which the second

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projection onto C is replaced by a projection onto a half-space:

$$\begin{cases} v^k = P_C(u^k - \ell A u^k), \\ C_k = \{v \in \mathcal{H} : \langle u^k - \ell A u^k - v^k, v - v^k \rangle \le 0\}, \\ u^{k+1} = P_{C_k}(u^k - \ell A v^k) \quad \forall k \ge 0, \end{cases}$$

with constant $\ell \in (0, \frac{1}{L})$ and L is the Lipschitz constant of A.

Let $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ be two nonlinear mappings. The general system of variational inequalities is the following problem of finding $(u^*, v^*) \in C \times C$ s.t.

$$\begin{cases} \langle \mu_1 B_1 v^* + u^* - v^*, u - u^* \rangle \ge 0 \quad \forall u \in C, \\ \langle \mu_2 B_2 u^* + v^* - u^*, v - v^* \rangle \ge 0 \quad \forall v \in C, \end{cases} \tag{1.1}$$

with constants $\mu_1, \mu_2 \in (0, \infty)$. In particular, if $B_1 = B_2 = A$ and $u^* = v^*$, then the GSVI (1.1) reduces to the classical VIP. Note that, problem (1.1) can be transformed into a fixed point problem in the following way.

Lemma 1.1 (see e.g., [22]). *For given $u^*, v^* \in C$, (u^*, v^*) is a solution of GSVI (1.1) if and only if $u^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)$, and $v^* = P_C(I - \mu_2 B_2) u^*$.*

Let $B_1, B_2 : C \to \mathcal{H}$ be two inverse-strongly monotone mappings and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_k\}$. Let the Ω denote the common solution set of the GSVI (1.1) and the fixed-point problem (FPP) of T. In 2018, using a modified extragradient method, Cai et al. [5] introduced a viscosity implicit rule for finding a solution of the GSVI (1.1) with the FPP constraint, that is, for any initial $x^1 \in C$, the sequence $\{x^k\}$ is generated by

$$\begin{cases} u^k = s_k x^k + (1 - s_k) p^k, \\ v^k = P_C(u^k - \mu_2 B_2 u^k), \\ p^k = P_C(v^k - \mu_1 B_1 v^k), \\ x^{k+1} = P_C[\alpha_k f(x^k) + (I - \alpha_k \rho F) T^k p^k], \end{cases} \tag{1.2}$$

where $f : C \to C$ is a δ-contraction with $\delta \in [0, 1)$, and $\{\alpha_k\}, \{s_k\} \subset (0, 1]$ are such that

(i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$;
(ii) $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} = 0$;
(iii) $0 < \varepsilon \le s_k \le 1$ and $\sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty$;
(iv) $\sum_{k=1}^{\infty} \|T^{k+1} p^k - T^k p^k\| < \infty$.

They proved the strong convergence of $\{x^k\}$ to an element $x^* \in \Omega$, which solves the VIP: $\langle (\rho F - f) x^*, x - x^* \rangle \ge 0 \ \forall x \in \Omega$. Subsequently, Ceng and Wen [24] proposed a hybrid extragradient-like implicit method with strong convergence for finding a solution of the GSVI (1.1) with the constraint of a common fixed point problem (CFPP). Very recently, Ceng et al. [18] suggested a modified inertial subgradient extragradient method for finding a common solution of the VIP with pseudomonotone and Lipschitz continuous mapping $A : \mathcal{H} \to \mathcal{H}$ and the CFPP of finitely many nonexpansive mappings $\{T_i\}_{i=1}^N$ on $\mathcal{H}$. Under some suitable conditions, they proved strong convergence of the constructed sequence to a common solution of the VIP and CFPP.

On the other hand, let $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ be a bifunction satisfying $\Phi(u, u) = 0\ \forall u \in C$. The equilibrium problem (shortly, EP(C, Φ)) for the bifunction Φ on the constraint domain C is to find $\hat{u} \in C$ such that

$$\Phi(\hat{u}, v) \geq 0 \quad \forall v \in C. \tag{1.3}$$

The solution set of EP(C, Φ) is denoted by Sol(C, Φ). It is worth pointing out that the EP (1.3) is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems and so forth. Many algorithms have been suggested and studied for solving the EP (1.3) and its extended versions; see [1, 5, 6, 8, 9, 11, 12, 15, 17, 21, 23, 24, 27, 28, 30, 31]and references therein. Very recently, Anh and An [1] introduced the monotone bilevel equilibrium problem (MBEP) with the fixed-point problem (FPP) constraint, i.e., a strongly monotone equilibrium problem EP(Ω, Ψ) over the common solution set Ω of another monotone equilibrium problem EP(C, Φ) and the fixed-point problem of $\mathcal{K}$-demicontractive mapping T:

$$\text{Find } u^* \in \Omega \text{ such that } \Psi(u^*, v) \geq 0 \quad \forall v \in \Omega, \tag{1.4}$$

where $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ such that $\Psi(u, u) = 0\ \forall u \in C$ and $\Omega = \text{Sol}(C, \Phi) \cap \text{Fix}(T)$.

Pick the parameter sequences $\{\lambda_k\}$ and $\{\beta_k\}$ such that

$$\begin{cases} \{\lambda_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}),\ \lim_{k\to\infty} \lambda_k = \lambda, \\ \beta_k \downarrow 0,\ 2\beta_k\eta - \beta_k^2 \Upsilon^2 < 1,\ \sum_{k=0}^{\infty} \beta_k = +\infty, \\ 0 < \tau < \min\{\eta, \Upsilon\},\ 0 < \beta_k < \min\{\frac{1}{\tau}, \frac{2\eta - 2\tau}{\Upsilon^2 - \tau^2}, \frac{2\eta}{\Upsilon^2}\}, \end{cases} \tag{1.5}$$

where Υ is a constant associated with Ψ. The following modified subgradient extragradient method is proposed in [1, Algorithm 4.1]for finding a unique element of Sol(Ω, Ψ).

Algorithm 1.1. Initial Step: Choose an initial point $u^0 \in C$ and $\{\alpha_k\} \subset [r, \bar{r}] \subset (0, 1 - \mathcal{K})$. The parameter sequences $\{\lambda_k\}$ and $\{\beta_k\}$ satisfy the conditions (1.5).

Iterative Steps: Compute u^{k+1} $(k \geq 0)$ as follows:

Step 1: Compute $v^k = \text{argmin}\{\lambda_k \Phi(u^k, v) + \frac{1}{2}\|v - u^k\|^2 : v \in C\}$ and $q^k = \text{argmin}\{\lambda_k \Phi(v^k, z) + \frac{1}{2}\|z - u^k\|^2 : z \in C_k\}$, where $C_k = \{y \in \mathcal{H} : \langle u^k - \lambda_k w^k - v^k, y - v^k \rangle \leq 0\}$ and $w^k \in \partial_2 \Phi(u^k, v^k)$.

Step 2: Compute $p^k = (1 - \alpha_k)q^k + \alpha_k T q^k$ and $u^{k+1} = \text{argmin}\{\beta_k \Psi(p^k, p) + \frac{1}{2}\|p - p^k\|^2 : p \in C\}$. Set $k := k + 1$ and return to Step 1.

It was proven in [1] that $\{u^k\}$ converges strongly to a unique element of Sol(Ω, Ψ) under the mild conditions. In what follows, let the CFPP indicate a common fixed-point problem of a countable family of uniformly Lipschitzian pseudocontractions and an asymptotically nonexpansive mapping. In this paper, we introduce two parallel Mann implicit subgradient extragradient algorithms for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. Some strong convergence results for the proposed algorithms are established under the suitable assumptions.

Finally, some applications for finding a common solution of the GSVI, VIP and FPP are also considered. Our results improve and extend some corresponding results in the very recent literature; see e.g., [1, 18, 24].

2. Preliminaries

Throughout this paper, we always assume that C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Given a sequence $\{u^k\} \subset \mathcal{H}$, we denote by $u^k \to u$ (resp., $u^k \rightharpoonup u$) the strong (resp., weak) convergence of $\{u^k\}$ to u. A bifunction $\Psi : C \times C \to \mathcal{R}$ is said to be

(i) η-strongly monotone, if $\Psi(u, v) + \Psi(v, u) \le -\eta\|u - v\|^2\ \forall u, v \in C$;
(ii) monotone, if $\Psi(u, v) + \Psi(v, u) \le 0\ \forall u, v \in C$;
(iii) Lipschitz-type continuous with constants $c_1, c_2 > 0$ (see [39]), if $\Psi(u, v) + \Psi(v, w) \ge \Psi(u, w) - c_1\|u - v\|^2 - c_2\|v - w\|^2\ \forall u, v, w \in C$.

Also, recall that a mapping $F : C \to \mathcal{H}$ is said to be

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t. $\|Fu - Fv\| \le L\|u - v\|\ \forall u, v \in C$;
(ii) monotone if $\langle Fu - Fv, u - v\rangle \ge 0\ \forall u, v \in C$;
(iii) pseudomonotone if $\langle Fu, v - u\rangle \ge 0 \Rightarrow \langle Fv, v - u\rangle \ge 0\ \forall u, v \in C$;
(iv) η-strongly monotone if $\exists \eta > 0$ s.t. $\langle Fu - Fv, u - v\rangle \ge \eta\|u - v\|^2\ \forall u, v \in C$;
(v) α-inverse-strongly monotone if $\exists \alpha > 0$ s.t. $\langle Fu - Fv, u - v\rangle \ge \alpha\|Fu - Fv\|^2\ \forall u, v \in C$.

It is clear that every inverse-strongly monotone mapping is monotone and Lipschitz continuous but the converse is not true. For each point $u \in \mathcal{H}$, we know that there exists a unique nearest point in C, denoted by $P_C u$, such that $\|u - P_C u\| \le \|u - v\|\ \forall v \in C$. The mapping P_C is said to be the metric projection of $\mathcal{H}$ onto C.

Lemma 2.1 (see [35])**.** *The following hold:*

(i) $\langle u - v, P_C u - P_C v\rangle \ge \|P_C u - P_C v\|^2\ \forall u, v \in \mathcal{H}$;
(ii) $\langle u - P_C u, v - P_C u\rangle \le 0\ \forall u \in \mathcal{H}, v \in C$;
(iii) $\|u - v\|^2 \ge \|u - P_C u\|^2 + \|v - P_C u\|^2\ \forall u \in \mathcal{H}, v \in C$;
(iv) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle\ \forall u, v \in \mathcal{H}$;
(v) $\|su + (1-s)v\|^2 = s\|u\|^2 + (1-s)\|v\|^2 - s(1-s)\|u - v\|^2\ \forall u, v \in \mathcal{H}, s \in [0, 1]$.

The following inequality is an immediate consequence of the subdifferential inequality of the function $\|\cdot\|^2/2$:

$$\|u + v\|^2 \le \|u\|^2 + 2\langle v, u + v\rangle \quad \forall u, v \in \mathcal{H}.$$

We now recall that a mapping $T : C \to C$ is said to be contractive if $\exists \alpha \in (0, 1)$ s.t. $\|Tu - Tv\| \le \alpha\|u - v\|\ \forall u, v \in C$. It is called pseudocontractive if $\langle Tu - Tv, u - v\rangle \le \|u - v\|^2\ \forall u, v \in C$. Also, it is called strongly pseudocontractive if $\exists \alpha \in (0, 1)$ s.t. $\langle Tu - Tv, u - v\rangle \le \alpha\|u - v\|^2\ \forall u, v \in C$. We shall use the following notion in the sequel.

Definition 2.2 (see [24])**.** Suppose that $\{T_k\}_{k=1}^\infty$ is a sequence of continuous pseudocontractive self-mappings on C. Then $\{T_k\}_{k=1}^\infty$ is called a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell > 0$ such that each T_k is ℓ-Lipschitz continuous.

We need the following Lemmas and propositions for proving our main results.

Lemma 2.3 (see [41])**.** *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, ...$ be a sequence of self-mappings on C. Suppose that $\sum_{k=1}^{\infty} \sup\{\|T_{k+1}x - T_k x\| : x \in C\} < \infty$. Then for each $x \in C$, $\{T_k x\}$ converges strongly to some point of C. Moreover, let T be a self-mapping on C defined by $Tx = \lim_{k\to\infty} T_k x$ $\forall x \in C$. Then $\lim_{k\to\infty} \sup\{\|T_k x - Tx\| : x \in C\} = 0$.*

It is easy to check that the following lemma is valid.

Lemma 2.4. *Let the mapping $B : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone. Then, for a given $\mu \geq 0$, $\|(I - \mu B)u - (I - \mu B)v\|^2 \leq \|u - v\|^2 - \mu(2\alpha - \mu)\|Bu - Bv\|^2$. In particular, if $0 \leq \mu \leq 2\alpha$, then $I - \mu B$ is nonexpansive.*

Utilizing Lemma 2.4, we immediately obtain the following lemma.

Lemma 2.5. *Let the mappings $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : \mathcal{H} \to C$ be defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. If $0 \leq \mu_1 \leq 2\alpha$ and $0 \leq \mu_2 \leq 2\beta$, then $G : \mathcal{H} \to C$ is nonexpansive.*

Lemma 2.6 (see [29, Lemma 2.1])**.** *Let $A : C \to \mathcal{H}$ be pseudomonotone and continuous. Given a point $u \in C$. Then $\langle Au, v - u\rangle \geq 0$ $\forall v \in C \Leftrightarrow \langle Av, v - u\rangle \geq 0$ $\forall v \in C$.*

Lemma 2.7 (see [26])**.** *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{u^k\}$ is a sequence in C such that $u^k \rightharpoonup u \in C$ and $(I - T)u^k \to 0$, then $(I - T)u = 0$, where I is the identity mapping of X.*

Lemma 2.8 (see [32])**.** *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C.*

The following lemma is very useful to analyze the convergence of the proposed algorithms in this paper.

