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# NOTE ON DIFFERENTIATION BASES IN $\mathbb{R}^n$

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Abstract. It is known that the Lebesgue differentiation theorem (LDT) is dependent on the geometric properties of the sets over which a function is integrated. If LDT holds over a collections of sets, then the collection of sets is said to "differentiate the integral." This paper composes of two parts. In the first part, it presents an elementary extension of the differentiation theorem from cubes to rectangles in $\mathbb{R}^n$. In the second part, it presents a solution to a question posed by C.A. Hayes and C.Y. Pauc in 1955: if a basis differentiates the integral for all measurable functions that is essentially bounded on every compact set, then this family of sets must have the property where it must almost cover every bounded measurable set (i.e. the set difference must have 0 measure), among other properties. [3]

## 1. Introduction

In 1910, Lebesgue extended the fundamental theorem of calculus to the Lebesgue integral with the Lebesgue differentiation theorem: let $f$ be a locally integrable function in $\mathbb{R}^n$. Then let $|\cdot|$ and $B(x,r)$ respectively denote the Lebesgue measure in $\mathbb{R}^n$ and the ball centered at $x$ with radius $r$. Then we have

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x) \text{ almost everywhere.}$$

In the case of the ball $B(x,r)$, one may say that the ball "contracts" to a point $x$. Hence we can equivalently write "$B(x,r) \to x$" to denote this process. However, as M. Guzman [6] notes in 1972, one observes that the definition of when a set "contracts" to a point is not immediately obvious for more general collections of sets: a sequence of sets that contain a point could converge to 0 in either measure or in diameter. For sets of fixed proportions such as balls or cubes, the two notions of "contraction" are equivalent. These two notions are not equivalent for sets such as intervals – parallelepipeds in $\mathbb{R}^n$ with sides parallel to the coordinate axes. In fact, one can construct a "counterexample" to the LDT if we consider the collection of rectangles with measure contracting to 0:

**Example 1.1** ("Counterexample" to LDT under Arbitrary Rectangles)**.** Consider the sequence of rectangles $\{R_k\}_{k=1}^{\infty}$ in $\mathbb{R}^2$, for $k > 1, k \in \mathbb{N}$:

$$R_k = [1,k] \times \left[-\frac{1}{k^2}, \frac{1}{k^2}\right]$$

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As $k \to \infty$, it can be readily seen that $\mu(R_k) \to 0$. Define $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \begin{cases} x^{-2} & \text{for } [1,\infty) \times [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Denote $\mathcal{L}(\mathbb{R}^2)$ as the set of integrable functions in $\mathbb{R}^2$. It can be verified that $f \in \mathcal{L}(\mathbb{R}^2)$ but $\lim_{k\to\infty} \int_{R_k} f d\mu \neq x$ for any $x \in \mathbb{R} \times [-1,1]$.

*Proof.* We first show that $f \in \mathcal{L}(\mathbb{R}^2)$. First note that $|A_j| = 4$ for every $j$. Then, we can compute:

$$\int_{\mathbb{R}^n} |f| d\mu \quad = \int_{(1,\infty)\times[-1,1]} x^{-2} d\mu = 2 \cdot 1 < \infty.$$

Now, we compute:

$$\begin{aligned} \int_{R_k} f d\mu \quad &= \tfrac{2}{k^2} \int_1^k x^{-2} \\ &= \tfrac{2}{k^2}(1 - \tfrac{1}{k}). \end{aligned}$$

Hence we easily obtain that

$$\lim_{k\to\infty} \frac{1}{|R_k|} \int_{R_k} f d\mu \quad = \lim_{k\to\infty} \tfrac{2}{k^2}(1 - \tfrac{1}{k}) = 0$$

Recall by the definition of $f(x)$, $f(x) \neq 0$ anywhere on $[1,\infty) \times [-1,1]$. Which shows that for any $x \in [1,\infty) \times [-1,1]$:

$$\lim_{k\to\infty} \frac{1}{|R_k|} \int_{R_k} f d\mu \neq f(x).$$

□

The theory of differentiation of integrals has been closely related with the covering properties of the family of sets. The example above shows that simply restricting the measure of sets is insufficient for the LDT to hold. Thus the established theory requires the stricter condition that restricts the diameter of a set. This is seen in the following definition of a differentiation basis, which is the subject of our study:

**Definition 1.2** (Differentiation Basis)**.** For every $x \in \beta^n$, let $\beta(x)$ be a collection of bounded measurable sets with nonzero measure that contains $x$. Then, if there exists a sequence $\{\gamma_k\} \subset \beta(x)$ such that the diameter $\delta(\gamma_k) \to 0$, then, the following collection

$$\beta = \bigcup_{x\in\mathbb{R}^n} \beta(x)$$

is a **differentiation basis**.

Hence we say that $\gamma_k \to x$ if each set in the sequence $\{\gamma_k\}$ contains $x$ and furthermore its diameter approaches 0. The basis above, subject to the additional constraints below, allows for the definition of a derivative of an integral:

**Definition 1.3** (Busemann-Feller Basis and Derivative)**.** A differentiation basis composed of opens sets is a **Busemann-Feller basis** when for every $\gamma \in \beta$ and $x \in \gamma$ we have $\gamma \in \beta(x)$. Hence, given a locally integrable $f$, a Busemann-Feller Basis $\beta$, and any sequence $\{\gamma_k\}$ such that $\gamma_k \to x$ as $k \to \infty$, we define the **derivative** with respect to the basis $\beta$:

$$D\left(x, \int f\right) = \lim_{\substack{\gamma_k \to \infty \\ x \in \gamma_k \in \beta}} \int_{\gamma_k} f.$$

**Remark 1.4.** If $D\left(x, \int f\right) = f(x)$, then we say that the basis $\beta$ differentiates the integral $\int f$ at $x$.

Thus the definition above generalizes Lebesgue's original definition of the derivative of integrals. Using elementary methods, one can already show that this generalization of Lebesgue's Differentiation Theorem holds for a wider class of sets – rectangles in $\mathbb{R}^n$ that can be rotated, scaled, and translated – in the following section.

This paper is based on my senior thesis at Georgetown University where I surveyed the topic of differentiation bases in relation to Lebesgue's Differentiation Theorem. It is a topic that has important and interesting applications in harmonic analysis, differential equations, and economics and has attracted much scholarly interest [7] [8] [2].

I am grateful for the assistance I have received from the faculty of the Department of Mathematics and Statistics at Georgetown. I would especially like to thank my teacher and advisor, Professor Der-Chen Chang, for not only introducing me to this research topic but also providing his guidance, knowledge, and encouragement throughout the entire process. I would also like to thank professor Nate Strawn for his assistance in the reviewing and correcting process.

