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NEW ALGORITHM FOR NON-MONOTONE AND NON-LIPSCHITZ EQUILIBRIUM PROBLEM OVER FIXED POINT SET OF A QUASI-NONEXPANSIVE NON-SELF MAPPING IN HILBERT SPACE

YITONG SHI, ZIQI ZHU, AND YUEYAO ZHANG

Abstract. In this paper, we propose a new projection algorithm for solving an equilibrium problem over the fixed point set of a quasi-nonexpansive non-self mapping in Hilbert spaces. The bifunction involved in the equilibrium problem is not required to satisfy any monotone or Lipschitz continuous property. We prove the weak convergence for the proposed algorithm. A numerical example is given to illustrate the effectiveness of our algorithm.

1. Introduction

Let C is a nonempty closed convex subset in a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for all $x \in C$. Consider the following equilibrium problem: find a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C, \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $EP(f, C)$.

The equilibrium problem was initially introduced by Nikaido and Isoda [18] for studying the Nash equilibrium problem and has been applied to a large of theoretical and practical problems such as complementarity problems, optimization problems, feasibility problems, variational inequality problems, and fixed point problems, signal recovery problems, image processing problems, finance problem and so on [6,17].

The bifunction f is said to be Lipschitz-type continuous [19] on C if

$$f(x, y) + f(y, z) \geq f(x, z) - c_1(x - y)^2 - c_2(y - z)^2, \ \forall x, y, z \in C.$$

For solving the problem (1.1) with the Lipschitz-type continuous bifunction f, Quoc et al [21] proposed the following extragradient method in Euclidean spaces: $x_0 \in C$ and

$$\begin{cases} y_n = \operatorname{argmin} \{\rho f(x_n, y) + G(x_n, y) : y \in C\}, \\ x_{n+1} = \operatorname{argmin} \{\rho f(y_n, y) + G(x_n, y) : y \in C\}, \ n \geq 0, \end{cases} \tag{1.2}$$

where ρ is a suitable parameter and $G(x, y)$ is a given Bregman distance function defined on $C \times C$. The authors proved that the sequence $\{x_n\}$ obtained by (1.2)

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converges to a point $x^* \in EP(f,C)$. In recent years, the various modified extragradient methods for solving the problem (1.1) involved the Lipschitz-type continuous bifunction have been intensively studied and extended to Hilbert spaces; see, e.g., [3, 9–13, 15, 16, 22, 25].

The methods in the above mentioned literature are proposed for solving the equilibirum problems with Lipschitz-type continuous bifunctions. However, when the Lipschitz constants c_1 and c_2 may be unknown or hard to compute, the methods can be applied. For overcoming the difficulty, some authors investigated the equilibrium problems with non-Lipschitz-type continuous bifunctions. For example, Santos and Scheimberg [23] proposed the following inexact projected subgradient method :

$$\begin{cases} x_0 \in C, \\ v_k \in \partial f(x_k, \cdot)(x_k), \\ x_{k+1} = P_C\left(x_k - \dfrac{\beta_k}{\max\{\rho_k, \|v_k\|\}} v_k\right), \end{cases} \tag{1.3}$$

where $\partial f(x,\cdot)(x)$ is the subdiffential at x, $\{\beta_k\} \subset (0,1)$ satisfies the conditions that $\sum_{k=0}^{\infty} \beta_k = \infty$ and $\sum_{k=0}^{\infty} \beta_k^2 < \infty$, and f is a pseudomonotone and non-Lipschitz-type continuous bifunction. The authors proved that the sequence $\{x_k\}$ generate by (1.3) converges weakly to the solution of the problem (1.1). On other methods for solving the equilibrium problems with non-Lipschitz-type continuous bifunctions, the readers may refer to [2, 8].