Lemma 2.9 (see [37])**.** *Let $\{\Gamma_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$ which satisfies $\Gamma_{k_j} < \Gamma_{k_j+1}$ for each integer $j \geq 1$. Define the sequence $\{\tau(k)\}_{k\geq k_0}$ of integers as follows:*

$$\tau(k) = \max\{j \leq k : \Gamma_j < \Gamma_{j+1}\},$$

where integer $k_0 \geq 1$ such that $\{j \leq k_0 : \Gamma_j < \Gamma_{j+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(k_0) \leq \tau(k_0 + 1) \leq \cdots$ *and* $\tau(k) \to \infty$;

(ii) $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$ *and* $\Gamma_k \leq \Gamma_{\tau(k)+1}$ $\forall k \geq k_0$.

On the other hand, the normal cone $N_C(u)$ of C at $u \in C$ is defined as $N_C(u) = \{w \in \mathcal{H} : \langle w, v - u\rangle \leq 0 \ \forall v \in C\}$. The subdifferential of a convex function $g : C \to \mathcal{R} \cup \{+\infty\}$ at $u \in C$ is defined by

$$\partial g(u) = \{w \in \mathcal{H} : g(v) - g(u) \geq \langle w, v - u\rangle \ \forall v \in C\}.$$

In this paper, we are devoted to finding a solution $x^* \in \mathrm{Sol}(\Omega, \Psi)$ of the problem $\mathrm{EP}(\Omega, \Psi)$, where $\Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi)$ with $T_0 := T$. We assume always that the following hold:

- $\{T_k\}_{k=1}^{\infty}$ is a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C and $T : \mathcal{H} \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_k\}$.
- $\sum_{k=1}^{\infty} \sup_{x\in D} \|T_{k+1}x - T_k x\| < \infty$ for any bounded subset D of C, and $\mathrm{Fix}(\widehat{T}) = \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$, where $\widehat{T}$ is defined as $\widehat{T}x = \lim_{k\to\infty} T_k x \ \forall x \in C$.
- $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $G : \mathcal{H} \to C$ is defined as $G := P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)$ where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$.

Choose the sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}$, $\{s_k\}$ such that

(H1) $\beta_k + \gamma_k + \delta_k = 1 \ \forall k \geq 1$, $0 < \liminf_{k\to\infty} \beta_k$ and $0 < \liminf_{k\to\infty} \delta_k$;
(H2) $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$ and $0 < \liminf_{k\to\infty} \varepsilon_k$;
(H3) $\sum_{k=1}^{\infty} s_k = \infty$, $\lim_{k\to\infty} s_k = 0$, $\lim_{k\to\infty} \theta_k / s_k = 0$ and $\sum_{k=1}^{\infty} \theta_k < \infty$;
(H4) $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ and $\lim_{k\to\infty} \alpha_k = \tilde{\alpha}$;
(H5) $2s_k\nu - s_k^2 \Upsilon^2 < 1$, $0 < \lambda < \min\{\nu, \Upsilon\}$ and $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu-2\lambda}{\Upsilon^2-\lambda^2}, \frac{2\nu}{\Upsilon^2}\}$.

Algorithm 2.1.

Initial Step: Given $x^1 \in C$ arbitrarily. The sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}, \{s_k\}$ satisfy the conditions (H1)-(H5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1: Compute

$$u^k = \varepsilon_k x^k + (1 - \varepsilon_k)\hat{p}^k,$$
$$v^k = P_C(u^k - \mu_2 B_2 u^k).$$

Step 2: Compute

$$\hat{p}^k = P_C(v^k - \mu_1 B_1 v^k),$$
$$y^k = \mathrm{argmin}\left\{\alpha_k \Phi(\hat{p}^k, y) + \frac{1}{2}\|y - \hat{p}^k\|^2 : y \in C\right\}.$$

Step 3: Choose $\hat{w}^k \in \partial_2 \Phi(\hat{p}^k, y^k)$, and compute

$$C_k = \{v \in \mathcal{H} : \langle \hat{p}^k - \alpha_k \hat{w}^k - y^k, v - y^k\rangle \leq 0\},$$
$$z^k = \mathrm{argmin}\left\{\alpha_k \Phi(y^k, z) + \frac{1}{2}\|z - \hat{p}^k\|^2 : z \in C_k\right\}.$$

Step 4: Compute

$$\hat{q}^k = \beta_k x^k + \gamma_k T_k \hat{q}^k + \delta_k T^k z^k,$$
$$x^{k+1} = \mathrm{argmin}\left\{s_k \Psi(\hat{q}^k, t) + \frac{1}{2}\|t - \hat{q}^k\|^2 : t \in C\right\}.$$

Set $k := k + 1$ and return to Step 1.

We need the following technical propositions.

Proposition 2.10 (see [4, Theorem 2.1.3])**.** *Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathcal{R} \cup \{+\infty\}$ be subdifferentiable. Then, $\bar{u}$ is a solution to the following convex minimization problem*

$$\min\{g(u) : u \in C\}$$

if and only if $0 \in \partial g(\bar{u}) + N_C(\bar{u})$, where ∂g denotes the subdifferential of g.

Proposition 2.11 (see [3, Proposition 23])**.** *Let X and Y be two sets, $\mathcal{G}$ be a set-valued map from Y to X, and W be a real valued function defined on $X \times Y$. The marginal function M is defined as*

$$M(v) = \{u^* \in \mathcal{G}(v) : W(u^*, v) = \sup\{W(u, v) : u \in \mathcal{G}(v)\}\}.$$

If W and $\mathcal{G}$ are continuous, then M is upper semicontinuous.

Next, we assume that two bifunctions $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ and $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ satisfy the following conditions:

- $\mathbf{Ass}_\Phi$:

(Φ_1) $\Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi) \neq \emptyset$ with $T_0 := T$.

(Φ_2) Φ is monotone and Lipschitz-type continuous with constants $c_1, c_2 > 0$, and Φ is weakly continuous, i.e., $\{u^k \rightharpoonup \hat{u}$ and $v^k \rightharpoonup \hat{v}\} \Rightarrow \{\Phi(u^k, v^k) \to \Phi(\hat{u}, \hat{v})\}$.

- $\mathbf{Ass}_\Psi$:

(Ψ_1) Ψ is ν-strongly monotone and weakly continuous.

(Ψ_2) There exist the mappings $\bar{\Psi}_i : C \times C \to \mathcal{H}$ and $\hat{\psi}_i : C \to \mathcal{H}$ for each $i \in \{1, ..., m\}$ such that $\bar{\Psi}_i(x, y) + \bar{\Psi}_i(y, x) = 0,\ \|\bar{\Psi}_i(x, y)\| \leq \bar{L}_i\|x - y\|$ and $\|\hat{\psi}_i(x) - \hat{\psi}_i(y)\| \leq \hat{L}_i\|x - y\|$ for all $x, y \in C$, and

$$\Psi(x, y) + \Psi(y, z) \geq \Psi(x, z) + \sum_{i=1}^{m} \langle \bar{\Psi}_i(x, y), \hat{\psi}_i(y - z) \rangle \quad \forall x, y, z \in C.$$

(Ψ_3) For any sequence $\{y^k\} \subset C$ such that $y^k \to d$, we have $\limsup_{k \to \infty} \frac{|\Psi(d, y^k)|}{\|y^k - d\|} < +\infty$.

Remark 2.12. Suppose that the bifunction Ψ satisfies the condition $\mathbf{Ass}_\Psi(\Psi_2)$. Then

$$\begin{aligned}
\Psi(x, y) + \Psi(y, z) &\geq \Psi(x, z) + \sum_{i=1}^{m} \langle \bar{\Psi}_i(x, y), \hat{\psi}_i(y - z) \rangle \\
&\geq \Psi(x, z) - \sum_{i=1}^{m} |\langle \bar{\Psi}_i(x, y), \hat{\psi}_i(y - z) \rangle| \\
&\geq \Psi(x, z) - \sum_{i=1}^{m} \|\bar{\Psi}_i(x, y)\| \|\hat{\psi}_i(y - z)\| \\
&\geq \Psi(x, z) - \sum_{i=1}^{m} \bar{L}_i \hat{L}_i \|x - y\| \|y - z\| \\
&\geq \Psi(x, z) - (\frac{1}{2}\sum_{i=1}^{m} \bar{L}_i \hat{L}_i)\|x - y\|^2 - \left(\frac{1}{2}\sum_{i=1}^{m} \bar{L}_i \hat{L}_i\right) \|y - z\|^2.
\end{aligned}$$

Thus, Ψ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\sum_{i=1}^{m} \bar{L}_i \hat{L}_i$.

3. Main results

In this section, let the CFPP indicate the common fixed point problem of a countable family of ℓ-uniformly Lipschitzian pseudocontraction mappings $\{T_i\}_{i=1}^{\infty}$ and an asymptotically nonexpansive mapping T. We introduce and analyze two parallel Mann implicit subgradient extragradient algorithms for solving the MBEP with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem EP(Ω, Ψ) over the common solution set Ω of another monotone equilibrium problem EP(C, Φ), the GSVI (1.1) and the CFPP, where $\Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi)$ with $T_0 := T$. We are now in a position to state and prove the first main result in this paper.

Theorem 3.1. *Assume that $\{x^k\}$ is the sequence constructed by Algorithm 2.1. Let the bifunctions Ψ, Φ satisfy the assumptions $\mathbf{Ass}_{\Phi}$-$\mathbf{Ass}_{\Psi}$. Then, under the conditions (H1)-(H5), the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω, Ψ) provided $T^k x^k - T^{k+1} x^k \to 0$.*

Proof. First of all, we note that the mapping $G : \mathcal{H} \to C$ is defined as $G = P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)$, where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Then, by Lemma 2.5, we know that G is nonexpansive. Hence, for each $k \geq 1$ there exists a unique element $u^k \in C$ such that $u^k = \varepsilon_k x^k + (1 - \varepsilon_k) G u^k$. Meantime, from Lemma 2.8 it can be readily seen that for each $k \geq 1$ there exists a unique element $\hat{q}^k \in C$ such that

$$\hat{q}^k = \beta_k x^k + \gamma_k T_k \hat{q}^k + \delta_k T_k z^k \tag{3.1}$$

Choose an element $q \in \Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi)$ arbitrarily. Since $\lim_{k\to\infty} \frac{\theta_k}{s_k} = 0$, we may assume, without loss of generality, that $\theta_k \leq \frac{1}{2}\lambda s_k$ for all $k \geq 1$. We divide the proof into several steps as follows:

Step 1. We show that the following inequality holds

$$\|z^k - q\|^2 \leq \|\hat{p}^k - q\|^2 - (1 - 2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 - (1 - 2\alpha_k c_2)\|z^k - y^k\|^2 \quad \forall k \geq 1.$$

Indeed, by Proposition 2.10, we know that for $y^k = \mathrm{argmin}\{\alpha_k \Phi(\hat{p}^k, y) + \frac{1}{2}\|y - \hat{p}^k\|^2 : y \in C\}$, there exists $\hat{w}^k \in \partial_2 \Phi(\hat{p}^k, y^k)$ such that

$$\alpha_k \hat{w}^k + y^k - \hat{p}^k \in -N_C(y^k),$$

which hence yields

$$\langle \alpha_k \hat{w}^k + y^k - \hat{p}^k, x - y^k \rangle \geq 0 \quad \forall x \in C.$$

From the definition of $\hat{w}^k \in \partial_2 \Phi(\hat{p}^k, y^k)$, it follows that

$$\alpha_k [\Phi(\hat{p}^k, x) - \Phi(\hat{p}^k, y^k)] \geq \langle \alpha_k \hat{w}^k, x - y^k \rangle \quad \forall x \in \mathcal{H}. \tag{3.2}$$

Adding the last two inequalities, we get

$$\alpha_k [\Phi(\hat{p}^k, x) - \Phi(\hat{p}^k, y^k)] + \langle y^k - \hat{p}^k, x - y^k \rangle \geq 0 \quad \forall x \in C. \tag{3.3}$$

It follows from $z^k \in C_k$ and the definition of C_k that

$$\langle \hat{p}^k - \alpha_k \hat{w}^k - y^k, v - y^k \rangle \leq 0,$$

and hence

$$\alpha_k\langle \hat{w}^k, z^k - y^k\rangle \geq \langle \hat{p}^k - y^k, z^k - y^k\rangle. \tag{3.4}$$

Putting $x = z^k$ in (3.2), we get

$$\alpha_k[\Phi(\hat{p}^k, z^k) - \Phi(\hat{p}^k, y^k)] \geq \alpha_k\langle \hat{w}^k, z^k - y^k\rangle.$$

Adding (3.4) and the last inequality, we have

$$\alpha_k[\Phi(\hat{p}^k, z^k) - \Phi(\hat{p}^k, y^k)] \geq \langle \hat{p}^k - y^k, z^k - y^k\rangle. \tag{3.5}$$

By Proposition 2.10, we know that for $z^k = \operatorname{argmin}\{\alpha_k\Phi(y^k, y) + \frac{1}{2}\|y - \hat{p}^k\|^2 : y \in C_k\}$, there exist $\hat{h}^k \in \partial_2\Phi(y^k, z^k)$ and $t^k \in N_{C_k}(z^k)$ such that

$$\alpha_k\hat{h}^k + z^k - \hat{p}^k + t^k = 0.$$

So, we infer that $\alpha_k\langle \hat{h}^k, y - z^k\rangle \geq \langle \hat{p}^k - z^k, y - z^k\rangle\ \forall y \in C_k$, and

$$\Phi(y^k, y) - \Phi(y^k, z^k) \geq \langle \hat{h}^k, y - z^k\rangle \quad \forall y \in \mathcal{H}.$$

Putting $y = q \in C \subset C_k$ in two last inequalities and later adding them, we get

$$\alpha_k[\Phi(y^k, q) - \Phi(y^k, z^k)] \geq \langle \hat{p}^k - z^k, q - z^k\rangle.$$

By the monotonicity of Φ, $q \in \mathrm{Sol}(C, \Phi)$ and $y^k \in C$, we get

$$\Phi(y^k, q) \leq -\Phi(q, y^k) \leq 0.$$

Therefore,

$$-\alpha_k\Phi(y^k, z^k) \geq \langle \hat{p}^k - z^k, q - z^k\rangle.$$

Combining this and the following Lipschitz-type continuity of Φ

$$\Phi(\hat{p}^k, y^k) + \Phi(y^k, z^k) \geq \Phi(\hat{p}^k, z^k) - c_1\|\hat{p}^k - y^k\|^2 - c_2\|y^k - z^k\|^2,$$

we obtain that

$$\begin{aligned}\langle \hat{p}^k - z^k, z^k - q\rangle &\geq \alpha_k\Phi(y^k, z^k)\\ &\geq \alpha_k[\Phi(\hat{p}^k, z^k) - \Phi(\hat{p}^k, y^k)]\\ &\quad - \alpha_k c_1\|\hat{p}^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2.\end{aligned}$$

This together with (3.5), implies that

$$\langle \hat{p}^k - z^k, z^k - q\rangle \geq \langle \hat{p}^k - y^k, z^k - y^k\rangle - \alpha_k c_1\|\hat{p}^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2. \tag{3.6}$$

Therefore, applying the equality

$$\langle u, v\rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2) \quad \forall u, v \in \mathcal{H}, \tag{3.7}$$

for $\langle \hat{p}^k - z^k, z^k - q\rangle$ and $\langle y^k - \hat{p}^k, z^k - y^k\rangle$ in (3.6), we obtain the desired result.