## 2. Differentiation of integrals with the rectangular differentiation basis

The typical approach to showing Lebesgue's Differentiation Theorem is to approximate an arbitrary measurable function $f$ with a sequence of continuous functions $C_k$ and show that the set where $|f - C_k|$ does not approach 0 has measure zero. The critical lemma in the proof is Vitali's Covering Lemma, which, in line with the textbook approach [1] [11], can be simplified such that it is still suffiencient in showing Lebesgue's Differentiation Theorem:

**Lemma 2.1.** *Let $E \subset \mathbb{R}^n$ be a set where $|E|_e < +\infty$, that is, $E$ has finite outer measure. Then let $K = \{Q\}$ be a collection of cubes $Q$ such that $Q$ covers $E$. Then, there exists a finite collection of disjoint cubes $\{Q_j\}_{j=1}^N$ such that*

$$\sum_{j=1}^{N} |Q_j|_e \geq 5^{-n}|E|_e. \tag{2.1}$$

The proof of this Theorem is well known and readily available in standard textbooks [11]. The critical fact that involves the geometry of Differentiation Bases is that it requires the following covering property for cubes and balls in $\mathbb{R}^n$:

Let $Q_1, Q_2 \subset \mathbb{R}^n$ be nondisjoint cubes such that $Q_2$ is scaled compared to $Q_1$ by some constant $0 < c < 2$. Then $Q_2 \subset 5(Q_1)$.

This property of well-behaved shapes can then be extended with modifications to rectangles up to rotations, translations, and scaling. We begin with defining these sets. In $\mathbb{R}^n$, a basic and commonly accepted definition of an interval is defined as the cross product $I_1 \times I_2 \times ... \times I_n$. This set is also commonly called a rectangle. However, various results in this section, including the Vitali Covering Lemma, requires the scaling (i.e., shrinking and expanding) of rectangles. The common approach to scaling shapes is to fix the center and expand each interval. Hence, we define the following:

**Definition 2.2.** We denote a rectangle in $\mathbb{R}^n$ centered at $x$ as $R(x) = I_1 \times I_2 \times ... \times I_n$, where each interval $I_j$ has a center $x_j \in \mathbb{R}$ such that:

$$I_j = \left[x_j - \frac{|I_j|}{2}, x_j + \frac{|I_j|}{2}\right]$$

Where $|I_j|$ denotes the measure of $I_j$. Let $x = x_1e_1 + x_2e_2 + ... + x_ne_n$. We say that $x$ is the center of $R(x)$.

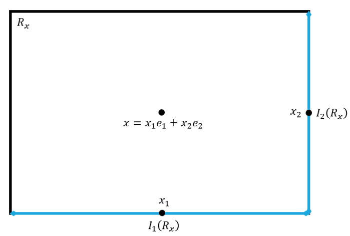


FIGURE 1. A Visualization of the rectangle centered at $x$.

**Definition 2.3.** For any rectangle $R(x) = \prod_{j=1}^n I_j$, the scaled rectangle $cR(x)$ where $0 < c < \infty$ is defined as the following:

$$cR(x) = \prod_{j=1}^n cI_j = \prod_{j=1}^n \left[x_j - \frac{c|I_j|}{2}, x_j + \frac{c|I_j|}{2}\right].$$

In other words, the "scaling" of $R$ by $c$ multiplies the length of each interval by $c$ while preserving its center $x$ and also preserving the ratio between the edges of the rectangle. Besides scaling, we would also like to "translate", i.e. move the rectangle in $\mathbf{R}_n$, which leads to the following definition:

**Definition 2.4.** For any rectangle $R(x)$, $R(x) + \mathbf{y} \in \mathbb{R}^n$ is defined as the rectangle with center $x + \mathbf{y}$. That is, for $1 \leq j \leq n$:

$$R(x) + \mathbf{y} = \prod_{j=1}^N \left[x_j + y_j - \frac{c|I_j|}{2}, x_j + y_j + \frac{c|I_j|}{2}\right].$$

We call this the translation of $R(x)$ by $\mathbf{y}$.

**Definition 2.5.** Let $R(x) \subset \mathbb{R}^n$ be a rectangle centered at $x$. Then, a rotated rectangle $\rho R(x)$ satisfies the following two conditions: (1) the distance between each point in $R(x)$ is preserved; (2) $x$ is fixed.

Note that in this definition, we allow for the transformation of a "flip" so long as it preserves the center. Although that the definition for rotation is much less explicit than the previous definition of translations, the properties above is sufficient to proving the critical result in our case - the Vitali Covering Lemma, and by extension, the Hardy-Littlewood Maximal Inequality [9].

**Example 2.6.** Let $R(x) \in \mathbb{R}^2$ be centered at the origin that is aligned with the axes. Then the rectangle rotated by $\theta$ is given by the set

$$\rho R(x) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{p} : \mathbf{p} \in R(x) \right\}.$$

We introduce the diagonal length of a rectangle, which provides the essential way of controlling and bounding the rectangle for this section:

**Definition 2.7.** Let $R(x) = \prod_{j=1}^{N} I_j$ be a rectangle in $\mathbf{R}^n$. Its diagonal length, $\ell[R(x)]$, is given by

$$\ell[R(x)] = \sqrt{\sum_{j=1}^{N} |I_i|^2}.$$

**Remark 2.8.** For any rectangle $R \subset \mathbb{R}^n$, observe that $|x - y| \leq \ell(R)/2$ for all $x, y \in R$.

*Proof.* Let $y_i \in I_i \subset \mathbb{R}$ and $x_i$ be the center of $I_i$, $1 \leq i \leq n$. Clearly we have $|x_i - y_i| \leq |I_i|/2$. This implies that

$$\begin{aligned} |x - y| &= \sqrt{\textstyle\sum_i^n (x-y)^2} \\ &\leq \sqrt{\textstyle\sum_i^n \frac{|I_i|^2}{2^2}} = \frac{\ell(R)}{2}. \end{aligned}$$

□

In other words, the distance between every point in a rectangle $R$ is bounded by $\ell(R)$.