Another interesting problem is to find a common element of set of solutions of the equilibrium problem and set of fixed points of a nonlinear mapping. Anh [1] proposed the following iterative algorithm: $x_0 \in C$ and

$$\begin{cases} y_n = \operatorname{argmin}\{\lambda_n f(x_n, y) + \dfrac{1}{2}\|y - x_n\|^2 : y \in C\}, \\ t_n = \operatorname{argmin}\{\lambda_n f(y_n, y) + \dfrac{1}{2}\|y - x_n\|^2 : y \in C\}, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) T t_n, \end{cases} \tag{1.4}$$

where $T : C \to C$ is a nonexpansive mapping and $\{\alpha_n\}, \{\lambda_n\} \subset (0,1)$. The author proved the strong convergence of $\{x_n\}$ generated by (1.4) provided $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and some other assumptions on $\{\alpha_n\}$ and $\{\lambda_n\}$. For the more methods on equilibrium problems and fixed points problems, the interested readers many refer to [24].

In this paper, inspired by the method (1.3) of Santos and Scheimberg [23], we introduce a method for solving a non-monotone and non-Lipschitz equilibrium problem over the set of fixed point problem of a quasi-nonexpansive non-self mapping in Hilbert space. Under some mixed conditions, the weak convergence is proved for the proposed algorithm. Finally, a numerical example is given to illustrate our algorithm.

2. Preliminaries and notation

Let C be a nonempty closed convex subset of a real Hilbert space H. For a sequence $\{x_k\} \subset C$, we use $x_k \rightharpoonup x$ to denote that $\{x_k\}$ converges weakly to x as $k \to \infty$. Let $T : C \to H$ be a mapping and $Fix(T)$ denote the set of fixed points of T. Let I be the identity mapping on H. The mapping $T : C \to H$ is said to be nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C$$

and quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \ \forall x \in C, \ \forall y \in Fix(T).$$

The mapping T is said to be demiclosed demiclosed at 0 if $x_k \rightharpoonup x$ with $\{x_k\} \subset C$ and $x \in H$ and $(I - T)x_k \to 0$ implies that $x = Tx$. It is known that $Fix(T)$ is closed and convex if $T : C \to H$ is a quasi-nonexpansive mapping and $Fix(T) \neq \emptyset$; see [4].

Let $g : C \times C \to \mathbb{R}$ be a convex function. For each $x \in C$, by $\partial g(x)$ we denote the subdifferential of the function $g(\cdot)$ at x, i.e.,

$$\partial g(x) = \{w \in H : g(y) - g(x) \geq \langle y - x, w\rangle, \ \forall y \in C\}.$$

If $\partial g(\cdot)$ at $x \in C$ is nonempty, g is said to be subdifferentiable at x. If $\partial g(\cdot)$ at every $x \in C$ is nonempty, g is said to be subdifferentiable on C.

For any $x \in H$, there exists a unique element $z \in C$, denoted by $P_C(x)$, such that $\|z - x\| = \inf_{y \in C} \|y - x\|$. The mapping $P_C : H \to C$ is called a metric projection from H onto C. It is known that P_C is nonexpansive. The following lemma characterizes the part properties of P_C.

Lemma 2.1 ([7]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $x \in H$, the following hold:*

(i) *$z = P_C(x)$ if and only if $\langle x - z, z - y\rangle \geq 0, \ \forall y \in C$;*
(ii) *$\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2, \forall y \in C$.*

Lemma 2.2 ([20]). *Let $\{a_k\}$ and $\{b_k\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{k+1} \leq a_k + b_k, \ \forall k \geq 1,$$

where $\sum b_k < \infty$. Then $\lim_{k\to\infty} a_k$ exists.

Lemma 2.3 ([5]). *Let D be a nonempty set of H and $\{x_k\}$ be a sequence in H such that the following two conditions hold:*

(i) *for all $x \in D$, $\lim_{k\to\infty} \|x_k - x\|$ exists;*
(ii) *every sequential weak cluster point of $\{x_k\}$ is in D.*

Then the sequence $\{x_k\}$ converges weakly to a point in D.

3. Main results

In this section, let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for all $x \in C$ and $T : C \to H$ be a quasi-nonexpansive mapping. We consider the following problem: find a point $\bar{x} \in Fix(T)$ such that

$$(\Gamma) \qquad f(\bar{x}, y) \geq 0, \ \forall y \in Fix(T).$$

Denote the set of solutions of the above problem by Ω.

Throughout this section, we assume that the following conditions hold:

(A1) $\Lambda \neq \emptyset$, where $\Lambda = Fix(T) \cap \{z \in C : f(y,z) \leq 0,\ \forall y \in C\}$.