Step 2. We show that the following inequality holds

$$\|x^{k+1} - x\|^2 \leq \|\hat{q}^k - x\|^2 - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x) - \Psi(\hat{q}^k, x^{k+1})] \quad \forall x \in C.$$

Indeed, since $x^{k+1} = \operatorname{argmin}\{s_k\Psi(\hat{q}^k, t) + \frac{1}{2}\|t - \hat{q}^k\|^2 : t \in C\}$, there exists $\hat{m}^k \in \partial_2\Psi(\hat{q}^k, x^{k+1})$ such that

$$0 \in s_k\hat{m}^k + x^{k+1} - \hat{q}^k + N_C(x^{k+1}).$$

By the definition of normal cone N_C and the subgradient $\hat{m}^k$, we get

$$\langle s_k\hat{m}^k + x^{k+1} - \hat{q}^k, x - x^{k+1}\rangle \geq 0 \quad \forall x \in C,$$
$$s_k[\Psi(\hat{q}^k, x) - \Psi(\hat{q}^k, x^{k+1})] \geq \langle s_k\hat{m}^k, x - x^{k+1}\rangle \quad \forall x \in C.$$

Adding the last two inequalities, we get

$$2s_k[\Psi(\hat{q}^k, x) - \Psi(\hat{q}^k, x^{k+1})] + 2\langle x^{k+1} - \hat{q}^k, x - x^{k+1}\rangle \geq 0 \quad \forall x \in C. \tag{3.8}$$

Putting $u = x^{k+1} - \hat{q}^k$ and $v = x - x^{k+1}$ in (3.7), we get

$$2s_k[\Psi(x^{k+1}, x) - \Psi(\hat{q}^k, x^{k+1})] + \|\hat{q}^k - x\|^2 - \|x^{k+1} - \hat{q}^k\|^2 - \|x^{k+1} - x\|^2 \geq 0 \quad \forall x \in C.$$

This attains the desired result.

Step 3. We show that if x^* is a solution of the MBEP with the GSVI and CFPP constraints, then

$$\|x^{k+1} - \hat{q}_*^k\| \leq \eta_k\|\hat{q}^k - x^*\| \leq (1 - \lambda s_k)\|\hat{q}^k - x^*\|,$$

where $\hat{q}_*^k = \operatorname{argmin}\{s_k\Psi(x^*, v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\}$, $\eta_k = \sqrt{1 - 2s_k\nu + s_k^2\Upsilon^2}$, $0 < \lambda < \min\{\nu, \Upsilon\}$, $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{\Upsilon^2 - \lambda^2}\}$, and $\Upsilon = \sum_{i=1}^m \bar{L}_i\hat{L}_i$.

Indeed, put $\hat{q}_*^k = \operatorname{argmin}\{s_k\Psi(x^*, v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\}$. By the similar arguments to those of (3.8), we also get

$$s_k[\Psi(x^*, x) - \Psi(x^*, \hat{q}_*^k)] + \langle \hat{q}_*^k - x^*, x - \hat{q}_*^k\rangle \geq 0 \quad \forall x \in C. \tag{3.9}$$

Setting $x = \hat{q}_*^k \in C$ in (3.8) and $x = x^{k+1} \in C$ in (3.9), respectively, we obtain that

$$s_k[\Psi(\hat{q}^k, \hat{q}_*^k) - \Psi(\hat{q}^k, x^{k+1})] + \langle x^{k+1} - \hat{q}^k, \hat{q}_*^k - x^{k+1}\rangle \geq 0,$$
$$s_k[\Psi(x^*, x^{k+1}) - \Psi(x^*, \hat{q}_*^k)] + \langle \hat{q}_*^k - x^*, x^{k+1} - \hat{q}_*^k\rangle \geq 0.$$

Adding the last two inequalities, we have

$$\begin{aligned} 0 &\leq 2s_k[\Psi(\hat{q}^k, \hat{q}_*^k) - \Psi(\hat{q}^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, \hat{q}_*^k)] \\ &\quad + 2\langle x^{k+1} - \hat{q}^k - \hat{q}_*^k + x^*, \hat{q}_*^k - x^{k+1}\rangle \\ &= 2s_k[\Psi(\hat{q}^k, \hat{q}_*^k) - \Psi(\hat{q}^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, \hat{q}_*^k)] \\ &\quad + \|\hat{q}^k - x^*\|^2 - \|x^{k+1} - \hat{q}^k - \hat{q}_*^k + x^*\|^2 - \|x^{k+1} - \hat{q}_*^k\|^2, \end{aligned} \tag{3.10}$$

where the last equality follows directly from (3.7).

Note that, under assumption $\mathbf{Ass}_\Psi(\Psi_2)$, it follows that

$$\Psi(\hat{q}^k, \hat{q}_*^k) - \Psi(x^*, \hat{q}_*^k) \leq \Psi(\hat{q}^k, x^*) - \sum_{i=1}^m \langle \bar{\Psi}_i(\hat{q}^k, x^*), \hat{\psi}_i(x^* - \hat{q}_*^k)\rangle,$$
$$\Psi(x^*, x^{k+1}) - \Psi(\hat{q}^k, x^{k+1}) \leq \Psi(x^*, \hat{q}^k) - \sum_{i=1}^m \langle \bar{\Psi}_i(x^*, \hat{q}^k), \hat{\psi}_i(\hat{q}^k - x^{k+1})\rangle.$$

Therefore, we have

$$\begin{aligned} &\Psi(\hat{q}^k, \hat{q}_*^k) - \Psi(\hat{q}^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, \hat{q}_*^k) \\ &\quad \leq \Psi(\hat{q}^k, x^*) + \Psi(x^*, \hat{q}^k) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} \langle \bar{\Psi}_i(\hat{q}^k, x^*), \hat{\psi}_i(x^* - \hat{q}^k_*) \rangle \\
&- \sum_{i=1}^{m} \langle \bar{\Psi}_i(x^*, \hat{q}^k), \hat{\psi}_i(\hat{q}^k - x^{k+1}) \rangle.
\end{aligned}
$$

Then, using $\mathbf{Ass}_\Psi(\Psi_2)$, and the strong monotonicity of Ψ in $\mathbf{Ass}_\Psi(\Psi_1)$ that $\Psi(x,y) + \Psi(y,x) \le -\nu\|x-y\|^2 \ \forall x, y \in C$, we get

$$
\begin{aligned}
&\Psi(\hat{q}^k, \hat{q}^k_*) - \Psi(\hat{q}^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, \hat{q}^k_*) \\
&\le -\nu\|\hat{q}^k - x^*\|^2 + \sum_{i=1}^{m} \langle \bar{\Psi}_i(\hat{q}^k, x^*), \hat{\psi}_i(\hat{q}^k - x^{k+1}) - \hat{\psi}_i(x^* - \hat{q}^k_*) \rangle \\
&\le -\nu\|\hat{q}^k - x^*\|^2 + \sum_{i=1}^{m} \|\bar{\Psi}_i(\hat{q}^k, x^*)\| \|\hat{\psi}_i(\hat{q}^k - x^{k+1}) - \hat{\psi}_i(x^* - \hat{q}^k_*)\| \qquad (3.11)\\
&\le -\nu\|\hat{q}^k - x^*\|^2 + \sum_{i=1}^{m} \bar{L}_i \hat{L}_i \|\hat{q}^k - x^*\| \|\hat{q}^k - x^{k+1} - x^* + \hat{q}^k_*\| \\
&= -\nu\|\hat{q}^k - x^*\|^2 + \Upsilon \|\hat{q}^k - x^*\| \|\hat{q}^k - x^{k+1} - x^* + \hat{q}^k_*\|.
\end{aligned}
$$

Combining (3.10) and (3.11), we get

$$
\begin{aligned}
0 &\le (1 - 2s_k\nu)\|\hat{q}^k - x^*\|^2 + 2s_k \Upsilon \|\hat{q}^k - x^*\| \|\hat{q}^k - x^{k+1} - x^* + \hat{q}^k_*\| \\
&\quad - \|x^{k+1} - \hat{q}^k - \hat{q}^k_* + x^*\|^2 - \|x^{k+1} - \hat{q}^k_*\|^2 \\
&= (1 - 2s_k\nu)\|\hat{q}^k - x^*\|^2 - (\|x^{k+1} - \hat{q}^k - \hat{q}^k_* + x^*\| - s_k \Upsilon \|\hat{q}^k - x^*\|)^2 \\
&\quad + s_k^2 \Upsilon^2 \|\hat{q}^k - x^*\|^2 - \|x^{k+1} - \hat{q}^k_*\|^2 \\
&\le (1 - 2s_k\nu + s_k^2 \Upsilon^2)\|\hat{q}^k - x^*\|^2 - \|x^{k+1} - \hat{q}^k_*\|^2.
\end{aligned}
$$

From

$$
0 < \lambda < \min\{\nu, \Upsilon\} \text{ and } 0 < s_k < \min\left\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{\Upsilon^2 - \lambda^2}\right\},
$$

it follows that $0 \le \eta_k = \sqrt{1 - 2s_k\nu + s_k^2 \Upsilon^2} < 1 - \lambda s_k$. This ensures the desired result.

Step 4. We show that the sequence $\{x^k\}$ is bounded. In fact, putting

$$
\begin{aligned}
&X := C, \ Y := [0,1], \ \mathcal{G}(s) := C \ \forall s \in Y, \\
&s := s_k, \ W(x,s) := -s\Psi(x^*, x) - \frac{1}{2}\|x - x^*\|^2 \ \forall (x,s) \in X \times Y,
\end{aligned}
$$

we have that

$$
\begin{aligned}
M(s_k) &= \operatorname{argmax}\{W(x, s_k) : x \in C\} \\
&= \operatorname{argmin}\left\{ s_k \Psi(x^*, x) + \frac{1}{2}\|x - x^*\|^2 : x \in C \right\} \\
&= \{\hat{q}^k_*\}.
\end{aligned}
$$

Note that M is continuous and $\lim_{k\to\infty}\hat{q}_*^k = x^*$. Since Ψ is continuous on C, we get $\lim_{k\to\infty}\Psi(x^*,\hat{q}_*^k) = \Psi(x^*,x^*) = 0$. In terms of $\mathbf{Ass}_\Psi(\Psi_3)$, there exists a constant $\hat{M}(x^*) > 0$ such that

$$|\Psi(x^*,\hat{q}_*^k)| \le \hat{M}(x^*)\|\hat{q}_*^k - x^*\| \quad \forall k \ge 1.$$

Putting $x = x^*$ in (3.9) and using $\Psi(x^*,x^*) = 0$, we get

$$-s_k\Psi(x^*,\hat{q}_*^k) + \langle \hat{q}_*^k - x^*, x^* - \hat{q}_*^k\rangle \ge 0,$$

which hence yields

$$\|\hat{q}_*^k - x^*\|^2 \le s_k[-\Psi(x^*,\hat{q}_*^k)] \le s_k\hat{M}(x^*)\|\hat{q}_*^k - x^*\| \quad \forall k \ge 1.$$

This immediately implies that

$$\|\hat{q}_*^k - x^*\| \le s_k\hat{M}(x^*) \quad \forall k \ge 1.$$

Also, according to Lemma 2.4 we know that $I-\mu_1B_1$ and $I-\mu_2B_2$ are nonexpansive mappings, where $\mu_1 \in (0,2\alpha)$ and $\mu_2 \in (0,2\beta)$. Moreover, by Lemma 2.5, we know that G is nonexpansive. We write $y^* = P_C(I-\mu_2B_2)x^*$. Then, by Lemma 1.1, we get $x^* = P_C(I-\mu_1B_1)y^* = Gx^*$. So it follows that

$$\begin{aligned}\|\hat{p}^k - x^*\| &= \|G(\varepsilon_k x^k + (1-\varepsilon_k)\hat{p}^k) - x^*\| \\ &\le \|\varepsilon_k(x^k - x^*) + (1-\varepsilon_k)(\hat{p}^k - x^*)\| \\ &\le \varepsilon_k\|x^k - x^*\| + (1-\varepsilon_k)\|\hat{p}^k - x^*\|,\end{aligned} \tag{3.12}$$

which immediately leads to $\|\hat{p}^k - x^*\| \le \|x^k - x^*\|$. Thus,

$$\begin{aligned}\|\hat{p}^k - x^*\| \le \|u^k - x^*\| &= \|\varepsilon_k(x^k - x^*) + (1-\varepsilon_k)(\hat{p}^k - x^*)\| \\ &\le \varepsilon_k\|x^k - x^*\| + (1-\varepsilon_k)\|\hat{p}^k - x^*\| \\ &\le \varepsilon_k\|x^k - x^*\| + (1-\varepsilon_k)\|x^k - x^*\| = \|x^k - x^*\|.\end{aligned} \tag{3.13}$$