**Proposition 2.9.** *Let $R_1 = \prod_{j=1}^{n} I_j$ be a rectangle centered at $x$. Then let $R_2 = \rho(cR_1) + \mathbf{y}$, $0 < c < \infty$. That is, $R_2$ is some rectangle that is scaled, rotated, and translated compared to $R_1$. Define*

$$r^* = \max_{1 \leq j \leq n} \left\{ \frac{\ell(R_1)}{|I_j(R_1)|} \right\} = \frac{\ell(R_1)}{\min_{1 \leq j \leq n}\{|I_j(R_1)|\}}$$

*Then, if $R_1$ and $R_2$ intersect, we have $(2cr^* + 1)R_1 \supseteq R_2$.*

*Proof.* First suppose we have $R_1(x)$ and $cR_1(x)$. Let $B$ be the ball with the same center $x$ and radius $c\ell(R_1)/2$. By Remark 2.8 and the property that a ball centered at $x$ contains all points $p \in \mathbb{R}$ s.t. $|x - p| < r$, we have:

$$p \in cR_1(x) \implies p \in B\left(x, \frac{c\ell(R_1)}{2}\right).$$

Then, since the rotation $\rho$ preserves the norm between each point in $cR_1(x)$, we similarly have $\rho[cR_1(x)] \subset B(x, \ell(R_2)/2)$. With this result, we now translate $\rho[cR_1(x)]$ by $y$ to obtain

$$R_2 \subset \rho[cR_1(x)] + y.$$

Observe that the ball given above recentered to $x + y$ similarly contains $R_2$. Hence, to prove our proposition, it suffices to show that

$$(2cr^* + 1)R_1 \supset B(x + y, c\ell(R_1)/2).$$

We first write that:

$$\begin{aligned}(2cr^* + 1)R_1 &= \prod_{j=1}^{N} \left[x_j - \tfrac{2cr^*|I_j|}{2} - \tfrac{|I_j|}{2}, x_j + \tfrac{2cr^*|I_j|}{2} + \tfrac{|I_j|}{2}\right] \\ &\supset \prod_{j=1}^{N} \left[x_j - c\ell(R_1) - \tfrac{|I_j|}{2}, x_j + c\ell(R_1) + \tfrac{|I_j|}{2}\right] \\ &= \prod_{j=1}^{N} \left[x_j - \ell(R_2) - \tfrac{|I_j|}{2}, x_j + \ell(R_2) + \tfrac{|I_j|}{2}\right]\end{aligned}$$

We then suppose that $B(x + y, \ell(R_2)) \cap R_1 \neq \emptyset$ and suppose that point $p \notin (2cr^* + 1)R_1$ and $p \in B(x + y, \ell(R_2))$. Hence for $1 \leq j \leq n$, since $p_j$ is in the Ball, we have

$$p \in B(x + y, \ell(R_2)/2) \quad \implies |p_j - x_j - y_j| < \tfrac{\ell(R_2)}{2}$$

On the other hand:

$$p \notin (2cr^* + 1)R_1 \quad \implies p_j \notin \left[x_j - \ell(R_2) - \tfrac{|I_j|}{2}, x_j + \ell(R_2) + \tfrac{|I_j|}{2}\right]$$

The above implies that $|y_j| > \ell(R_2)/2 + |I_j/2|, 1 \leq j \leq n$. But if that is the case, $B(x + y, \ell(R_2)/2)$ and $R_1$ cannot overlap since each component has been moved outside of the interval $I_j$. This gives us a contradiction and hence shows that we have the following containment:

$$(2cr^* + 1)R_1 \supset B(x + y, c\ell(R_1)/2) \supset \rho[cR_1(x)] = R_2.$$

□

As a note, the Proposition above implies Lemma 2.1.
With the result above, we can then construct an analogous version of the Hardy-Littlewood Maximal Function and hence the Hardy-Littlewood Maximal Inequality.

**Definition 2.10** (Hardy-Littlewood Maximal Operator)**.** Let $f$ be a function in $\mathbb{R}^n$ such that $f \in L(R)$ for every rectangle up to rotation and scaling, $R \subset \mathbb{R}^n$. Then we define the Hardy-Littlewood Maximal Operator $M$ for $f$ as

$$Mf(x) = \sup_{\substack{R \\ x \in R}} \left\{ \frac{1}{|R|} \int_R |f(t)| dt \right\}.$$

For a given $f$, we call $Mf(x)$ the Hardy-Littlewood Maximal Function. It is clear that

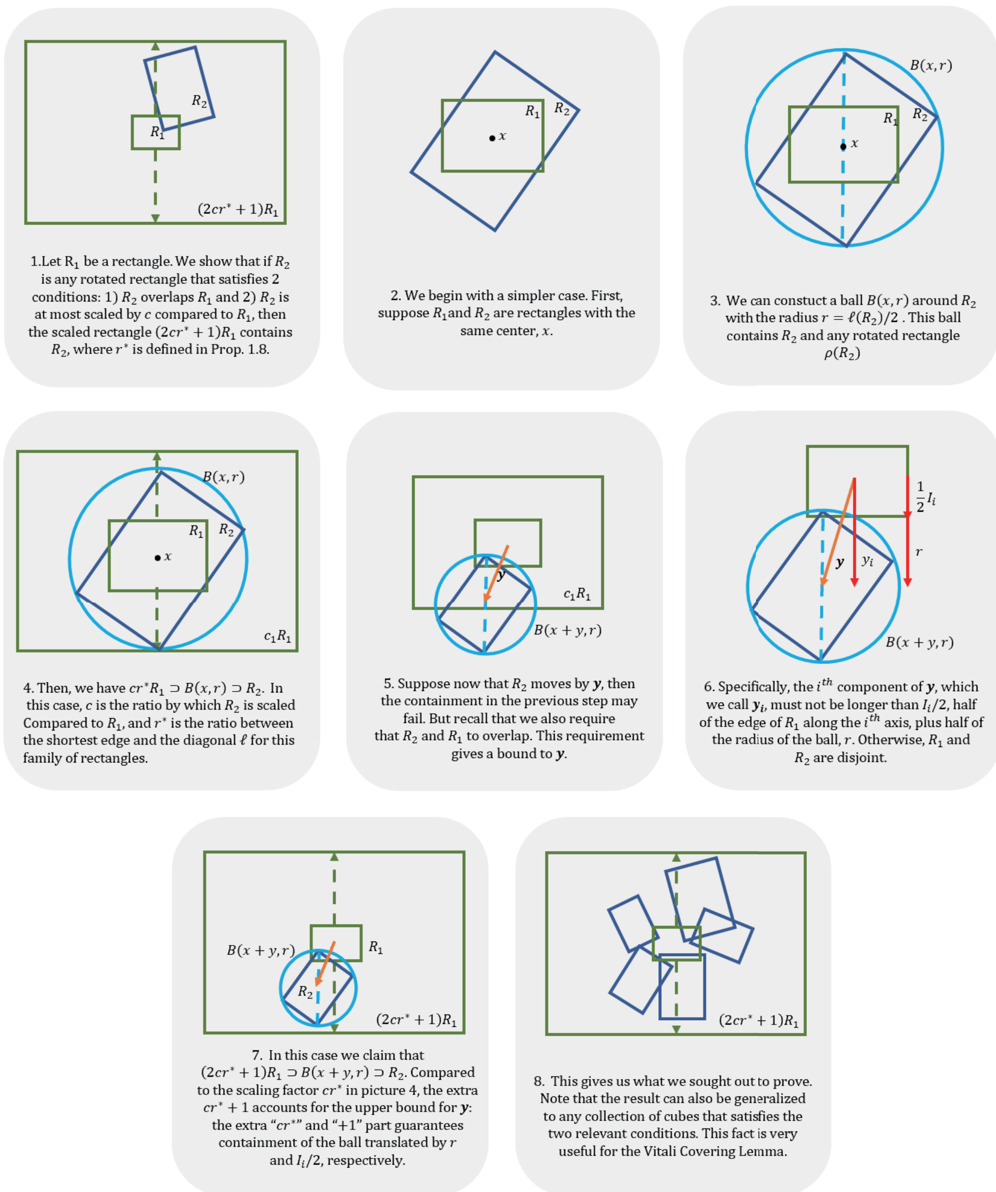


FIGURE 2. A Visual Illustration of Proposition 2.9

**Lemma 2.11** (Hardy-Littlewood Maximal Inequality). *Suppose $f \in L(\mathbb{R}^n)$ and $Mf(x)$ is defined over a basis of Rectangles where the largest rectangle is scaled by at most $c$ compared to the smallest rectangle. Then:*