(A2) $I - T$ is demiclosed at 0.

(A3) for each $x \in C$, $f(x,\cdot)$ is convex and subdifferentialbe on C, and the operator $\partial f(x,\cdot)(x)$ is bounded on the bounded subsets of C.

(A4) $f(\cdot,y)$ is weakly upper semicontinuous on C for each $y \in C$.

We present the following algorithm to approximate a solution of the problem (Γ).

Algorithm 3.1. Initialization Choose the initial point $x_1 \in C$ and the sequences $\{\beta_k\} \subset [0,\beta']$ with $\beta' < 1$, $\{\gamma_k\} \subset (0,1)$ and $\{\alpha_k\} \subset (0,+\infty)$ satisfying

$$\sum_{k=1}^{\infty} \alpha_k \gamma_k = \infty \text{ and } \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \tag{3.1}$$

Set $C_0 = C$ and $k = 1$.

Step 1: For each $k \geq 1$ and the current itertate x_k, compute

$$z_k = \beta_k x_k + (1-\beta_k) T x_k$$

and construct the subset

$$C_k = \{z \in C : \|z_k - z\| \leq \|x_k - z\|\}.$$

Step 2: Compute $w_k \in \partial f(x_k,\cdot)(x_k)$ and set

$$\eta_k = \max\{1, \|w_k\|\}.$$

Compute

$$\begin{cases} y_k = P_{C_{k-1}-x_k}\left(-\dfrac{\alpha_k}{\eta_k} w_k\right), \\ x_{k+1} = P_{C_k}(x_k + \gamma_k y_k). \end{cases}$$

Step 3: If $w_k = 0, x_k = Tx_k$ holds or $y_k = 0, x_k = Tx_k$ holds, then the algorithm stops and $x_k \in \Omega$; otherwise, set $k = k+1$ and go to **Step 1**.

The following remark shows that the stopping criterion of Algorithm 3.1 can well work.

Remark 3.1 Assume that $y_k = 0$ and $x_k = Tx_k$ for some $k \in \mathbb{N}$. For any $y \in Fix(T)$, since $y - x_k \in C_{k-1} - x_k$ which will be obtained by the following Lemma 3.1, by the definition of y_k and Lemma 2.1 (i) we have

$$-\frac{\alpha_k}{\eta_k}\langle w_k, x_k - y\rangle \geq 0 \tag{3.2}$$

and hence $\langle w_k, y - x_k\rangle \geq 0$. On the other hand, by the definition of w_k we have

$$\langle w_k, y - x_k\rangle \leq f(x_k, y) - f(x_k, x_k) = f(x_k, y),\ \forall y \in C. \tag{3.3}$$

From (3.2) and (3.3) it follows $f(x_k, y) \geq 0$ for all $y \in Fix(T)$, which still holds when $w_k = 0$. Furthermore, if $x_k = Tx_k$ it follows that $x_k \in Fix(T)$. Therefore, $x_k \in \Omega$.

Assume that the stopping criterion of Algorithm 3.1 does not hold and hence $\{x_k\}$ is an infinite sequence.

Lemma 3.1. *For each $n \in \mathbb{N}$, the set C_k is nonempty closed and convex, and hence the sequences $\{x_k\}$ and $\{y_k\}$ are well defined.*

Proof. From the definition C_k it is obvious that C_k is closed and convex for each $n \in \mathbb{N}$. For any $y \in Fix(T)$, since T is quasi-nonexpansive, we have

$$\|z_k - y\| \leq \beta_k \|x_k - y\| + (1 - \beta_k)\|Tx_k - y\| \leq \|x_k - y\|,$$

which implies that $y \in C_k$. It follows that

$$Fix(T) \subset C_k, \ \forall k \in \mathbb{N}. \tag{3.2}$$

Therefore, the sequences $\{x_k\}$ and $\{y_k\}$ are well defined. This completes the proof. □