Utilizing the result in Step 1, from (3.13) we get

$$\|z^k - x^*\| \le \|\hat{p}^k - x^*\| \le \|u^k - x^*\| \le \|x^k - x^*\| \quad \forall k \ge 1. \tag{3.14}$$

Since each T_k is a pseudocontraction mapping, and T is asymptotically nonexpansive, we deduce from (3.14) that

$$\begin{aligned}\|\hat{q}^k - x^*\|^2 &= \langle \beta_k(x^k - x^*) + \gamma_k(T_k\hat{q}^k - x^*) + \delta_k(T^kz^k - x^*), \hat{q}^k - x^*\rangle \\ &= \beta_k\langle x^k - x^*, \hat{q}^k - x^*\rangle + \gamma_k\langle T_k\hat{q}^k - x^*, \hat{q}^k - x^*\rangle \\ &\quad + \delta_k\langle T^kz^k - x^*, \hat{q}^k - x^*\rangle \\ &\le \beta_k\|x^k - x^*\|\|\hat{q}^k - x^*\| + \gamma_k\|\hat{q}^k - x^*\|^2 \\ &\quad + \delta_k(1+\theta_k)\|z^k - x^*\|\|\hat{q}^k - x^*\| \\ &\le \beta_k(1+\theta_k)\|x^k - x^*\|\|\hat{q}^k - x^*\| \\ &\quad + \gamma_k\|\hat{q}^k - x^*\|^2 + \delta_k(1+\theta_k)\|x^k - x^*\|\|\hat{q}^k - x^*\| \\ &= (1-\gamma_k)(1+\theta_k)\|x^k - x^*\|\|\hat{q}^k - x^*\| + \gamma_k\|\hat{q}^k - x^*\|^2,\end{aligned}$$

which hence yields

$$\|\hat{q}^k - x^*\| \le (1+\theta_k)\|x^k - x^*\|.$$

Consequently,

$$\begin{aligned}\|x^{k+1}-x^*\| &\le \|x^{k+1}-\hat{q}_*^k\| + \|\hat{q}_*^k - x^*\| \\ &\le (1-\lambda s_k)\|\hat{q}^k - x^*\| + \|\hat{q}_*^k - x^*\| \\ &\le (1-\lambda s_k)(1+\theta_k)\|x^k - x^*\| + s_k\hat{M}(x^*) \\ &\le [(1-\lambda s_k)+\theta_k]\|x^k - x^*\| + s_k\hat{M}(x^*) \\ &\le [(1-\lambda s_k)+\frac{1}{2}\lambda s_k]\|x^k - x^*\| + s_k\hat{M}(x^*) \\ &\le \max\left\{\|x^k - x^*\|, \frac{2\hat{M}(x^*)}{\lambda}\right\}.\end{aligned} \tag{3.15}$$

By induction, we get $\|x^k - x^*\| \le \max\{\|x^1 - x^*\|, \frac{2\hat{M}(x^*)}{\lambda}\}\ \forall k \ge 1$. Thus, $\{x^k\}$ is bounded, and so are the sequences $\{\hat{p}^k\}, \{\hat{q}^k\}, \{y^k\}, \{z^k\}, \{u^k\}, \{v^k\}$.

Step 5. We show that if $x^{k_i} \rightharpoonup \hat{x}$, $\hat{p}^{k_i} - x^{k_i} \to 0$ and $\hat{p}^{k_i} - y^{k_i} \to 0$ for $\{k_i\} \subset \{k\}$, then $\hat{x} \in \mathrm{Sol}(C, \Phi)$.

Indeed, noticing $\hat{p}^{k_i} - x^{k_i} \to 0$ and $\hat{p}^{k_i} - y^{k_i} \to 0$, we get

$$\|x^{k_i} - y^{k_i}\| \le \|x^{k_i} - \hat{p}^{k_i}\| + \|\hat{p}^{k_i} - y^{k_i}\| \to 0 \quad (i \to \infty). \tag{3.16}$$

So it follows from $x^{k_i} \rightharpoonup \hat{x}$ that $\hat{p}^{k_i} \rightharpoonup \hat{x}$ and $y^{k_i} \rightharpoonup \hat{x}$. Since $\{y^k\} \subset C$, $y^{k_i} \rightharpoonup \hat{x}$ and C is weakly closed, we know that $\hat{x} \in C$. By (3.3), we have

$$\alpha_{k_i}\Phi(\hat{p}^{k_i}, x) \ge \alpha_{k_i}\Phi(\hat{p}^{k_i}, y^{k_i}) + \langle y^{k_i} - \hat{p}^{k_i}, y^{k_i} - x\rangle \quad \forall x \in C.$$

Taking the limit as $i \to \infty$ and using the assumptions that $\lim_{k\to\infty}\alpha_k = \tilde{\alpha} > 0$, $\Phi(\hat{x}, \hat{x}) = 0$, $\{y^{k_i}\}$ is bounded and Φ is weakly continuous, we obtain that $\tilde{\alpha}\Phi(\hat{x}, x) \ge 0\ \forall x \in C$. This implies that $\hat{x} \in \mathrm{sol}(C, \Phi)$.

Step 6. We show that $x^k \to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraints. Indeed, set $\varGamma_k = \|x^k - x^*\|^2$. Since each T_k is pseudocontractive and T is asymptotically nonexpansive, by Lemma 2.1 (iv) we obtain

$$\begin{aligned}\|\hat{q}^k - x^*\|^2 &= \beta_k\langle x^k - x^*, \hat{q}^k - x^*\rangle + \gamma_k\langle T_k\hat{q}^k - x^*, \hat{q}^k - x^*\rangle \\ &\quad + \delta_k\langle T^k z^k - x^*, \hat{q}^k - x^*\rangle \\ &\le \frac{\beta_k}{2}[\|x^k - x^*\|^2 + \|\hat{q}^k - x^*\|^2 - \|x^k - \hat{q}^k\|^2] + \gamma_k\|\hat{q}^k - x^*\|^2 \\ &\quad + \frac{\delta_k}{2}[\|T^k z^k - x^*\|^2 + \|\hat{q}^k - x^*\|^2 - \|T^k z^k - \hat{q}^k\|^2] \\ &= \frac{\beta_k}{2}\|x^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|\hat{q}^k - x^*\|^2 + \frac{\delta_k}{2}\|T^k z^k - x^*\|^2 \\ &\quad - \frac{\beta_k}{2}\|x^k - \hat{q}^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - \hat{q}^k\|^2 \\ &\le \frac{\beta_k}{2}\|x^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|\hat{q}^k - x^*\|^2 + \frac{\delta_k(1+\theta_k)^2}{2}\|z^k - x^*\|^2 \\ &\quad - \frac{\beta_k}{2}\|x^k - \hat{q}^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - \hat{q}^k\|^2\end{aligned}$$

$$\leq \frac{\beta_k}{2}\|x^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|\hat{q}^k - x^*\|^2 + \frac{\delta_k}{2}\|z^k - x^*\|^2 + \frac{\theta_k \widetilde{M}}{2} - \frac{\beta_k}{2}\|x^k - \hat{q}^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - \hat{q}^k\|^2,$$

where $\sup_{k\geq 1}(2+\theta_k)\|x^k - x^*\|^2 \leq \widetilde{M}$ for some $\widetilde{M} > 0$. This implies that

$$\|\hat{q}^k - x^*\|^2 \leq \frac{1}{1-\gamma_k}[\beta_k\|x^k - x^*\|^2 + \delta_k\|z^k - x^*\|^2 + \theta_k\widetilde{M} - \beta_k\|x^k - \hat{q}^k\|^2 - \delta_k\|T^k z^k - \hat{q}^k\|^2]. \tag{3.17}$$

By the results in Steps 1 and 2 we deduce from (3.14) and (3.17) that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|\hat{q}^k - x^*\|^2 - \|x^{k+1} - \hat{q}^k\|^2 \\
&\quad + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \frac{1}{1-\gamma_k}[\beta_k\|x^k - x^*\|^2 + \delta_k\|z^k - x^*\|^2 \\
&\quad + \theta_k\widetilde{M} - \beta_k\|x^k - \hat{q}^k\|^2 - \delta_k\|T^k z^k - \hat{q}^k\|^2] \\
&\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \frac{1}{1-\gamma_k}\{\beta_k\|x^k - x^*\|^2 + \delta_k[\|\hat{p}^k - x^*\|^2 \\
&\quad - (1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\
&\quad + \theta_k\widetilde{M} - \beta_k\|x^k - \hat{q}^k\|^2 - \delta_k\|T^k z^k - \hat{q}^k\|^2\} \\
&\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \|x^k - x^*\|^2 \\
&\quad - \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 + (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\
&\quad + \frac{\theta_k\widetilde{M}}{1-\gamma_k} - \frac{1}{1-\gamma_k}[\beta_k\|x^k - \hat{q}^k\|^2 + \delta_k\|T^k z^k - \hat{q}^k\|^2] \\
&\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \|x^k - x^*\|^2 - \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 \\
&\quad + (1-2\alpha_k c_2)\|z^k - y^k\|^2] + \frac{\theta_k\widetilde{M}}{1-\gamma_k} \\
&\quad - \frac{1}{1-\gamma_k}[\beta_k\|x^k - \hat{q}^k\|^2 + \delta_k\|T^k z^k - \hat{q}^k\|^2] - \|x^{k+1} - \hat{q}^k\|^2 + s_k K,
\end{aligned} \tag{3.18}$$

where $\sup_{k\geq 1}\{2|\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})|\} \leq K$ for some $K > 0$.

Finally, we show the convergence of $\{\Gamma_k\}$ to zero by the following two cases:

Case 1. Suppose that there exists an integer $k_0 \geq 1$ such that $\{\Gamma_k\}$ is non-increasing. Then the limit $\lim_{k\to\infty}\Gamma_k = \hbar < +\infty$ and

$$\Gamma_k - \Gamma_{k+1} \to 0 \quad (k \to \infty).$$

From (3.18), we get

$$
\begin{aligned}
&\delta_k[(1-2\alpha_k c_1)\|y^k-\hat{p}^k\|^2+(1-2\alpha_k c_2)\|z^k-y^k\|^2] \\
&\quad+\beta_k\|x^k-\hat{q}^k\|^2+\delta_k\|T^k z^k-\hat{q}^k\|^2+\|x^{k+1}-\hat{q}^k\|^2 \\
&\quad\le \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k-\hat{p}^k\|^2+(1-2\alpha_k c_2)\|z^k-y^k\|^2] \\
&\qquad+\frac{1}{1-\gamma_k}[\beta_k\|x^k-\hat{q}^k\|^2+\delta_k\|T^k z^k-\hat{q}^k\|^2]+\|x^{k+1}-\hat{q}^k\|^2 \\
&\quad\le \Gamma_k-\Gamma_{k+1}+\frac{\theta_k\widetilde{M}}{1-\gamma_k}+s_kK.
\end{aligned}
\tag{3.19}
$$

Since $s_k\to 0,\ \theta_k\to 0,\ \Gamma_k-\Gamma_{k+1}\to 0,\ 0<\liminf_{k\to\infty}\beta_k,\ 0<\liminf_{k\to\infty}\delta_k$ and $0<\liminf_{k\to\infty}(1-\gamma_k)$, we obtain from $\{\alpha_k\}\subset(\underline{\alpha},\overline{\alpha})\subset(0,\min\{\frac{1}{2c_1},\frac{1}{2c_2}\})$ that

$$
\lim_{k\to\infty}\|x^k-\hat{q}^k\|=\lim_{k\to\infty}\|T^k z^k-\hat{q}^k\|=0,
\tag{3.20}
$$

$$
\lim_{k\to\infty}\|y^k-\hat{p}^k\|=\lim_{k\to\infty}\|z^k-y^k\|=\lim_{k\to\infty}\|x^{k+1}-\hat{q}^k\|=0.
\tag{3.21}
$$

We now show that $\|u^k-\hat{p}^k\|\to 0$ as $k\to\infty$. Indeed, we set $y^*=P_C(x^*-\mu_2B_2x^*)$. Note that $v^k=P_C(u^k-\mu_2B_2u^k)$ and $\hat{p}^k=P_C(v^k-\mu_1B_1v^k)$. Then $\hat{p}^k=Gu^k$. By Lemma 2.4 we have

$$
\|v^k-y^*\|^2\le\|u^k-x^*\|^2-\mu_2(2\beta-\mu_2)\|B_2u^k-B_2x^*\|^2,
\tag{3.22}
$$

$$
\|\hat{p}^k-x^*\|^2\le\|v^k-y^*\|^2-\mu_1(2\alpha-\mu_1)\|B_1v^k-B_1y^*\|^2.
\tag{3.23}
$$

Substituting (3.22) for (3.23), by (3.14) we get

$$
\begin{aligned}
\|\hat{p}^k-x^*\|^2\le\|x^k-x^*\|^2&-\mu_2(2\beta-\mu_2)\|B_2u^k-B_2x^*\|^2 \\
&-\mu_1(2\alpha-\mu_1)\|B_1v^k-B_1y^*\|^2.
\end{aligned}
\tag{3.24}
$$