$$|\{x \in \mathbb{R}^n : |Mf(x)| > \alpha\}| \leq \frac{1}{(2cr^* + 1)^n \alpha} \int_{\mathbb{R}^n} |f(x)| dx$$

.

*Proof.* Let $\alpha > 0$ and $E = \{x : |Mf(x)| > \alpha\}$. Let $x \in E$. Then, by the definition of $E$, there exists a $R$ up to rotation and translation containing $x$ such that:

$$\frac{1}{|R|}\int_R |f(y)|dy > \alpha.$$

the existence of such a R(x) is guaranteed since $Mf$ takes the supremum across all cubes and $Mf(x) > \alpha$ in $E$. We can rearrange the above, which leads to

$$|R| < \frac{1}{\alpha}\int_R |f(y)|dy \tag{2.2}$$

After establishing the inequality above, let $\{R\}$ be a collection of cubes that covers $E$. Then, the set $E_k = E \cap \{x : ||x|| < k\}$ are also covered by $\{R\}$. Since each ball $\{x : ||x|| < k\}$ has finite measure, $E_k$ also has finite measure. Then, by Lem. 2.1 (Vitali), there exists a finite number of points $\{x_j^{(k)}\}_j \subset E$ such that the collection of cubes centered at these points $\{R_j^{(k)}\}^N$ are disjoint and covers $E_k$, and that

$$|E_k| \leq \frac{1}{(2cr^* + 1)^n}\sum_{j=1}|R_{x_j^{(k)}}|$$

But due to Equation 2.2, we also have

$$\begin{aligned} \frac{1}{(2cr^*+1)^n}\sum_{j=1}|R_{x_j^{(k)}}| &< \tfrac{1}{(2cr^*+1)^n}\textstyle\sum_{j=1}\tfrac{1}{\alpha}\int_{R_{x_j^{(k)}}}|f| \\ &= \tfrac{1}{(2cr^*+1)^n\alpha}\textstyle\int_{\bigcup_{j=1}^N R_{x_j^{(k)}}}|f| \\ &\leq \tfrac{1}{(2cr^*+1)^n\alpha}\textstyle\int_{\mathbb{R}^n}|f| \end{aligned}$$

Where we arrive at the second equality since integration is additive for disjoint sets. The last inequality is due to the monotonicity of integration. Since the above inequality is true for all $k$ and $\bigcup_{k=1}^{\infty}$, we have

$$|E| < \frac{1}{(2cr^*+1)^n\alpha}\int_{\mathbb{R}^n}|f|.$$

□

**Theorem 2.12** (Lebesgue Differentiation Theorem for Rectangular Basis)**.** *Let* $\mathbf{R}^*$ *be a Busemann-Feller Basis of rectangles in* $\mathbb{R}^n$ *that can be scaled, rotated, and translated. Let* $\{R_x\} \subset \mathbf{R}^*$ *be a set of rectangles that contain* $x$*. Then we have:*

$$\frac{1}{|R_x|}\int_{R_x \to x} f d\mu = f(x)$$

*almost everywhere. Note that by* $R_x \to x$*, we mean the process where we restrict* $R_x$ *so that* $\ell(R_x) \to 0$*.*

*Proof.* In line with the standard approach to prove Lebesgue's Differentiation Theorem, we construct an alternate version of the Simplified Vitali Lemma (2.1) and the Hardy Littlewood Maximal Inequality (2.11) to arrive at the proof for the theorem above.

In the standard approach, recall that the Vitali Covering Lemma is dependent on the fact that if two rectangles, $Q_1$ and $Q_2$ overlap and $Q_2$ is at most scaled by 2 compared to $Q_1$, then $5Q_1 \supset Q_2$. This directly leads to the 5 in the constant $5^{-n}$. This result is then used to show that the Hardy-Littlewood Maximal Inequality to complete the proof of LDT. For rectangles in the collection $\mathbf{R}^*$ we have, by Proposition 2.9:

$$(2cr^* + 1)R_1 \supset R_2$$

if $R_1, R_2$ are nondisjoint rectangles with the same ratio between the edges. In the proof of the Vitali Covering Lemma, cubes are recursively chosen such that each cubes is at least 1/2 of the supremum of lengths. Hence if we similarly choose rectangles $R_i$ such that

$$\ell(R_i) \leq \frac{1}{2}\sup\{\ell(R_\lambda)\}$$

Where $R_\lambda$ is any element of some covering $K_1$ of some set $E$. In this case, $R_\lambda$ is scaled by at most 2 compared with $R_j$. By Proposition 2.9, we have

$$(2 \cdot 2 \cdot r^* + 1)R_j = (4r^* + 1)R_j \supset R_\lambda$$

By invoking the modified Hardy-Littlewood Inequality, this gives us the following:

$$|\{x \in \mathbb{R}^n : |Mf(x)| > \alpha\}| \leq \frac{1}{(4r^* + 1)^n \alpha} \int_{\mathbb{R}^n} |f(x)|dx.$$

The general idea of the following proof is to partition $\mathbb{R}^n$ into $B$ and $\mathbb{R}^n \backslash B$. It is sufficient to show that the measure of $\mathbb{R}^n \backslash B$ is 0 and that $f$ is differentiable everywhere in $B$.