Lemma 3.2. *The limit of $\{\|x_k - z\|^2\}$ exists for any $z \in \Lambda$.*

Proof. By the definition of y_k we have

$$\begin{aligned} y_k &= P_{C_{k-1}-x_k}\left(-\frac{\alpha_k}{\eta_k} w_k\right) \\ &= \operatorname{argmin}_{v \in C_{k-1}-x_k} \frac{1}{2}\left\| v - (-\frac{\alpha_k}{\eta_k} w_k) \right\|^2 \\ &= \operatorname{argmin}_{v \in C_{k-1}-x_k} \left\{ \frac{\alpha_k}{\eta_k} \langle w_k, v \rangle + \frac{1}{2}\left(\|v\|^2 + \frac{\alpha_k^2}{\eta_k^2}\|w_k\|^2 \right) \right\} \\ &= \operatorname{argmin}_{v \in C_{k-1}-x_k} \left\{ \frac{\alpha_k}{\eta_k} \langle w_k, v \rangle + \frac{\|v\|^2}{2} \right\}, \ \forall k \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Note that from $x_k \in C_{k-1}$ it follows that $0 \in C_{k-1} - x_k$ for each $k \in \mathbb{N}$. So by (3.3) we get

$$\frac{\alpha_k}{\eta_k} \langle w_k, y_k \rangle + \frac{\|y_k\|^2}{2} \leq 0, \ \forall k \in \mathbb{N}. \tag{3.4}$$

From (3.4) it follows that

$$\begin{aligned} \|y_k\|^2 &\leq -\frac{2\alpha_k}{\eta_k} \langle w_k, y_k \rangle \leq \frac{2\alpha_k}{\eta_k} \|w_k\| \|y_k\| \\ &\leq 2\alpha_k \|y_k\|, \end{aligned}$$

which implies that

$$\|y_k\| \leq 2\alpha_k, \ \forall k \in \mathbb{N}. \tag{3.5}$$

Let $u_k = x_k + \gamma_k y_k$ for each $k \in \mathbb{N}$. By (3.5) it is easy to obtain that

$$\|u_k - x_k\| = \gamma_k \|y_k\| \leq 2\alpha_k, \ \forall k \in \mathbb{N}. \tag{3.6}$$

For any $y \in Fix(T)$, since $y - x_k \in C_{k-1} - x_k$, by Lemma 2.1(i) we have

$$\left\langle -\frac{\alpha_k}{\eta_k} w_k - y_k, y - x_k - y_k \right\rangle \leq 0$$

and hence

$$\langle y_k, x_k - y \rangle \leq -\|y_k\|^2 + \frac{\alpha_k}{\eta_k}\langle w_k, y - x_k \rangle - \frac{\alpha_k}{\eta_k}\langle w_k, y_k \rangle, \ \forall k \in \mathbb{N}. \tag{3.7}$$

By (3.2), (3.5)-(3.7) we get

$$\begin{aligned}
\|x_{k+1}-y\|^2 &= \|P_{C_k}u_k - P_{C_k}y\|^2 \le \|u_k - y\|^2 \\
&= \|u_k - x_k\|^2 + \|x_k - y\|^2 + 2\langle u_k - x_k, x_k - y\rangle \\
&= \|u_k - x_k\|^2 + \|x_k - y\|^2 + 2\gamma_k\langle y_k, x_k - y\rangle \\
&\le 4\alpha_k^2 + \|x_k - y\|^2 + 2\gamma_k\left[\frac{\alpha_k}{\eta_k}\langle w_k, y - x_k\rangle - \frac{\alpha_k}{\eta_k}\langle w_k, y_k\rangle\right] \\
&\le 4\alpha_k^2 + \|x_k - y\|^2 + \frac{2\gamma_k\alpha_k}{\eta_k}\langle w_k, y - x_k\rangle + \frac{2\gamma_k\alpha_k}{\eta_k}\|w_k\|\|y_k\| \\
&\le 4\alpha_k^2 + \|x_k - y\|^2 + \frac{2\gamma_k\alpha_k}{\eta_k}\langle w_k, y - x_k\rangle + 4\gamma_k\alpha_k^2 \\
&\le \|x_k - y\|^2 + \frac{2\gamma_k\alpha_k}{\eta_k}\langle w_k, y - x_k\rangle + 8\alpha_k^2, \forall k \in \mathbb{N}.
\end{aligned} \tag{3.8}$$