Also, substituting (3.24) for (3.18), we get

$$
\begin{aligned}
\|x^{k+1}-x^*\|^2 &\le \frac{1}{1-\gamma_k}[\beta_k\|x^k-x^*\|^2+\delta_k\|z^k-x^*\|^2+\theta_k\widetilde{M}]+s_kK \\
&\le \frac{1}{1-\gamma_k}\{\beta_k\|x^k-x^*\|^2+\delta_k[\|x^k-x^*\|^2 \\
&\quad-\mu_2(2\beta-\mu_2)\|B_2u^k-B_2x^*\|^2 \\
&\qquad-\mu_1(2\alpha-\mu_1)\|B_1v^k-B_1y^*\|^2]+\theta_k\widetilde{M}\} \\
&\quad+s_kK \\
&=\|x^k-x^*\|^2-\frac{\delta_k}{1-\gamma_k}[\mu_2(2\beta-\mu_2)\|B_2u^k-B_2x^*\|^2 \\
&\quad+\mu_1(2\alpha-\mu_1)\|B_1v^k-B_1y^*\|^2]+\frac{\theta_k\widetilde{M}}{1-\gamma_k}+s_kK,
\end{aligned}
$$

which immediately yields

$$\frac{\delta_k}{1-\gamma_k}[\mu_2(2\beta-\mu_2)\|B_2u^k-B_2x^*\|^2+\mu_1(2\alpha-\mu_1)\|B_1v^k-B_1y^*\|^2]$$
$$\leq \mathit{\Gamma}_k-\mathit{\Gamma}_{k+1}+\frac{\theta_k\widetilde{M}}{1-\gamma_k}+s_kK.$$

Since $\mu_1\in(0,2\alpha),\ \mu_2\in(0,2\beta),\ s_k\to 0,\ \theta_k\to 0,\ \mathit{\Gamma}_k-\mathit{\Gamma}_{k+1}\to 0,\ \liminf_{k\to\infty}\delta_k>0$ and $\liminf_{k\to\infty}(1-\gamma_k)>0$, we get

$$\lim_{k\to\infty}\|B_2u^k-B_2x^*\|=0 \quad \text{and} \quad \lim_{k\to\infty}\|B_1v^k-B_1y^*\|=0. \tag{3.25}$$

On the other hand, from Lemma 2.1 (i), (iv), we get

$$\begin{aligned}\|\hat{p}^k-x^*\|^2 &\leq \langle v^k-y^*,\hat{p}^k-x^*\rangle+\mu_1\langle B_1y^*-B_1v^k,\hat{p}^k-x^*\rangle\\ &\leq \frac{1}{2}[\|v^k-y^*\|^2+\|\hat{p}^k-x^*\|^2-\|v^k-\hat{p}^k+x^*-y^*\|^2]\\ &\quad +\mu_1\|B_1y^*-B_1v^k\|\|\hat{p}^k-x^*\|.\end{aligned}$$

This ensures that

$$\|\hat{p}^k-x^*\|^2\leq\|v^k-y^*\|^2-\|v^k-\hat{p}^k+x^*-y^*\|^2+2\mu_1\|B_1y^*-B_1v^k\|\|\hat{p}^k-x^*\|. \tag{3.26}$$

Similarly, we get

$$\|v^k-y^*\|^2\leq\|u^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2+2\mu_2\|B_2x^*-B_2u^k\|\|v^k-y^*\|. \tag{3.27}$$

Combining (3.26) and (3.27), by (3.14) we have

$$\begin{aligned}\|\hat{p}^k-x^*\|^2 &\leq \|x^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2\\ &\quad -\|v^k-\hat{p}^k+x^*-y^*\|^2\\ &\quad +2\mu_1\|B_1y^*-B_1v^k\|\|\hat{p}^k-x^*\|\\ &\quad +2\mu_2\|B_2x^*-B_2u^k\|\|v^k-y^*\|.\end{aligned} \tag{3.28}$$

Substituting (3.28) for (3.18), we get

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\leq \frac{1}{1-\gamma_k}[\beta_k\|x^k-x^*\|^2+\delta_k\|z^k-x^*\|^2+\theta_k\widetilde{M}]+s_kK\\ &\leq \frac{1}{1-\gamma_k}\{\beta_k\|x^k-x^*\|^2+\delta_k[\|x^k-x^*\|^2\\ &\qquad -\|u^k-v^k+y^*-x^*\|^2-\|v^k-\hat{p}^k+x^*-y^*\|^2\\ &\qquad +2\mu_1\|B_1y^*-B_1v^k\|\|\hat{p}^k-x^*\|\\ &\qquad +2\mu_2\|B_2x^*-B_2u^k\|\|v^k-y^*\|]+\theta_k\widetilde{M}\}\\ &\quad +s_kK\\ &\leq \|x^k-x^*\|^2-\frac{\delta_k}{1-\gamma_k}[\|u^k-v^k+y^*-x^*\|^2\\ &\qquad +\|v^k-\hat{p}^k+x^*-y^*\|^2]\\ &\qquad +\frac{1}{1-\gamma_k}[2\mu_1\|B_1y^*-B_1v^k\|\|\hat{p}^k-x^*\|\\ &\qquad +2\mu_2\|B_2x^*-B_2u^k\|\|v^k-y^*\|+\theta_k\widetilde{M}]\end{aligned}$$

$$+ s_k K.$$

This immediately leads to

$$\frac{\delta_k}{1-\gamma_k}[\|u^k - v^k + y^* - x^*\|^2 + \|v^k - p^k + x^* - y^*\|^2]$$
$$\leq \varGamma_k - \varGamma_{k+1} + \frac{2\mu_1\|B_1 y^* - B_1 v^k\|\|\hat{p}^k - x^*\| + 2\mu_2\|B_2 x^* - B_2 u^k\|\|v^k - y^*\| + \theta_k \widetilde{M}}{1-\gamma_k} + s_k K.$$

Since $s_k \to 0,\ \theta_k \to 0,\ \varGamma_k - \varGamma_{k+1} \to 0,\ \liminf_{k\to\infty} \delta_k > 0$ and $\liminf_{k\to\infty}(1-\gamma_k) > 0$, we deduce from (3.25) that

$$\lim_{k\to\infty} \|u^k - v^k + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|v^k - \hat{p}^k + x^* - y^*\| = 0.$$

Thus,

$$\|u^k - Gu^k\| = \|u^k - \hat{p}^k\| \leq \|u^k - v^k + y^* - x^*\| + \|v^k - \hat{p}^k + x^* - y^*\| \to 0 \quad (k\to\infty). \tag{3.29}$$

Since $u^k = \varepsilon_k x^k + (1-\varepsilon_k)\hat{p}^k$ and $\liminf_{k\to\infty} \varepsilon_k > 0$, we have

$$\|x^k - u^k\| = \frac{1-\varepsilon_k}{\varepsilon_k}\|\hat{p}^k - u^k\| \leq \frac{1}{\varepsilon_k}\|\hat{p}^k - u^k\| \to 0 \quad (k\to\infty). \tag{3.30}$$

By (3.29) and (3.30), we get

$$\|x^k - \hat{p}^k\| \leq \|x^k - u^k\| + \|u^k - \hat{p}^k\| \to 0 \quad (k\to\infty). \tag{3.31}$$

Combining (3.30) and (3.31), we have

$$\begin{aligned}\|x^k - Gx^k\| &\leq \|x^k - Gu^k\| + \|Gu^k - Gx^k\| \\ &\leq \|x^k - \hat{p}^k\| + \|u^k - x^k\| \to 0 \quad (k\to\infty).\end{aligned} \tag{3.32}$$

We claim that $\|Tx^k - x^k\| \to 0$ and $\|T_k x^k - x^k\| \to 0$ as $k \to \infty$. In fact, since $\hat{q}^k = \beta_k x_k + \gamma_k T_k \hat{q}^k + \delta_k T^k z^k$ and $0 < \liminf_{k\to\infty} \gamma_k$, we obtain from (3.20) that

$$\begin{aligned}\|T_k\hat{q}^k - \hat{q}^k\| &= \frac{1}{\gamma_k}\|\beta_k(x_k - \hat{q}^k) + \delta_k(T^k z^k - \hat{q}^k)\| \\ &\leq \frac{1}{\gamma_k}[\|x_k - \hat{q}^k\| + \|T^k z^k - \hat{q}^k\|] \to 0 \quad (k\to\infty).\end{aligned}$$

Since $\{T_k\}_{k=1}^{\infty}$ is ℓ-uniformly Lipschitzian, we deduce from (3.20) and (3.33) that

$$\begin{aligned}\|T_k x^k - x^k\| &\leq \|T_k x^k - T_k\hat{q}^k\| + \|T_k\hat{q}^k - \hat{q}^k\| + \|\hat{q}^k - x^k\| \\ &\leq (\ell+1)\|x^k - \hat{q}^k\| + \|T_k\hat{q}^k - \hat{q}^k\| \to 0 \quad (k\to\infty).\end{aligned} \tag{3.33}$$

From (3.20), (3.21) and (3.31), we conclude that

$$\|z^k - x^k\| \leq \|z^k - y^k\| + \|y^k - \hat{p}^k\| + \|\hat{p}^k - x^k\| \to 0 \quad (k\to\infty), \tag{3.34}$$

and hence

$$\begin{aligned}\|T^k x^k - x^k\| &\leq \|T^k x^k - T^k z^k\| + \|T^k z^k - \hat{q}^k\| + \|\hat{q}^k - x^k\| \\ &\leq (1+\theta_k)\|x^k - z^k\| + \|T^k z^k - \hat{q}^k\| + \|\hat{q}^k - x^k\| \to 0 \quad (k\to\infty).\end{aligned} \tag{3.35}$$

This together with the assumption $\|T^k x^k - T^{k+1}x^k\| \to 0$, implies that

$$\begin{aligned} \|x^k - Tx^k\| &\le \|x^k - T^k x^k\| + \|T^k x^k - T^{k+1}x^k\| + \|T^{k+1}x^k - Tx^k\| \\ &\le (2+\theta_1)\|x^k - T^k x^k\| + \|T^k x^k - T^{k+1}x^k\| \to 0 \quad (k \to \infty). \end{aligned} \tag{3.36}$$

In addition, from (3.20) and 3.21) it is easy to see that

$$\|x^k - x^{k+1}\| \le \|x^k - \hat{q}^k\| + \|\hat{q}^k - x^{k+1}\| \to 0 \quad (k \to \infty). \tag{3.37}$$

We claim that $\|x^k - \overline{T}x^k\| \to 0$ as $k \to \infty$ where $\overline{T} := (2I - \widehat{T})^{-1}$. In fact, it is first clear that $\widehat{T} : C \to C$ is pseudocontractive and ℓ-Lipschitzian where

$$\widehat{T}x = \lim_{k\to\infty} T_k x \quad \forall x \in C. \tag{3.38}$$

We claim that $\lim_{k\to\infty} \|\widehat{T}x^k - x^k\| = 0$. Using the boundedness of $\{x^k\}$ and putting $D = \overline{\text{conv}}\{x^k : k \ge 1\}$ (the closed convex hull of the set $\{x^k : k \ge 1\}$), by the assumption we have $\sum_{k=1}^{\infty} \sup_{x\in D} \|T_{k+1}x - T_k x\| < \infty$. Hence, by Lemma 2.3 we get $\lim_{k\to\infty} \sup_{x\in D} \|T_k x - \widehat{T}x\| = 0$, which immediately yields

$$\lim_{k\to\infty} \|T_k x^k - \widehat{T}x^k\| = 0. \tag{3.39}$$

Thus, combining (3.33) and (3.39) we have

$$\|x^k - \widehat{T}x^k\| \le \|x^k - T_k x^k\| + \|T_k x^k - \widehat{T}x^k\| \to 0 \quad (k \to \infty). \tag{3.40}$$

Now, let us show that if we define $\overline{T} := (2I - \widehat{T})^{-1}$, then $\overline{T} : C \to C$ is nonexpansive, $\text{Fix}(\overline{T}) = \text{Fix}(\widehat{T}) = \bigcap_{k=1}^{\infty} \text{Fix}(T_k)$ and $\lim_{k\to\infty} \|x^k - \overline{T}x^k\| = 0$. In fact, put $\overline{T} := (2I - \widehat{T})^{-1}$, where I is the identity mapping of $\mathcal{H}$. Then it is known that $\overline{T}$ is nonexpansive and $\text{Fix}(\overline{T}) = \text{Fix}(\widehat{T}) = \bigcap_{k=1}^{\infty} \text{Fix}(T_k)$ as a consequence of Theorem 6 of [38]. From (3.40) it follows that

$$\begin{aligned} \|x^k - \overline{T}x^k\| &= \|\overline{T}\,\overline{T}^{-1}x^k - \overline{T}x^k\| \\ &\le \|\overline{T}^{-1}x^k - x^k\| \\ &= \|(2I - \widehat{T})x^k - x^k\| = \|x^k - \widehat{T}x^k\| \to 0 \quad (k \to \infty). \end{aligned} \tag{3.41}$$

Next we claim that $\lim_{k\to\infty} \|x^k - x^*\| = 0$. In fact, since the sequences $\{\hat{q}^k\}$ and $\{x^k\}$ are bounded, we know that there exists a subsequence $\{\hat{q}^{k_i}\}$ of $\{\hat{q}^k\}$ converging weakly to $\hat{x} \in C$ and satisfying the equality

$$\liminf_{k\to\infty} [\Psi(x^*, \hat{q}^k) + \Psi(\hat{q}^k, x^{k+1})] = \lim_{i\to\infty} [\Psi(x^*, \hat{q}^{k_i}) + \Psi(\hat{q}^{k_i}, x^{k_i+1})]. \tag{3.42}$$

From (3.20) and (3.21) it follows that $x^{k_i} \rightharpoonup \hat{x}$ and $x^{k_i+1} \rightharpoonup \hat{x}$. Then, by the result in Step 5, we deduce that $\hat{x} \in \text{Sol}(C, \Phi)$.