Let $\varepsilon > 0$. Then there exists by a continuous $C_k$ for some $k \in \mathbb{Z}^+$ such that

$$||f - C_k||_1 < \frac{\varepsilon}{[(4r^* + 1)^n + 1]k2^k}.$$

where $n$ is equivalent to the $n$ in $\mathbb{R}^n$. Then let

$$B_k = \left\{ b \in \mathbb{R}^n : |f(b) - C_k(b)| \leq \frac{1}{k} \text{ and } M(f - C_k)[b] \leq \frac{1}{k} \right\}.$$

Then,

$$\begin{aligned} \mathbb{R}^n \backslash B_k &= \left\{ b \in \mathbb{R}^n : |f(b) - C_k(b)| > \tfrac{1}{k} \right\} \\ &\cup \left\{ b \in \mathbb{R}^n : M(f - C_k)(b) > \tfrac{1}{k} \right\}. \end{aligned} \tag{2.3}$$

The equation above gives a central idea for the proof of LDT. Note that the RHS of $\mathbb{R}^n \backslash B_k$ is a union of 2 sets. We apply Markov's inequality to the set on the left side of the union:

$$\begin{aligned} \left| \left\{ b : |f(b) - C_k(b)| > \frac{1}{k} \right\} \right| &\leq k \textstyle\int_{\mathbb{R}^n} |f(x) - C_k(x)|dx \\ &= k||f - C_k||_1 \\ &< \tfrac{\varepsilon}{((4r^*+1)^n+1)2^k}. \end{aligned}$$

For the second set, we apply the modified Hardy-Littlewood inequality and obtain:

$$\left|\left\{b : M(f - C_k)(b) > \frac{1}{k}\right\}\right| \leq \frac{(4r^* + 1)^n}{k} ||f - C_k||_1 < \frac{(4r^* + 1)^n \varepsilon}{((4r^* + 1)^n + 1)2^k}.$$

Combining the two inequalities above with the identity given in Equation 2.3, we then have

$$|\mathbb{R}^n \backslash B_k| < (4r^* + 1)^n \frac{\varepsilon}{[(4r^* + 1)^n + 1]2^k} + \frac{\varepsilon}{[(4r^* + 1]^n + 1)2^k} = \frac{\varepsilon}{2^k}.$$

Now, let $B = \bigcap_{k=1}^{\infty} B_k$. Then we have

$$|\mathbb{R}^n \backslash B| = \left|\bigcup_{k=1}^{\infty} \mathbb{R}^n \backslash B_k\right| \leq \sum_{k=1}^{\infty} |\mathbb{R}^n \backslash B_k| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Since for every $\varepsilon$, we may select a continuous $C_k$ such that $|\mathbb{R}^n \backslash B| < \varepsilon$, we have $|\mathbb{R}^n \backslash B| = 0$ as a result, which completes the first part of the proof. Now, it is sufficient to show that $f$ is differentiable in the set $B$. Suppose $b \in B$. Then, for all $k \in \mathbb{Z}^+$, there exists a continuous $C_k$ such that:

$$\begin{aligned} \frac{1}{|R|} \int_R |f - f(b)| &\leq \tfrac{1}{|R|} \textstyle\int_R (|f - C_k| + |C_k - C_k(b)| + |C_k(b) - f(b)|) \\ &\leq \tfrac{1}{k} + \tfrac{1}{|R|} \textstyle\int_R |C_k - C_k(b)| + \tfrac{1}{k}. \\ &= \tfrac{1}{|R|} \textstyle\int_R |C_k - C_k(b)| + \tfrac{2}{k}. \end{aligned}$$

Hence, we can select a sequence of $C_k$ where as $k \to \infty$, we have

$$\begin{aligned} \frac{1}{|R|} \int_R |f - f(b)| &\leq \tfrac{1}{|R|} \textstyle\int_R |C_k - C_k(b)| + \tfrac{2}{k} \\ &= \tfrac{1}{|R|} \textstyle\int_R |C_k - C_k(b)|. \end{aligned}$$

By the inequality above, we have, for the family of cubes $\{R_b\}$ where $R_b \subset B$:

$$\lim_{R_b \to b} \frac{1}{|R_b|} \int_{R_b} |f - f(b)| \leq \lim_{R_b \to b} \frac{1}{|R_b|} \int_{R_b} |C_k - C_k(b)| = 0.$$

Where the above follows from the definition of continuous functions. □

For the Vitali Covering Lemma, if rotations are forbidden, then it can be shown that the $r^*$ factor can be set to 1, which gives us the familiar constant 5. As another example, if we consider cubes up to rotation, the constant becomes

$$4r^* + 1 = 4\sqrt{2} + 1.$$

## 3. Abstract differentiation bases

Whereas the section above deals with a specific case of differentiation bases that has a clear geometric definition, there is a wide range of literature that deal with differentiation bases in abstract. De Possel's work [5] on the differentiation bases problem concerns the following property:

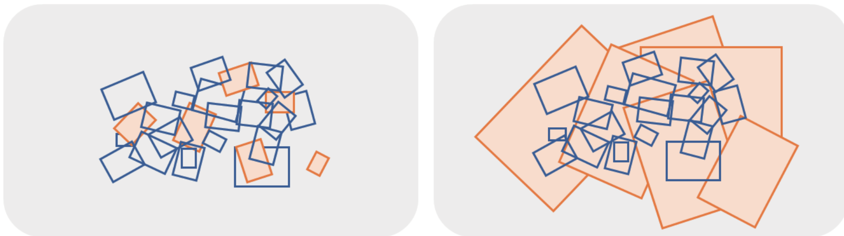

FIGURE 3. An illustration of the modified Vitali Covering Lemma.

**Definition 3.1** (Vitali Covering Property)**.** Given a measurable set $E$ in $\mathbb{R}^n$ where $V \subset \beta$, we say that $V$ is an $\beta$**-Vitali Covering** of $E$ if for every $x \in E$, there exists a sequence $\{\gamma_k\} \subset V$ such that $\gamma_k \in \beta(x)$ for each $k$ and $\gamma_k \to x$ as $k \to \infty$. A basis $\beta$ has the **Vitali covering property** $V_q$ if for each bounded measurable set $E$, for each $\varepsilon > 0$ and each $\beta$-Vitali Covering $V$ of $E$, there exists a sequence $\{\gamma_k\} \subset V$ such that

(1) $|E \backslash \bigcup_k \gamma_k| = 0$
(2) $|\bigcup_k \gamma_k \backslash E| < \varepsilon$
(3) $\left\| \sum_k \chi_{\gamma_k} - \chi_{\bigcup \gamma_k} \right\|_q < \varepsilon.$

For a differentiation basis $\beta$, de Possel [5] proved that the following two statements are equivalent:

(1) $\beta$ Differentiates $\int f$ for all $f \in L^\infty_{loc}(\mathbb{R}^n)$, which is the space of all measurable functions which is essentially bounded on every compact set.
(2) $\beta$ has the property $V_1$.