Note that for any $z \in \Lambda$, we have $f(x_k, z) \le 0$. Then by the definition of w_k we get

$$\langle w_k, z - x_k\rangle \le f(x_k, z) - f(x_k, x_k) = f(x_k, z) \le 0.$$

Substituting this inequality into (3.8) with $y = z \in \Lambda$ we have

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 8\alpha_k^2, \forall k \in \mathbb{N}. \tag{3.9}$$

Applying Lemma 2.2 and (3.1) to (3.9), we obtain that the limit of $\{\|x_k - z\|\}$ exists. This completes the proof. $\square$

Lemma 3.3. *It holds that* $\lim_{k\to\infty}\|z_k - x_k\| = 0$.

Proof. From Lemma 3.2 it follows that $\{x_k\}$ is bounded. Fix $y \in \Lambda$. Let $M > 0$ such that $\sup_{k\in\mathbb{N}}\|x_k - y\| < M$. By Lemma 2.1 (ii) and (3.6) we have

$$\begin{aligned}
\|x_{k+1}-y\|^2 &= \|P_{C_k}u_k - y\|^2 \\
&\le \|u_k - y\|^2 - \|x_{k+1} - u_k\|^2 \\
&= \|u_k - x_k\|^2 + \|x_k - y\|^2 + 2\langle u_k - x_k, x_k - y\rangle - \|x_{k+1} - u_k\|^2 \\
&\le 4\alpha_k^2 + \|x_k - y\|^2 + 4M\alpha_k - \|x_{k+1} - u_k\|^2
\end{aligned}$$

and hence

$$\|x_{k+1} - u_k\|^2 \le 4\alpha_k^2 + 4M\alpha_k + \|x_k - y\|^2 - \|x_{k+1} - y\|^2,\ \forall k \in \mathbb{N}. \tag{3.10}$$

Note that the limit of $\{\|x_k - y\|^2\}$ exists by Lemma 3.2. Letting $k \to \infty$ in (3.10) , by (3.1) we get

$$\lim_{k\to\infty}\|x_{k+1} - u_k\| = 0. \tag{3.11}$$

Furthermore, from (3.6) and (3.11) it follows that

$$\|x_{k+1}-x_k\| \le \|x_{k+1}-u_k\| + \|u_k - x_k\| \le \|x_{k+1}-u_k\| + 2\alpha_k \to 0,\ \text{as } k \to \infty. \tag{3.12}$$

Since $x_{k+1} \in C_k$, by (3.12) we have

$$\|z_k - x_{k+1}\| \le \|x_k - x_{k+1}\| \to 0,\ \text{as } k \to \infty. \tag{3.13}$$

Combining (3.12) and (3.13) we obtain

$$\|z_k - x_k\| \le \|z_k - x_{k+1}\| + \|x_k - x_{k+1}\| \to 0,\ \text{as } k \to \infty. \tag{3.14}$$

This completes the proof. □

Theorem 3.4. *The sequence $\{x_k\}$ converges weakly to a point $\bar{x}$ in Ω.*

Proof. For any $y \in Fix(T)$, by (3.8) we have

$$\frac{2\alpha_k\gamma_k}{\eta_k}\langle w_k, x_k - y\rangle \leq \|x_k - y\|^2 - \|x_{k+1} - y\|^2 + 8\alpha_k^2,\ \forall k \in \mathbb{N}. \tag{3.15}$$

Adding with k from 1 to l in (3.15), we get

$$\begin{aligned} 2\sum_{k=1}^{l}\frac{\alpha_k\gamma_k}{\eta_k}\langle w_k, x_k - y\rangle &\leq \|x_1 - y\|^2 - \|x_{l+1} - y\|^2 + 8\sum_{k=1}^{l}\alpha_k^2 \\ &\leq \|x_1 - y\|^2 + 8\sum_{k=1}^{l}\alpha_k^2. \end{aligned} \tag{3.16}$$

Letting $l \to \infty$ in (3.16), by (3.1) we obtain

$$2\sum_{k=1}^{\infty}\frac{\alpha_k\gamma_k}{\eta_k}\langle w_k, x_k - y\rangle < \infty. \tag{3.17}$$