It is clear from (3.36) that $x^{k_i} - Tx^{k_i} \to 0$. Note that Lemma 2.7 guarantees the demiclosedness of $I - T$ at zero. So, we know that $\hat{x} \in \text{Fix}(T)$. Also, note that Lemma 2.7 guarantees the demiclosedness of both $I - \overline{T}$ and $I - G$ at zero. Since $\lim_{k\to\infty} \|x^k - \overline{T}x^k\| = 0$ (due to (3.41)), we infer from $x^{k_i} \rightharpoonup \hat{x}$ that $\hat{x} \in \text{Fix}(\overline{T})$, which hence yields $\hat{x} \in \bigcap_{k=0}^{\infty} \text{Fix}(T_k)$. Meantime, from $x^{k_i} \rightharpoonup \hat{x}$ and $x^k - Gx^k \to 0$

(due to (3.32)) it follows that $\hat{x} \in \mathrm{Fix}(G) = \mathrm{GSVI}(C, B_1, B_2)$. Consequently, $\hat{x} \in \bigcap_{j=0}^{\infty} \mathrm{Fix}(T_j) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi) = \varOmega$. In terms of (3.42), we have

$$\liminf_{k\to\infty} [\Psi(x^*, \hat{q}^k) + \Psi(\hat{q}^k, x^{k+1})] = \Psi(x^*, \hat{x}) \geq 0. \tag{3.43}$$

Since Ψ is ν-strongly monotone, we have

$$\limsup_{k\to\infty} [\Psi(x^*, \hat{q}^k) + \Psi(\hat{q}^k, x^*)] \leq \limsup_{k\to\infty} (-\nu \|\hat{q}^k - x^*\|^2) = -\nu\hbar. \tag{3.44}$$

Combining (3.43) and (3.44), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} [\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&= \limsup_{k\to\infty} [\Psi(\hat{q}^k, x^*) + \Psi(x^*, \hat{q}^k) - \Psi(x^*, \hat{q}^k) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \limsup_{k\to\infty} [\Psi(\hat{q}^k, x^*) + \Psi(x^*, \hat{q}^k)] + \limsup_{k\to\infty} [-\Psi(x^*, \hat{q}^k) - \Psi(\hat{q}^k, x^{k+1})] \\
&= \limsup_{k\to\infty} [\Psi(\hat{q}^k, x^*) + \Psi(x^*, \hat{q}^k)] - \liminf_{k\to\infty} [\Psi(x^*, \hat{q}^k) + \Psi(\hat{q}^k, x^{k+1})] \\
&\leq -\nu\hbar.
\end{aligned} \tag{3.45}$$

We now claim that $\hbar = 0$. On the contrary, we assume $\hbar > 0$. Without loss of generality we may assume that $\exists k_0 \geq 1$ s.t.

$$\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1}) \leq -\frac{\nu\hbar}{2} \quad \forall k \geq k_0, \tag{3.46}$$

which together with (3.18), implies that for all $k \geq k_0$,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 \\
&\quad - \frac{\delta_k}{1-\gamma_k} [(1 - 2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 + (1 - 2\alpha_k c_2)\|z^k - y^k\|^2] \\
&\quad + \frac{\theta_k \widetilde{M}}{1-\gamma_k} - \frac{1}{1-\gamma_k} [\beta_k \|x^k - \hat{q}^k\|^2 + \delta_k \|T^k z^k - \hat{q}^k\|^2] \\
&\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k [\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \|x^k - x^*\|^2 + \frac{\theta_k \widetilde{M}}{1-\gamma_k} + 2s_k [\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\leq \|x^k - x^*\|^2 + \frac{\theta_k \widetilde{M}}{1-\gamma_k} - s_k \nu\hbar.
\end{aligned} \tag{3.47}$$

So it follows that for all $k \geq k_0$,

$$\varGamma_k - \varGamma_{k_0} \leq \widetilde{M} \sum_{j=k_0}^{k-1} \frac{\theta_j}{1-\gamma_j} - \nu\hbar \sum_{j=k_0}^{k-1} s_j. \tag{3.48}$$

Since $\sum_{j=1}^{\infty} s_j = \infty$, $\sum_{j=1}^{\infty} \theta_j < \infty$, $0 < \liminf_{k\to\infty}(1-\gamma_k)$ and $\lim_{k\to\infty} \Gamma_k = \hbar$, taking the limit in (3.48) as $k \to \infty$ we get

$$\begin{aligned} -\infty &< \hbar - \Gamma_{k_0} = \lim_{k\to\infty}(\Gamma_k - \Gamma_{k_0}) \\ &\le \lim_{k\to\infty}[\widetilde{M}\sum_{j=k_0}^{k-1} \tfrac{\theta_j}{1-\gamma_j} - \nu\hbar\sum_{j=k_0}^{k-1} s_j] = -\infty. \end{aligned}$$

This reaches a contradiction. Therefore, $\lim_{k\to\infty} \Gamma_k = 0$ and hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem $\mathrm{EP}(\Omega, \Psi)$.

Case 2. Suppose that $\exists\{\Gamma_{k_j}\} \subset \{\Gamma_k\}$ s.t. $\Gamma_{k_j} < \Gamma_{k_j+1}\ \forall j \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau : \mathcal{N} \to \mathcal{N}$ by

$$\tau(k) := \max\{j \le k : \Gamma_j < \Gamma_{j+1}\}.$$

By Lemma 2.9, we get

$$\Gamma_{\tau(k)} \le \Gamma_{\tau(k)+1} \quad \text{and} \quad \Gamma_k \le \Gamma_{\tau(k)+1}. \tag{3.49}$$

Utilizing the same inferences as in (3.21) and (3.37), we can obtain that

$$\lim_{k\to\infty} \|x^{\tau(k)+1} - \hat{q}^{\tau(k)}\| = \lim_{k\to\infty} \|\hat{p}^{\tau(k)} - y^{\tau(k)}\| = \lim_{k\to\infty} \|y^{\tau(k)} - z^{\tau(k)}\| = 0, \tag{3.50}$$

$$\lim_{k\to\infty} \|x^{\tau(k)+1} - x^{\tau(k)}\| = 0. \tag{3.51}$$

Since $\{\hat{q}^k\}$ is bounded, there exists a subsequence of $\{\hat{q}^{\tau(k)}\}$ converging weakly to $\hat{x}$. Without loss of generality, we may assume that $\hat{q}^{\tau(k)} \rightharpoonup \hat{x}$. Then, utilizing the same inferences as in Case 1, we can obtain that $\hat{x} \in \Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi)$. From $\hat{q}^{\tau(k)} \rightharpoonup \hat{x}$ and (3.50), we get $x^{\tau(k)+1} \rightharpoonup \hat{x}$. Using the condition $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$, we have $1 - 2\alpha_{\tau(k)}c_1 > 0$ and $1 - 2\alpha_{\tau(k)}c_2 > 0$. So it follows from (3.18) that

$$\begin{aligned} &2s_{\tau(k)}[\Psi(\hat{q}^{\tau(k)}, x^{\tau(k)+1}) - \Psi(\hat{q}^{\tau(k)}, x^*)] \le \Gamma_{\tau(k)} - \Gamma_{\tau(k)+1} \\ &\quad - \frac{\delta_{\tau(k)}}{1-\gamma_{\tau(k)}}[(1-2\alpha_{\tau(k)}c_1)\|y^{\tau(k)} - \hat{p}^{\tau(k)}\|^2 + (1-2\alpha_{\tau(k)}c_2)\|z^{\tau(k)} - y^{\tau(k)}\|^2] \\ &\quad + \frac{\theta_{\tau(k)}\widetilde{M}}{1-\gamma_{\tau(k)}} - \frac{1}{1-\gamma_{\tau(k)}}[\beta_{\tau(k)}\|x^{\tau(k)} - \hat{q}^{\tau(k)}\|^2 + \delta_{\tau(k)}\|T^{\tau(k)}z^{\tau(k)} - \hat{q}^{\tau(k)}\|^2] \\ &\quad - \|x^{\tau(k)+1} - \hat{q}^{\tau(k)}\|^2 \\ &\le \frac{\theta_{\tau(k)}\widetilde{M}}{1-\gamma_{\tau(k)}}, \end{aligned}$$

which hence leads to

$$\Psi(\hat{q}^{\tau(k)}, x^{\tau(k)+1}) - \Psi(\hat{q}^{\tau(k)}, x^*) \le \frac{\theta_{\tau(k)}}{s_{\tau(k)}} \cdot \frac{\widetilde{M}}{2(1-\gamma_{\tau(k)})}. \tag{3.52}$$

Since Ψ is ν-strongly monotone on C, we get

$$\nu\|\hat{q}^{\tau(k)} - x^*\|^2 \le -\Psi(\hat{q}^{\tau(k)}, x^*) - \Psi(x^*, \hat{q}^{\tau(k)}). \tag{3.53}$$

Combining (3.52) and (3.53), we deduce from $\mathbf{Ass}_\Psi(\Psi_1)$ and $\hat{x} \in \Omega$ that

$$\begin{aligned}\nu \limsup_{k\to\infty} \|\hat{q}^{\tau(k)} - x^*\|^2 &= \limsup_{k\to\infty} [-\frac{\theta_{\tau(k)}}{s_{\tau(k)}} \cdot \frac{\widetilde{M}}{2(1-\gamma_{\tau(k)})} + \nu\|\hat{q}^{\tau(k)} - x^*\|^2] \\ &\le \limsup_{k\to\infty} [-\Psi(\hat{q}^{\tau(k)}, x^{\tau(k)+1}) - \Psi(x^*, \hat{q}^{\tau(k)})] \\ &= -\Psi(\hat{x}, \hat{x}) - \Psi(x^*, \hat{x}) \\ &\le 0.\end{aligned}$$

Hence, $\limsup_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 \le 0$. Thus, we get $\lim_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 = 0$. From (3.51), we get

$$\begin{aligned}\|x^{\tau(k)+1} - x^*\|^2 - \|x^{\tau(k)} - x^*\|^2 &= 2\langle x^{\tau(k)+1} - x^{\tau(k)}, x^{\tau(k)} - x^*\rangle + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2 \\ &\le 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| \\ &\quad + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2 \to 0 \quad (k\to\infty).\end{aligned}$$

Owing to $\Gamma_k \le \Gamma_{\tau(k)+1}$, we get

$$\begin{aligned}\|x^k - x^*\|^2 &\le \|x^{\tau(k)+1} - x^*\|^2 \\ &\le \|x^{\tau(k)} - x^*\|^2 + 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2.\end{aligned}$$

So it follows from (3.51) that $x^k \to x^*$ as $k \to \infty$. This completes the proof. □

Next, we introduce another parallel Mann implicit subgradient extragradient algorithm.

Algorithm 3.1. Initial Step: Given $x^1 \in C$ arbitrarily. The sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}, \{s_k\}$ satisfy the conditions (H1)-(H5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1. Compute

$$u^k = \varepsilon_k x^k + (1-\varepsilon_k)\hat{p}^k,$$
$$v^k = P_C(u^k - \mu_2 B_2 u^k).$$

Step 2. Compute

$$\hat{p}^k = P_C(v^k - \mu_1 B_1 v^k),$$
$$y^k = \operatorname{argmin}\{\alpha_k \Phi(\hat{p}^k, y) + \frac{1}{2}\|y - \hat{p}^k\|^2 : y \in C\}.$$

Step 3. Choose $\hat{w}^k \in \partial_2 \Phi(\hat{p}^k, y^k)$, and compute

$$C_k = \{v \in \mathcal{H} : \langle \hat{p}^k - \alpha_k \hat{w}^k - y^k, v - y^k\rangle \le 0\},$$
$$z^k = \operatorname{argmin}\{\alpha_k \Phi(y^k, z) + \frac{1}{2}\|z - \hat{p}^k\|^2 : z \in C_k\}.$$

Step 4. Compute

$$\hat{q}^k = \beta_k \hat{p}^k + \gamma_k T_k \hat{q}^k + \delta_k T^k z^k,$$
$$x^{k+1} = \operatorname{argmin}\{s_k \Psi(\hat{q}^k, t) + \frac{1}{2}\|t - \hat{q}^k\|^2 : t \in C\}.$$

Set $k := k+1$ and return to Step 1.

Theorem 3.2. *Assume that $\{x^k\}$ is the sequence constructed by Algorithm 3.1. Let the bifunctions Ψ, Φ satisfy the assumptions $\mathbf{Ass}_\Phi$-$\mathbf{Ass}_\Psi$. Then, under the conditions (H1)-(H5), the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω, Ψ) provided $T^k x^k - T^{k+1}x^k \to 0$.*

Proof. In terms of Lemma 2.5, we know that G is nonexpansive. Hence, for each $k \geq 1$ there exists a unique element $u^k \in C$ such that $u^k = \varepsilon_k x^k + (1-\varepsilon_k)Gu^k$. Also, by Lemma 2.8 we know that for each $k \geq 1$ there exists a unique element $\hat{q}^k \in C$ such that

$$\hat{q}^k = \beta_k \hat{p}^k + \gamma_k T_k \hat{q}^k + \delta_k T^k z^k. \tag{3.54}$$

Choose an element $q \in \Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Sol}(C, \Phi)$ arbitrarily. Noticing $\lim_{k\to\infty} \frac{\theta_k}{s_k} = 0$, we might assume that $\theta_k \leq \frac{1}{2}\lambda s_k$ for all $k \geq 1$. We divide the proof into several steps as follows:

Steps 1-3. We show that the results in Steps 1-3 of the proof of Theorem 3.1 are still valid. In fact, using the same arguments as in the proof of Theorem 3.1, we obtain the desired results.