Based on the above theorem, Hayes and Pauc posed in 1955 the problem of whether the following statements are equivalent [10]:

$$\beta \text{ differentiates } \int f \text{ for every} f \in L^p_{loc}(\mathbb{R}^n) \text{where} 1 < p < \infty. \tag{3.1}$$

$$\beta \text{ has the Vitali covering property} V_q, 1/p + 1/q = 1. \tag{3.2}$$

This is a question attracted much attention. It is known that statement (3.2) implies statement (3.1) . De Guzman [6] showed instead that statement (3.1) implies the following statement:

$$\beta \text{ has the Vitali covering property } V_{q_1} \text{ for all } q_1 < q.$$

In 1976, Cordoba [4] obtained (3.1) $\implies$ (3.2) under the assumption that the basis $\beta$ is translation invariant. It is shown in this section that (3.1) $\implies$ (3.2) for any Busemann-Feller Basis $\beta$ with the Vitali Covering Property. We first state our main theorem below:

**Theorem 3.2.** *Let $R$ be a Busemann-Feller basis in $\mathbb{R}^n$. Then, $R$ differentiates $\int f$ for every $f \in L^p_{loc}(\mathbb{R}^n)$ if and only if $R$ has the covering property $V_q$ where $1/p + 1/q = 1$ and $1 < p < \infty$.*

We begin by stating several lemmas. Let $V = \{\gamma_k\}$ be a countably subfamily of $R$ and let

$$\phi_V(x) = \sum_{k=1}^{\infty} \chi_{\gamma_k}(x) \text{ ,and}$$
$$\psi_V(x) = [\phi_V(x) - 1]\chi_{\bigcup \gamma_k}$$

Then we have the following lemma:

**Lemma 3.3.** *If $V$ is a countable subfamily of $R$, then*

$$0 \le \int_{\mathbb{R}^n} \phi_V^q(x)dx \le 2^q \int_{\mathbb{R}^n} \psi_V^q(x)dx + \left|\bigcup_k \gamma_k\right|$$

*Proof.* Let $A = \{x : \phi_V(x) = 1\}, B = \{x : \phi_V(x) \ge 2\}$. Clearly, $A \cup B = \bigcup_k^{\infty} \gamma_k$ and for $x \in B$, $\phi_V(x) = \psi_V(x) + 1 \le 2\psi_V(x)$. Therefore

$$\begin{aligned} 0 \le \int_{\mathbb{R}^n} \phi_V^q(x)dx &= \int_B \phi_V^q(x)dx + \int_A \phi_V^q(x)dx \\ &\le 2^q \int_B \psi_V^q(x)dx + |A| \\ &\le 2^q \int_{\mathbb{R}^n} \psi_V^q(x)dx + \left|\bigcup_k^{\infty} \gamma_k\right|. \end{aligned}$$

□

**Lemma 3.4.** *Let $V$ be a countable subfamily of $R$ for which $\int_{\mathbb{R}^n} \phi_V^q(x)dx$ is finite. Then, if $W$ is an element in $R$ and $\beta = V \cup \{W\}$, then we have*

$$0 \le \int_{\mathbb{R}^n} \psi_\beta^q(x)dx \le \int_{\mathbb{R}^n} \psi_V^q(x)dx + q\int_W \phi_V^{q-1}(x)dx.$$

*Proof.* Let $A = \bigcup_{k=1}^{\infty} \gamma_k$. We make three observations:

(1) $\psi_\beta(x) = \psi_V(x)$ if $x \in (A\backslash W)$
(2) $\psi_\beta(x) = 0$ if $x \in (W\backslash A)$
(3) $\psi_\beta = \phi_V(x)$ if $x \in A \cap W$

And hence, we have

$$\begin{aligned} 0 \le \int_{\mathbb{R}^n} \psi_\beta^q(x)dx &= \int_\beta \psi^q(x)dx \\ &= \int_{A\backslash W} \psi_\beta^q(x)dx + \int_{W\backslash A} \psi_\beta^q(x)dx + \int_{A\cap W} \psi_\beta^q(x)dx \\ &= \int_{A\backslash W} \psi_V^q(x)dx + \int_{A\cap W} \psi_V^q(x)dx \end{aligned}$$

$$= \int_A \psi_V^q x)dx - \int_{A\cap W} \psi_V^q(x)dx + \int_{A\cap W} \phi_V^q(x)dx$$
$$= \int_A \psi_V^q(x)dx + \int_{A\cap W} [\phi_V^q(x) - \psi_V^q(x)]dx$$
$$\leq \int_A \psi_V^q(x)dx + \int_W [\phi_V^q(x) - \psi_V^q(x)]dx$$
$$\leq \int_{\mathbb{R}^n} \psi_V^q(x)dx + q\int_W \phi_V^{q-1}(x)dx$$

The last inequality follows from the fact that $(x+1)^q \leq x^1 + q(x+1)^{q-1}$ for any $x \geq 0$, which is a consequence of the mean value theorem. □

**Lemma 3.5.** *Let $E \subset \mathbb{R}^n$ with $|E| < \infty$ and $X \supset E$ with $|X| < \infty$. Let $\tilde{V}$ be an R-Vitali covering of $E$ that differentiates $\int f$ for all $f \in L^p_{loc}(\mathbb{R}^n)$. Then, if $0 < \varepsilon < 1$ and $V = \{\gamma_k\}$ is a countable subfamily of $\tilde{V}$ satisfying the following conditions, where we denote $A = \gamma_k$:*

$$\int_{\mathbb{R}^n} \psi_V^q(x)dx \leq \varepsilon|X \cap A| \tag{3.3}$$

$$(1-\varepsilon)\sum_{\gamma_k \in V} |\gamma_k| < |X \cap A| \tag{3.4}$$

$$|X \backslash A| > 0 \tag{3.5}$$

*Then, there exists a set $W$ depending on $\varepsilon$ such that*

$$W \in \tilde{V} \text{ and } \int_W \phi_V^{q-1}(x)dx + |W - A| \leq \frac{\varepsilon}{2q}|W| \tag{3.6}$$

*Proof.* We begin with the proof of (3.6). By the assumption in (3.3) and (3.4) and the finiteness of $|X|$, we know that $\int_{\mathbb{R}^n} \psi_V^q(x)dx$ and $|A|$ are also finite. Then, by Lemma 3.3, $\int_{\mathbb{R}^n} \phi_V^q(x)dx < \infty$, that is, $\phi_v \in L^Q(\mathbb{R}^n)$. Thus, $R$ differentiates the integral of $\phi_v^{q-1}(x)$. By the hypothesis, $R$ also differentiates the integral of the characteristic function of $X^C = \mathbb{R}^n \backslash X$. Thus, as $k \to \infty$, we have, for every $y \in X$

$$\frac{1}{|\gamma_k(y)|}\int_{\gamma_k(y)} \phi_V^{q-1}(x)dx + \frac{|\gamma_k(y) - X|}{|\gamma_k(y)|} \to \phi_V^{q-1}(y) + \chi_{X^C}(y).$$

Since $|X\backslash A| > 0$, there exists a $y \in X\backslash A$, so we can take $y \in W \in \tilde{V}$ such that:

$$\frac{1}{|W|}\int_W \phi)V^{q-1}(x)dx + \frac{|W - X|}{|W|} \leq \frac{\varepsilon}{2q}$$

which gives us

$$\int_W \phi_V^{q-1}(x)dx + |W - X| \leq \frac{\varepsilon}{2q}|W|.$$

□

*Moreover, if* $W$ *is any set satisfying* (3.3) – (3.6)*, then we have*

$$\int_{\mathbb{R}^n} \psi_\beta^q(x)dx \le \varepsilon|X \cap \tilde{\beta}| \tag{3.7}$$

$$(1-\varepsilon)\sum_{B\in\beta}|B| < |X \cap \tilde{\beta}| \tag{3.8}$$

*where* $\beta = V \cup \{W\}$ *and* $\tilde{\beta} = (\bigcup \gamma_k) \cup W = A \cup W$.