Since $\{x_k\}$ is bounded, from (A3) it holds that $\{w_k\}$ is bounded, which leads to $\{\eta_k\}$ is also bounded. So there exists $M' > 0$ such that $\sup_{k\geq 1}\eta_k < M'$. It follows that

$$\sum_{k=1}^{\infty}\frac{\alpha_k\gamma_k}{\eta_k} \geq \sum_{k=1}^{\infty}\frac{\alpha_k\gamma_k}{M'},$$

which together with (3.1) implies that

$$\sum_{k=1}^{\infty}\frac{\alpha_k\gamma_k}{\eta_k} = \infty.$$

This together with (3.17) leads to

$$\liminf_{k\to\infty}\langle w_k, x_k - y\rangle \leq 0. \tag{3.18}$$

By the definition of w_k we have $-f(x_k, y) = f(x_k, x_k) - f(x_k, y) \leq \langle w_k, x_k - y\rangle$. This fact with (3.18) leads to

$$\limsup_{k\to\infty} f(x_k, y) \geq 0. \tag{3.19}$$

Since $\{x_k\}$ is bounded, there exists a subsequence x_{k_j} of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \bar{x}$ with $j \to \infty$. Without loss of generality, we may assume that

$$\limsup_{k\to\infty} f(x_k, y) = \lim_{j\to\infty} f(x_{k_j}, y).$$

Since $f(\cdot, y)$ is weakly upper simicontinuous on C, by (3.19) we get

$$f(\bar{x}, y) \geq \limsup_{j\to\infty} f(x_{k_j}, y) = \lim_{j\to\infty} f(x_{k_j}, y) = \limsup_{k\to\infty} f(x_k, y) \geq 0. \tag{3.20}$$

Next we prove that $\bar{x} \in Fix(T)$. By the definition of z_k and (3.14) we have

$$\|x_{k_j} - Tx_{k_j}\| = \frac{1}{1-\beta_{k_j}}\|x_{k_j} - z_{k_j}\| \leq \frac{1}{1-\beta'}\|x_{k_j} - z_{k_j}\| \to 0,\ \text{as } j \to \infty. \tag{3.21}$$

From (3.21) and (A2) we get $\bar{x} \in Fix(T)$, which together with the arbitrariness of $y \in Fix(T)$ and (3.21) implies that $\bar{x} \in \Omega$.

Finally, we prove that $\{x_k\}$ converges weakly to $\bar{x}$. In fact, by the argument above, we have shown that every sequential weak cluster point of $\{x_k\}$ is in Ω. Hence by Lemma 3.2 and Lemma 2.3 with $D = \Omega$ we obtain that $\{x_k\}$ converges weakly to $\bar{x}$. This completes the proof. □

4. Applications

In this section, we apply the result in last section to a variational inequality problem on the set of fixed points of a quasi-nonexpansive mapping and to a constrained optimization problem, respectively.

The first application Let $A : C \to H$ be a mapping and $T : C \to H$ be a quasi-nonexpansive mapping. We consider the following problem: find a point $\bar{x} \in Fix(T)$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \ \forall y \in Fix(T). \tag{4.1}$$

Let $f(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. We see that $\partial f(x, \cdot)(x) = Ax$ for all $x \in C$ and the set Ω is the set of solutions of the problem (4.1), where Ω is defined as in Section 3. So by Theorem 3.4 we get the following result. Since the proof method is similar with the one of Theorem 3.4 with $f(x, y) = \langle Ax, y - x\rangle$, we only give the result without the proof process.

Theorem 4.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A : C \to H$ be a mapping and $T : C \to H$ be a quasi-nonexpansive mapping. Let the sequences $\{\gamma_k\} \subset (0, 1)$ and $\{\alpha_k\} \subset (0, +\infty)$ satisfy*

$$\sum_{k=1}^{\infty} \alpha_k \gamma_k = \infty \text{ and } \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

Set $C_0 = C$ and let $\{x_k\}$ be the sequence generated by the manner: $x_1 \in C$ and

$$\begin{cases} y_k = P_{C_{k-1} - x_k}\left(-\dfrac{\alpha_k}{\max\{1, \|Ax_k\|\}} Ax_k\right), \\ x_{k+1} = P_{C_k}(x_k + \gamma_k y_k), \end{cases} \tag{4.2}$$

If the following conditions hold:

(B1) *$\{z \in C : \langle Ay, z - y\rangle \leq 0, \ \forall y \in C\} \cap Fix(T) \neq \emptyset$;*

(B2) *$I - T$ is demiclosed at 0;*

(B3) *if $\{x_k\}$ is bounded, then $\{Ax_k\}$ is also bounded;*

(B4) *A is weakly upper semicontinuous on C,*

then the sequence $\{x_k\}$ generated by (4.3) *converges weakly to a solution of the problem* (4.1).