Step 4. We claim that the sequence $\{x^k\}$ is bounded. Indeed, using the same arguments as in the proof of Theorem 3.1, we know that the inequality (3.14) still holds. Since each T_k is a pseudocontraction mapping, and T is asymptotically nonexpansive, we deduce from (3.14) that

$$\begin{aligned}
\|\hat{q}^k - x^*\|^2 &= \langle \beta_k(\hat{p}^k - x^*) + \gamma_k(T_k\hat{q}^k - x^*) + \delta_k(T^k z^k - x^*), \hat{q}^k - x^* \rangle \\
&= \beta_k \langle \hat{p}^k - x^*, \hat{q}^k - x^* \rangle + \gamma_k \langle T_k \hat{q}^k - x^*, \hat{q}^k - x^* \rangle \\
&\quad + \delta_k \langle T^k z^k - x^*, \hat{q}^k - x^* \rangle \\
&\leq \beta_k \|\hat{p}^k - x^*\| \|\hat{q}^k - x^*\| + \gamma_k \|\hat{q}^k - x^*\|^2 \\
&\quad + \delta_k (1+\theta_k) \|z^k - x^*\| \|\hat{q}^k - x^*\| \\
&\leq \beta_k (1+\theta_k) \|\hat{p}^k - x^*\| \|\hat{q}^k - x^*\| + \gamma_k \|\hat{q}^k - x^*\|^2 \\
&\quad + \delta_k (1+\theta_k) \|\hat{p}^k - x^*\| \|\hat{q}^k - x^*\| \\
&\leq (1-\gamma_k)(1+\theta_k) \|x^k - x^*\| \|\hat{q}^k - x^*\| + \gamma_k \|\hat{q}^k - x^*\|^2,
\end{aligned}$$

which hence yields $\|\hat{q}^k - x^*\| \leq (1+\theta_k)\|x^k - x^*\|$. Consequently,

$$\begin{aligned}
\|x^{k+1} - x^*\| &\leq \|x^{k+1} - \hat{q}^k_*\| + \|\hat{q}^k_* - x^*\| \leq (1-\lambda s_k)\|\hat{q}^k - x^*\| + \|\hat{q}^k_* - x^*\| \\
&\leq (1-\lambda s_k)(1+\theta_k)\|x^k - x^*\| + s_k \hat{M}(x^*) \\
&\leq \max\left\{ \|x^k - x^*\|, \frac{2\hat{M}(x^*)}{\lambda} \right\}.
\end{aligned}$$

By induction, we get $\|x^k - x^*\| \leq \max\{\|x^1 - x^*\|, \frac{2\hat{M}(x^*)}{\lambda}\}\ \forall k \geq 1$. Thus, $\{x^k\}$ is bounded, and so are the sequences $\{\hat{p}^k\}, \{\hat{q}^k\}, \{y^k\}, \{z^k\}, \{u^k\}, \{v^k\}$.

Step 5. We show that if $x^{k_i} \rightharpoonup \hat{x}$, $\hat{p}^{k_i} - x^{k_i} \to 0$ and $\hat{p}^{k_i} - y^{k_i} \to 0$ for $\{k_i\} \subset \{k\}$, then $\hat{x} \in \mathrm{Sol}(C, \Phi)$. Indeed, using the same arguments as in the proof of Theorem 3.1, we obtain the desired result.

Step 6. We show that $x^k \to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraints.

Indeed, set $\varGamma_k = \|x^k - x^*\|^2$. Since each T_k is pseudocontractive and T is asymptotically nonexpansive, by Lemma 2.1 (iv) we obtain

$$\begin{aligned}\|\hat{q}^k - x^*\|^2 &= \beta_k\langle \hat{p}^k - x^*, \hat{q}^k - x^*\rangle + \gamma_k\langle T_k\hat{q}^k - x^*, \hat{q}^k - x^*\rangle \\ &\quad + \delta_k\langle T^k z^k - x^*, \hat{q}^k - x^*\rangle \\ &\le \frac{\beta_k}{2}[\|\hat{p}^k - x^*\|^2 + \|\hat{q}^k - x^*\|^2 - \|\hat{p}^k - \hat{q}^k\|^2] \\ &\quad + \gamma_k\|\hat{q}^k - x^*\|^2 + \frac{\delta_k}{2}[\|T^k z^k - x^*\|^2 \\ &\quad + \|\hat{q}^k - x^*\|^2 - \|T^k z^k - \hat{q}^k\|^2] \\ &= \frac{\beta_k}{2}\|\hat{p}^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|\hat{q}^k - x^*\|^2 \\ &\quad + \frac{\delta_k}{2}\|T^k z^k - x^*\|^2 - \frac{\beta_k}{2}\|\hat{p}^k - \hat{q}^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - \hat{q}^k\|^2 \\ &\le \frac{\beta_k}{2}\|\hat{p}^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|\hat{q}^k - x^*\|^2 + \frac{\delta_k(1+\theta_k)^2}{2}\|z^k - x^*\|^2 \\ &\quad - \frac{\beta_k}{2}\|\hat{p}^k - \hat{q}^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - \hat{q}^k\|^2 \\ &\le \frac{\beta_k}{2}\|\hat{p}^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|\hat{q}^k - x^*\|^2 + \frac{\delta_k}{2}\|z^k - x^*\|^2 + \frac{\theta_k\widetilde{M}}{2} \\ &\quad - \frac{\beta_k}{2}\|\hat{p}^k - \hat{q}^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - \hat{q}^k\|^2,\end{aligned}$$

where $\sup_{k\ge1}(2+\theta_k)\|x^k - x^*\|^2 \le \widetilde{M}$ for some $\widetilde{M} > 0$. This implies that

$$\begin{aligned}\|\hat{q}^k - x^*\|^2 \le \frac{1}{1-\gamma_k}&[\beta_k\|\hat{p}^k - x^*\|^2 + \delta_k\|z^k - x^*\|^2 \\ &+ \theta_k\widetilde{M} - \beta_k\|\hat{p}^k - \hat{q}^k\|^2 - \delta_k\|T^k z^k - \hat{q}^k\|^2].\end{aligned} \tag{3.55}$$

By the results in Steps 1 and 2 we deduce from (3.14) and (3.55) that

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le \|\hat{q}^k - x^*\|^2 - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\ &\le \frac{1}{1-\gamma_k}[\beta_k\|\hat{p}^k - x^*\|^2 + \delta_k\|z^k - x^*\|^2 \\ &\quad + \theta_k\widetilde{M} - \beta_k\|\hat{p}^k - \hat{q}^k\|^2 - \delta_k\|T^k z^k - \hat{q}^k\|^2] \\ &\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\ &\le \frac{1}{1-\gamma_k}\{\beta_k\|\hat{p}^k - x^*\|^2 + \delta_k[\|\hat{p}^k - x^*\|^2 \\ &\quad - (1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\ &\quad + \theta_k\widetilde{M} - \beta_k\|\hat{p}^k - \hat{q}^k\|^2 - \delta_k\|T^k z^k - \hat{q}^k\|^2\} \\ &\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})]\end{aligned}$$

$$
\begin{aligned}
&\le \|\hat{p}^k - x^*\|^2 - \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 \\
&\quad + (1-2\alpha_k c_2)\|z^k - y^k\|^2] + \frac{\theta_k \widetilde{M}}{1-\gamma_k} \\
&\quad - \frac{1}{1-\gamma_k}[\beta_k\|\hat{p}^k - \hat{q}^k\|^2 + \delta_k\|T^k z^k - \hat{q}^k\|^2] \\
&\quad - \|x^{k+1} - \hat{q}^k\|^2 + 2s_k[\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})] \\
&\le \|x^k - x^*\|^2 - \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 \\
&\quad + (1-2\alpha_k c_2)\|z^k - y^k\|^2] + \frac{\theta_k \widetilde{M}}{1-\gamma_k} \\
&\quad - \frac{1}{1-\gamma_k}[\beta_k\|\hat{p}^k - \hat{q}^k\|^2 + \delta_k\|T^k z^k - \hat{q}^k\|^2] - \|x^{k+1} - \hat{q}^k\|^2 \\
&\quad + s_k K,
\end{aligned} \tag{3.56}
$$

where $\sup_{k\ge 1}\{2|\Psi(\hat{q}^k, x^*) - \Psi(\hat{q}^k, x^{k+1})|\} \le K$ for some $K > 0$.

Finally, we show the convergence of $\{\varGamma_k\}$ to zero by the following two cases:

Case 1. Suppose that there exists an integer $k_0 \ge 1$ such that $\{\varGamma_k\}$ is non-increasing. Then the limit $\lim_{k\to\infty} \varGamma_k = \hbar < +\infty$ and $\lim_{k\to\infty}(\varGamma_k - \varGamma_{k+1}) = 0$. From (3.56), we get

$$
\begin{aligned}
&\frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - \hat{p}^k\|^2 + (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\
&\quad + \frac{1}{1-\gamma_k}[\beta_k\|\hat{p}^k - \hat{q}^k\|^2 + \delta_k\|T^k z^k - \hat{q}^k\|^2] + \|x^{k+1} - \hat{q}^k\|^2 \\
&\quad \le \varGamma_k - \varGamma_{k+1} + \frac{\theta_k \widetilde{M}}{1-\gamma_k} + s_k K,
\end{aligned} \tag{3.57}
$$

Since $s_k \to 0,\ \theta_k \to 0,\ \varGamma_k - \varGamma_{k+1} \to 0,\ 0 < \liminf_{k\to\infty} \beta_k,\ 0 < \liminf_{k\to\infty} \delta_k$ and $0 < \liminf_{k\to\infty}(1-\gamma_k)$, we obtain from $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ that

$$
\lim_{k\to\infty} \|\hat{p}^k - \hat{q}^k\| = \lim_{k\to\infty} \|T^k z^k - \hat{q}^k\| = 0, \tag{3.58}
$$

$$
\lim_{k\to\infty} \|y^k - \hat{p}^k\| = \lim_{k\to\infty} \|z^k - y^k\| = \lim_{k\to\infty} \|x^{k+1} - \hat{q}^k\| = 0. \tag{3.59}
$$

Next we show that $\lim_{k\to\infty} \|x^k - x^*\| = 0$. In fact, utilizing the same arguments as those of (3.24) we get

$$
\begin{aligned}
\|\hat{p}^k - x^*\|^2 &\le \|x^k - x^*\|^2 - \mu_2(2\beta - \mu_2)\|B_2 u^k - B_2 x^*\|^2 \\
&\quad - \mu_1(2\alpha - \mu_1)\|B_1 v^k - B_1 y^*\|^2.
\end{aligned} \tag{3.60}
$$

Also, substituting (3.60) for (3.56), we get

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &\le \|\hat{p}^k - x^*\|^2 + \frac{\theta_k \widetilde{M}}{1-\gamma_k} + s_k K \\
&\le \|x^k - x^*\|^2 - \mu_2(2\beta - \mu_2)\|B_2 u^k - B_2 x^*\|^2
\end{aligned}
$$

$$- \mu_1(2\alpha - \mu_1)\|B_1v^k - B_1y^*\|^2 + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_kK,$$

which immediately yields

$$\mu_2(2\beta - \mu_2)\|B_2u^k - B_2x^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1v^k - B_1y^*\|^2] \leq \Gamma_k - \Gamma_{k+1} + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_kK.$$

Since $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, $s_k \to 0$, $\theta_k \to 0$, $\Gamma_k - \Gamma_{k+1} \to 0$ and $\liminf_{n\to\infty}(1-\gamma_k) > 0$, we get

$$\lim_{k\to\infty} \|B_2u^k - B_2x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|B_1v^k - B_1y^*\| = 0. \tag{3.61}$$

On the other hand, utilizing the same arguments as those of (3.28) we get

$$\begin{aligned}\|\hat{p}^k - x^*\|^2 \leq \|x^k - x^*\|^2 &- \|u^k - v^k + y^* - x^*\|^2 - \|v^k - \hat{p}^k + x^* - y^*\|^2 \\ &+ 2\mu_1\|B_1y^* - B_1v^k\|\|\hat{p}^k - x^*\| \\ &+ 2\mu_2\|B_2x^* - B_2u^k\|\|v^k - y^*\|.\end{aligned} \tag{3.62}$$

Substituting (3.62) for (3.56), we get

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|\hat{p}^k - x^*\|^2 + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_kK \\ &\leq \|x^k - x^*\|^2 - \|u^k - v^k + y^* - x^*\|^2 - \|v^k - \hat{p}^k + x^* - y^*\|^2 \\ &\quad + 2\mu_1\|B_1y^* - B_1v^k\|\|\hat{p}^k - x^*\| + 2\mu_2\|B_2x^* - B_2u^k\|\|v^k - y^*\| \\ &\quad + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_kK.\end{aligned}$$

This immediately leads to

$$\begin{aligned}&\|u^k - v^k + y^* - x^*\|^2 + \|v^k - \hat{p}^k + x^* - y^*\|^2 \\ &\quad \leq \Gamma_k - \Gamma_{k+1} + 2\mu_1\|B_1y^* - B_1v^k\|\|\hat{p}^k - x^*\| + 2\mu_2\|B_2x^* - B_2u^k\|\|v^k - y^*\| \\ &\qquad + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_kK.\end{aligned}$$

Since $s_k \to 0$, $\theta_k \to 0$, $\Gamma_k - \Gamma_{k+1} \to 0$ and $0 < \liminf_{k\to\infty}(1-\gamma_k)$, we deduce from (3.61) that

$$\lim_{k\to\infty} \|u^k - v^k + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|v^k - \hat{p}^k + x^* - y^*\| = 0. \tag{3.63}$$

Hence,

$$\begin{aligned}\|u^k - Gu^k\| = \|u^k - \hat{p}^k\| &\leq \|u^k - v^k + y^* - x^*\| \\ &\quad + \|v^k - \hat{p}^k + x^* - y^*\| \to 0 \quad (k \to \infty).\end{aligned} \tag{3.64}$$

Utilizing the same arguments as those of (3.31) and (3.32), we get

$$\lim_{k\to\infty} \|x^k - \hat{p}^k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x^k - Gx^k\| = 0. \tag{3.65}$$