*Proof.* Now we prove statements (3.7) and (3.8). Consider an arbitrary set $W$ satisfying condition (3.6). Observe that

$$\begin{aligned}|W \cap (X \cap A^C)^C| = |W \cap (X^C \cup A)| &= |(W \cap X^C) \cup (W \cap A)| \\ &\le |W \cap X^C| + |W \cap A| \\ &\le |W \cap X^C| + \int_W \phi_V^{q-1}(x)dx \le \frac{\varepsilon}{2q}|W|.\end{aligned}$$

Hence, we have

$$\begin{aligned}|W| &\le |W \cap (X\backslash A)| + |W\backslash(X\backslash A)| \\ &\le |W \cap (X\backslash A)| + \frac{\varepsilon}{2q}|W|\end{aligned}$$

Which gives us the following inequalities:

$$(1-\frac{\varepsilon}{2q})|W| \le |W \cap (X\backslash A)| \tag{3.9}$$

$$|W| \le 2|W \cap (X\backslash A)| \tag{3.10}$$

From condition (3.6) and inequality 3.10 we obtain

$$\begin{aligned}\int_W \phi_V^{q-1}(x)dx &\le \int_W \phi_V^{q-1}(x)dx + |W\backslash X| \\ &\le \frac{\varepsilon}{2q}|W| \\ &\le \frac{\varepsilon}{q}|W \cap (X\backslash A)|\end{aligned}$$

Which gives us the third inequality

$$\int_W \phi_V^{q-1}(x)dx \le \frac{\varepsilon}{q}|W \cap (X\backslash A)| \tag{3.11}$$

From (3.3), 3.11, and Lemma 3.4, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \psi_\beta^q(x)dx &\le \int_{\mathbb{R}^n} \psi)V^q(x)dx + q\int_W \phi_V^{q-1}(x)dx \\ &\le \varepsilon[|X \cap A| + |W \cap (X\backslash A)|]\end{aligned}$$

Which establishes statement (3.7). Now, from statement (3.4) and Inequality , we obtain

$$(1-\varepsilon)\sum_{B\in\beta}|B| \le (1-\varepsilon)\sum_{\gamma_k\in V}|\gamma_k| + (1-\varepsilon)|W|$$

$$\leq |X \cap A| + (1 - \frac{\varepsilon}{2q})|W|$$
$$\leq |X \cap A| + |W \cap (X - A)| = |X \cap \tilde{B}|$$

which gives us the statement (3.8) of this lemma. □

Now we begin the proof of the main theorem. Recall that it is sufficient to show that statement in (3.2) implies (3.1). Let $E$ be a bounded measurable set in $\mathbb{R}^n$. Then, let $X$ be an open subset of $\mathbb{R}^n$ containing $E$ with $|X| < \infty$ with $|X \backslash E| < \varepsilon$, where $\varepsilon$ satisfies $0 < \varepsilon < 1$. Now, let $\tilde{V} = \{\gamma\}$ denote an $R$-Vitali Covering of $E$ in which every element $\gamma$ of $\tilde{V}$ is contained in $X$.
Because $R$ differentiates the integral of the characteristic function of the set complement $X^C$, there exists at least one point $y \in E$ for which

$$D\mu(y) = \chi_{X^C}(Y) = 0$$

where $\mu(\gamma) = \int_\gamma \chi_{X^C}(x)dx = |\gamma \cap X^C| = |\gamma \backslash X|$ for each element $\gamma \in R$. Thus, since $\tilde{V}$ is a Vitali covering of $E$, there must be at least one set $W \in \tilde{V}$ such that

$$|W \backslash X| \leq \frac{\varepsilon}{2q}|W|. \tag{3.12}$$

Then, let $C_1$ denote the family of sets $W \in \tilde{V}$ that satisfy the relation (3.12). This $C_1$ is nonempty, and further, it follows from (3.12) that if we have $W \in C_1$, then

$$(1 - \varepsilon)|W| \leq \frac{\varepsilon}{2q}|W|$$

Thus, if we set $\xi_1 = \sup_{w \in C_1} |W|$, it follows that $0 < \xi_1 < \infty$. If we choose a member $\gamma_1$ of $C_1$ with $|\gamma_1| > \xi_1/2$ and set $V_1 = \{\gamma_1\}, A_1 = \bigcup \gamma_1$ then $V_1$ satisfies the conditions (3.3) and (3.4) of Lemma 3.5:

(1) $\int_{\mathbb{R}^n} \psi^q_{V_1}(x)dx = \int_{\mathbb{R}^n} \psi^1_{\gamma_1} = - \leq \varepsilon|X \cap \gamma|$.
(2) $(1 - \varepsilon)\sum_{\gamma_k \in V_1} |\gamma_k| = (1 - \varepsilon)|\gamma_1| < |\gamma_1| < |X \cap \gamma_1|$

Then, we proceed inductively. Suppose that $k > 1$ and we have a family of sets $V_k = \{\gamma_1, \gamma_2, ..., \gamma_k\} \subset \tilde{V}$ that satisfies the conditions (3.3) and (3.4) of Lemma 3.5. Then let

$$A_k = \bigcup_{i=1}^{k} \gamma_i.$$

In the case where $|X \backslash A_k| = 0$, define $V_{\gamma+1} = V_k$ and $\bigcup_{i=1}^{k+1} \gamma_i = A_{k+1} = A_k$. Observe that $V_{k+1}$ satisfies the conditions (3.3) and (3.4) of Lemma 3.5 since they hold for $V_k$.