We claim that $\|Tx^k - x^k\| \to 0$ and $\|T_k x^k - x^k\| \to 0$ as $k \to \infty$. In fact, since $\hat{q}^k = \beta_k \hat{p}_k + \gamma_k T_k \hat{q}^k + \delta_k T^k z^k$ and $0 < \liminf_{k\to\infty} \gamma_k$, we obtain from (3.58) that

$$\begin{aligned} \|T_k \hat{q}^k - \hat{q}^k\| &= \frac{1}{\gamma_k} \|\beta_k(\hat{p}_k - \hat{q}^k) + \delta_k(T^k z^k - \hat{q}^k)\| \\ &\le \frac{1}{\gamma_k} [\|\hat{p}_k - \hat{q}^k\| + \|T^k z^k - \hat{q}^k\|] \to 0 \quad (k \to \infty). \end{aligned} \tag{3.66}$$

Since $\{T_k\}_{k=1}^\infty$ is ℓ-uniformly Lipschitzian, we deduce from (3.58), (3.65) and (3.66) that

$$\|x^k - \hat{q}^k\| \le \|x^k - \hat{p}^k\| + \|\hat{p}^k - \hat{q}^k\| \to 0 \quad (k \to \infty), \tag{3.67}$$

and hence

$$\begin{aligned} \|T_k x^k - x^k\| &\le \|T_k x^k - T_k \hat{q}^k\| + \|T_k \hat{q}^k - \hat{q}^k\| + \|\hat{q}^k - x^k\| \\ &\le (\ell + 1)\|x^k - \hat{q}^k\| + \|T_k \hat{q}^k - \hat{q}^k\| \to 0 \quad (k \to \infty). \end{aligned} \tag{3.68}$$

From (3.58), (3.59), (3.65) and (3.67), we conclude that

$$\|z^k - x^k\| \le \|z^k - y^k\| + \|y^k - \hat{p}^k\| + \|\hat{p}^k - x^k\| \to 0 \quad (k \to \infty), \tag{3.69}$$

and hence

$$\begin{aligned} \|T^k x^k - x^k\| &\le \|T^k x^k - T^k z^k\| + \|T^k z^k - \hat{q}^k\| + \|\hat{q}^k - x^k\| \\ &\le (1 + \theta_k)\|x^k - z^k\| + \|T^k z^k - \hat{q}^k\| + \|\hat{q}^k - x^k\| \to 0 \\ &\quad (k \to \infty). \end{aligned} \tag{3.70}$$

Utilizing the same arguments as those of (3.36) we have

$$\lim_{k\to\infty} \|x^k - Tx^k\| = 0. \tag{3.71}$$

Utilizing the same arguments as those of (3.41) we get

$$\lim_{k\to\infty} \|x^k - \overline{T} x^k\| = 0. \tag{3.72}$$

where $\overline{T} := (2I - \widehat{T})^{-1}$ and $\mathrm{Fix}(\overline{T}) = \mathrm{Fix}(\widehat{T}) = \bigcap_{k=1}^\infty \mathrm{Fix}(T_k)$. Further, utilizing the same arguments as in Case 1 of the proof of Theorem 3.1, we obtain that $\lim_{k\to\infty} \mathit{\Gamma}_k = 0$, and hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem $\mathrm{EP}(\mathit{\Omega}, \Psi)$.

Case 2. Suppose that $\exists\{\mathit{\Gamma}_{k_j}\} \subset \{\mathit{\Gamma}_k\}$ s.t. $\mathit{\Gamma}_{k_j} < \mathit{\Gamma}_{k_j+1}\ \forall j \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau : \mathcal{N} \to \mathcal{N}$ by

$$\tau(k) := \max\{j \le k : \mathit{\Gamma}_j < \mathit{\Gamma}_{j+1}\}.$$

In the remainder of the proof, utilizing the same arguments as in Case 2 of the proof of Theorem 3.1, we obtain the desired result. This completes the proof. □

Remark 3.3. Compared with the corresponding results in Ceng et al. [18], Anh and An [1] and Ceng and Wen [24], our results improve and extend them in the following aspects.

(i) The problem of finding a solution of the GSVI (1.1) with the CFPP constraint of a countable family of ℓ-uniformly Lipschitzian pseudocontractions and an asymptotically nonexpansive mapping in [24] is extended to develop our problem of finding a solution of the MBEP with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. The hybrid extragradient-like implicit method in [24] is extended to develop our parallel Mann implicit subgradient extragradient algorithms for solving the MBEP with the GSVI and CFPP constraints.

(ii) The problem of finding a solution of the MBEP with the FPP constraint in [1] is extended to develop our problem of finding a solution of the MBEP with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. The modified subgradient extragradient method for solving the MBEP with the FPP constraint in [1] is extended to develop our parallel Mann implicit subgradient extragradient algorithms for solving the MBEP with the GSVI and CFPP constraints.

(iii) The problem of finding a solution of the VIP with the CFPP constraint of finitely many nonexpansive mappings in [18] is extended to develop our problem of finding a solution of the MBEP with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP. The inertial subgradient extragradient method for solving the VIP with the CFPP constraint in [18] is extended to develop our parallel Mann implicit subgradient extragradient algorithms for solving the MBEP with the GSVI and CFPP constraints.

4. Applications to the GSVI, VIP and FPP

In this section, we consider the applications of Theorems 3.1 and 3.2 to finding a common solution of the GSVI, VIP and FPP. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $G : \mathcal{H} \to C$ be defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ where $0 < \mu_1 < 2\alpha$ and $0 < \mu_2 < 2\beta$. Let $T = P_C : \mathcal{H} \to C$ be the metric projection of $\mathcal{H}$ onto C, and $T_k = S : C \to C$ be a nonexpansive mapping for all $k \geq 1$. An operator $A : \mathcal{H} \to \mathcal{H}$ is said to be

(i) monotone if $\langle Ax - Ay, x - y\rangle \geq 0\ \forall x, y \in \mathcal{H}$;

(ii) L-Lipschitz continuous if $\exists L > 0$ s.t. $\|Ax - Ay\| \leq L\|x - y\|\ \forall x, y \in \mathcal{H}$.

The VIP for A is to find $x^* \in C$ s.t.

$$\langle Ax^*, x - x^*\rangle \geq 0 \quad \forall x \in C. \tag{4.1}$$

We denote by $\mathrm{VI}(C, A)$ the solution set of the problem (4.1). Let $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{VI}(C, A) \neq \emptyset$, and suppose that A satisfies the following conditions:

(B1) A is monotone;

(B2) A is weakly to strongly continuous, that is, $Ax^k \to Ax$ for each sequence $\{x^k\} \subset \mathcal{H}$ converging weakly to x;

(B3) A is L-Lipschitz continuous for some constant $L > 0$.

In addition, let the bifunction Ψ, and the positive sequences $\{\alpha_k\}, \{s_k\}$ and $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ be the same as in Algorithm 2.1. We define $\Phi(x, y) := \langle Ax, y - x\rangle$ for each $x, y \in \mathcal{H}$. Then the EP 1.3 becomes the VIP (4.1). It is easy to check that the bifunction $\Phi(x,y) = \langle Ax, y-x\rangle$ satisfies conditions $\mathbf{Ass}_{\mathbf{\Phi}}(\Phi_1)$-$\mathbf{Ass}_{\mathbf{\Phi}}(\Phi_2)$ where Φ is Lipschitz-type continuous with $c_1 = c_2 = L/2$. It follows from the definitions of y^k in Algorithm 2.1 and Φ that

$$\begin{aligned} y^k &= \operatorname{argmin}\{\alpha_k\langle A\hat{p}^k, y-\hat{p}^k\rangle + \tfrac{1}{2}\|\hat{p}^k - y\|^2 : y \in C\} \\ &= \operatorname{argmin}\{\tfrac{1}{2}\|y-(\hat{p}^k - \alpha_k A\hat{p}^k)\|^2 : y \in C\} - \tfrac{\alpha_k^2}{2}\|A\hat{p}^k\|^2 \\ &= P_C(\hat{p}^k - \alpha_k A\hat{p}^k), \end{aligned}$$

and similarly, z^k in Algorithm 2.1 reduces to

$$z^k = P_{C_k}(\hat{p}^k - \alpha_k Ay^k).$$

In terms of $\hat{w}^k \in \partial_2\Phi(\hat{p}^k, y^k)$ and the definition of the subdifferential of Φ, we have

$$\langle \hat{w}^k, z-y^k\rangle \le \langle A\hat{p}^k, z-\hat{p}^k\rangle - \langle A\hat{p}^k, y^k-\hat{p}^k\rangle = \langle A\hat{p}^k, z-y^k\rangle \quad \forall z \in \mathcal{H},$$

and hence

$$0 \le \langle A\hat{p}^k - \hat{w}^k, z-y^k\rangle \quad \forall z \in \mathcal{H}.$$

Thus

$$\begin{aligned} \langle \hat{p}^k - \alpha_k A\hat{p}^k - y^k, z-y^k\rangle &\le \langle \hat{p}^k - \alpha_k A\hat{p}^k - y^k, z-y^k\rangle + \alpha_k\langle A\hat{p}^k - \hat{w}^k, z-y^k\rangle \\ &= \langle(\hat{p}^k - \alpha_k\hat{w}^k) - y^k, z-y^k\rangle. \end{aligned}$$

Therefore, the parallel Mann implicit subgradient extragradient Algorithm 2.1 reduces to the following algorithm for solving the GSVI, VIP and FPP.

Algorithm 4.1. Initial Step: Given $x^1 \in C$ arbitrarily. The sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}$, $\{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}, \{s_k\}$ satisfy the conditions (H1)-(H5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1. Compute

$$\begin{aligned} u^k &= \varepsilon_k x^k + (1-\varepsilon_k)\hat{p}^k, \\ v^k &= P_C(u^k - \mu_2 B_2 u^k). \end{aligned}$$

Step 2. Compute

$$\begin{aligned} \hat{p}^k &= P_C(v^k - \mu_1 B_1 v^k), \\ y^k &= P_C(\hat{p}^k - \alpha_k A\hat{p}^k). \end{aligned}$$

Step 3. Choose $\hat{w}^k = A\hat{p}^k$, and compute

$$\begin{aligned} C_k &= \{v \in \mathcal{H} : \langle \hat{p}^k - \alpha_k\hat{w}^k - y^k, v-y^k\rangle \le 0\}, \\ z^k &= P_{C_k}(\hat{p}^k - \alpha_k Ay^k). \end{aligned}$$

Step 4. Compute

$$\begin{aligned} \hat{q}^k &= \beta_k x^k + \gamma_k T_k\hat{q}^k + \delta_k P_C z^k, \\ x^{k+1} &= \operatorname{argmin}\left\{ s_k\Psi(\hat{q}^k, t) + \frac{1}{2}\|t-\hat{q}^k\|^2 : t \in C\right\}. \end{aligned}$$

Set $k := k+1$ and return to Step 1.

Using Theorem 3.1 we obtain the following result.

Theorem 4.1. *Assume that* $\{x^k\}$ *is the sequence constructed by Algorithm 4.1. Then* $\{x^k\}$ *converges strongly to the unique solution* x^* *of the problem* $\mathrm{EP}(\Omega, \Psi)$, *where* $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{VI}(C, A)$.

In the same way, the parallel Mann implicit subgradient extragradient Algorithm 3.1 reduces to the following algorithm for solving the GSVI, VIP and FPP.

Algorithm 4.2. Initial Step: Given $x^1 \in C$ arbitrarily. The sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0, 1)$, and positive sequences $\{\alpha_k\}, \{s_k\}$ satisfy the conditions (H1)-(H5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1. Compute

$$\begin{aligned} u^k &= \varepsilon_k x^k + (1 - \varepsilon_k)\hat{p}^k, \\ v^k &= P_C(u^k - \mu_2 B_2 u^k). \end{aligned}$$

Step 2. Compute

$$\begin{aligned} \hat{p}^k &= P_C(v^k - \mu_1 B_1 v^k), \\ y^k &= P_C(\hat{p}^k - \alpha_k A\hat{p}^k). \end{aligned}$$

Step 3. Choose $\hat{w}^k = A\hat{p}^k$, and compute

$$\begin{aligned} C_k &= \{v \in \mathcal{H} : \langle \hat{p}^k - \alpha_k \hat{w}^k - y^k, v - y^k \rangle \leq 0\}, \\ z^k &= P_{C_k}(\hat{p}^k - \alpha_k A y^k). \end{aligned}$$

Step 4. Compute

$$\begin{aligned} \hat{q}^k &= \beta_k \hat{p}^k + \gamma_k T_k \hat{q}^k + \delta_k P_C z^k, \\ x^{k+1} &= \operatorname{argmin} \left\{ s_k \Psi(\hat{q}^k, t) + \frac{1}{2}\|t - \hat{q}^k\|^2 : t \in C \right\}. \end{aligned}$$

Set $k := k + 1$ and return to Step 1.

Using Theorem 3.2 we derive the following result.

Theorem 4.2. *Assume that* $\{x^k\}$ *is the sequence constructed by Algorithm 4.2. Then* $\{x^k\}$ *converges strongly to the unique solution* x^* *of the problem* $\mathrm{EP}(\Omega, \Psi)$, *where* $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{VI}(C, A)$.

5. Concluding remarks

In a real Hilbert space, let the GSVI and CFPP represent a general system of variational inequalities and a common fixed-point problem of a countable family of uniformly Lipschitzian pseudocontractions and an asymptotically nonexpansive mapping, respectively. In this article, we have suggested two new iterative algorithms with the parallel Mann implicit subgradient extragradient technique for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem over the common solution set of another monotone equilibrium problem, the GSVI and the CFPP.

The strong convergence results for the proposed algorithms to solve such a MBEP with the GSVI and CFPP constraints are established under some suitable assumptions. Furthermore, in the proposed method, the second minimization problem over a closed convex set is replaced by the subgradient projection onto some constructible half-space, and a new approach for solving the GSVI and CFPP via Mann implicit iterations is presented. As a consequence, we have obtained the iterative algorithms for solving the GSVI, VIP and FPP.

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