Now, we consider the cases of $|X \backslash A_k| > 0$. Let $C_{k+1}$ be the family of the sets $W \in \tilde{V}$ satisfying the relation

$$\int_W \phi^{q-1}_{V_k}(x)dx + |W \backslash A| \leq \frac{\varepsilon}{2q}|W| \tag{3.13}$$

By Lemma 3.5, $C_{k+q} \neq \emptyset$. From the inequality (3.13), it follows that $(1 - \frac{\varepsilon}{2q})|W| < |W \cap X|$, and hence $|W| \leq 2|W \cap X|$ whenever $W \in C_{k+1}$. Thus if we set $\xi_{k+1} = \sup_{W \in C_{k+1}} |W|$, it follows that $0 < \xi_{k+1} < \infty$. Now we select a member $\gamma_{k+1}$ of $C_{k+1}$ such that $|\gamma_{k+1}| > \frac{1}{2}\xi_{k+1}$ and we define $V_{k+1} = V_k \cup \{\gamma_{k+1}\}$ and $A_{k+1} = \bigcup_{i=1}^{k+1} \gamma_i$. By Lemma 3.4, we have

$$\int_{\mathbb{R}^n} \psi^q_{V_{k+1}}(x)dx \leq \varepsilon|X \cap A_{k+1}$$

and furthermore

$$(1-\varepsilon)\left(\sum_{\gamma_i \in V_{k+1}} |\gamma_i|\right) \leq |X \cap A_{k+1}| \tag{3.14}$$

Thus, in either case, we obtain a family $V_{k+1} \subset \tilde{V}$ that satisfies the relation (3.14). In this way, we obtain through induction a sequence $\{V_k\}$ of finite subfamilies of $\tilde{V}$ that satisfies (3.14). Now, we let $V = \bigcup_{k=1}^{\infty} V_k$, $A = \bigcup_{\gamma_k \in V} \gamma_k$. The monotone convergence theorem applied to (3.14) gives

$$\int_{\mathbb{R}^n} \psi^q_v(x)dx \leq \varepsilon|X \cap A| \leq \varepsilon|X| < \infty$$

and

$$(1-\varepsilon)|A| \leq (1-\varepsilon)\left(\sum_{\gamma_i \in V_{k+1}} |\gamma_i|\right) \leq |X \cap A| \leq |X| < \infty. \tag{3.15}$$

from the above, it follows that

$$|A \backslash X| \leq \varepsilon|A| \leq \frac{\varepsilon}{1-\varepsilon}|X| < \infty \tag{3.16}$$

Because this $\varepsilon$ is arbitrary for $0 < \varepsilon < 1$, it follows from statements (3.15) and (3.16) that $V$ can be chosen to satisfy statements 2 and 3 in Definition 3.1 of the Vitali Covering Property $V_q$.

Now, it remains to be shown that $V$ covers almost all of $E$. Assume for contradiction that $|E \backslash A| > 0$, which implies that $|X \backslash A| > 0$. This implies that $|X \backslash A_k| \geq |X \backslash A| > 0$ for $k = 0, 1, 2...$, which means that the inductive process does not stop producing new sets, and so $V$ consists of a countably infinite family of sets $\{\gamma_1, \gamma_2, ..., \gamma_k, ...\}$ chosen from $V$. The conditions (3.3), (3.4), and (3.5) of Lemma 3.5 are satisfied by $V$, and hence, by the same lemma, there exists a set $W \in \tilde{V}$ such that

$$\int_W \phi^{q-1}_v(x)dx + |W \backslash X| \leq \frac{\varepsilon}{2q}|W| \tag{3.17}$$

From (3.17) and the fact that $X_{V_k} \to X_V$ as $k \to \infty$, it follows that

$$\int_W \phi)V_k{}^{q-1}(x)dx + |W \backslash X| \leq \frac{\varepsilon}{2q}|W|$$

for each positive integer $k$, and hence, $w \in C_{k+1}$ for each such $k$. Hence for each $k$ we have $0 < |W| \leq \xi_{k+1} < 2|\gamma_{k+1}|$. However, from (3.15) we have

$$\sum_{\gamma_k \in V} |\gamma_k| = \sum_{i=1}^{\infty} |\gamma_k| \leq \frac{|X|}{1-\varepsilon} < \infty$$

which implies that $|\gamma_{k+1}| \to 0$ as $k \to \infty$, which contradicts the conclusion that $|\gamma_k| > 0$ for all positive integers $k$, and hence we have $|X\backslash A| = 0 \to |E\backslash A| = 0$, which completes the proof of the theorem.

**Corollary 3.6.** *If $f \in L^p_{loc}(\mathbb{R}^n), 1 < p < \infty$, then*

$$\lim_{\substack{\gamma\to x\\ \gamma\in R}} \frac{1}{|\gamma|}\int_\gamma |f(y) - f(x)|dy = 0 \text{ a.e. for } x \in \mathbb{R}^n$$

*If and only if $R$ has the Vitali covering property $V_q$, $1/p + 1/q = 1$.*

*Proof.* Note that for any $\gamma \in R$ we have

$$\frac{1}{|\gamma|}\int_\gamma f(y)dy - f(x) = \frac{1}{|\gamma|}\int_\gamma [f(y) - f(x)]dy \le \int_\gamma |f(y) - f(x)|dy$$

hence, by the assumption state above we have

$$\limsup_{\substack{\gamma\to x\\ \gamma\in R}} \frac{1}{|\gamma|}\int_\gamma [f(y) - f(x)]dy \le \limsup_{\substack{\gamma\to x\\ \gamma\in R}} \frac{1}{|\gamma|}\int_\gamma |f(y) - f(x)|dy = 0$$

a.e. for $x \in \mathbb{R}^n$ then we know that $R$ differentiates $\int f$ for every $f \in L^p_{loc}(\mathbb{R}^n), 1 < p < \infty$. By the main Theorem 3.2 it follows that $R$ has Vitali covering property $V_q$, $1/p + 1/q = 1$.

For the other direction, suppose $f \in L^p_{loc}(\mathbb{R}^n)$. Then $|f(x) - c| \le |f(x)| + |c|$ where $c$ is an arbitrary complex number, and

$$\int_k |f(x) - c|^p dx \le 2^{p-1}\int_k |f(x)|^p dx + 2^{p-1}|c|^p|k| < \infty$$

where $K$ is a compact set. If follows that $|f(x) - c| \in L^p_{loc}(\mathbb{R}^n)$.

Now let $\{r_k\}$ be a set of complex numbers with rational real and imaginary parts. Let $z_k$ be the set of all $x \in \mathbb{R}^n$ such that

$$\lim_{\substack{\gamma\to x\\ \gamma\in R}} \frac{1}{|\gamma|}\int_\gamma |f(y) - r_k|dy \ne |f(x) - r_k|.$$

By the main theorem we know that $z_k$ has measure zero. Define $Z = \bigcup Z_k$. It follows that $|Z| = 0$ and we have

$$\begin{aligned}\frac{1}{|\gamma|}\int_\gamma |f(y) - f(x)|dy &\le \frac{1}{|\gamma|}\int_\gamma |f(y) - r_k|dy + \frac{1}{|\gamma|}\int_\gamma |f(x) - r_k|dy\\ &= \frac{1}{|\gamma|}\int_\gamma |f(y) - r_k|dy + |f(x) - r_k|\end{aligned}$$

However, if $x \notin Z$, then for every $\varepsilon > 0$ we can choose $r_k$ such that $|f(x) - r_k| < \varepsilon/2$. In this case

$$\limsup_{\substack{\gamma\to x\\ \gamma\in R}} \frac{1}{|\gamma|}\int_\gamma |f(y) - f(x)|dy \le 2|f(x) - r_k| < \varepsilon$$

hence the result follows. □

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