The second application Let $h : C \to \mathbb{R}$ be a convex and differentiable function and $g : C \to \mathbb{R}$ be a convex function. Considering the constrained optimization problem:

$$\min_{x\in C} h(x) \quad \text{s.t. } g(x) \le 0. \tag{4.3}$$

Define a mapping $S_g : C \to H$ by

$$S_g x = \begin{cases} x - \dfrac{g(x)}{\|u\|^2}u, & \text{if } g(x) > 0, \\ x, & \text{otherwise,} \end{cases} \tag{4.4}$$

where u is any vector of $\partial g(x)$. Let $T = 2S_g - I$. It follows that T is a quasi-nonexpansive mapping and $Fix(T) = \{x \in C : g(x) \le 0\}$ (see [26]). Solving the problem (4.3) is equivalent to find a solution of the problem (4.1) with $A = \nabla h$ and $T = 2S_g - I$. So Theorem 4.1 can be applied to the problem (4.3) with $A = \nabla h$ and $T = 2S_g - I$.

5. Numerical experiment

In this section, we present a numerical example to illustrate the convergence of our algorithm. The code is written by Matlab 2016b and is performed on a PC Intel(R) Core (TM) i5-4260U CPU, 2.00 GHz, Ram 4.00 GB.

Example 5.1. Let $H = \mathbb{R}$ and $C = [0, +\infty)$ and $f(x, y) = y(x - y)^2$ for all $x, y \in C$. Clearly, f is not pseudomonotone on C. We show that f is not Lipschitz-type continuous by a contradiction. Assume that there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \ge f(x, z) - c_1(x - y)^2 - c_2(y - z)^2, \ \forall x, y, z \in C.$$

In particular, if $z = 2y = 2x$, it follows that

$$2y - 2 \le c_2, \ \forall y \in C.$$

Clearly, it is impossible. So f is not Lipschitz-type continuous.

Let $Tx = (\frac{1}{2}I - P_D)x$ for all $x \in C$, where $D = [0, 1]$ and $Ix = x$ for all $x \in H$. Obviously, $T : C \to H$ is quasi-nonexpansive and $Fix(T) = \{0\}$. It is easy to see that the solution set $\Omega = \{0\}$ for the problem (Γ) with the bifunction f and T defined in this example. Find that

1. $\{0\} = Fix(T) \cap \{z \in C : f(y, z) \le 0, \ \forall y \in C\}$;
2. $I - T$ is demiclosed at 0;
3. since $\{0\} = \partial f(x, \cdot)(x)$ for each $x \in C$, the operator $\partial f(x, \cdot)(x)$ is bounded on the bounded subsets of C;
4. $f(\cdot, y)$ is continuous on C for each $y \in C$.

It follows that the conditions (A1)-(A4) holds and so Algorithm 3.1 can be applied to this example.

Since $w_k = 0$, $\eta_k = 1$, and $0 \in C_{k-1} - x_k$, when performing Algorithm 3.1 for this example, we have

$$\begin{cases} y_k = P_{C_{k-1}-x_k}\left(-\dfrac{\alpha_k}{\eta_k} w_k\right) = P_{C_{k-1}-x_k} 0 = 0, \\ x_{k+1} = P_{C_k}(x_k + \gamma_k y_k) = P_{C_k} x_k. \end{cases}$$

By the definition of T, we have $Tx_k = (\frac{1}{2}I - P_D)x_k$ hence

$$z_k = \beta_k x_k + (1-\beta_k)Tx_k = \left(\frac{1}{2}(1+\beta_k)I - (1-\beta_k)P_D\right)x_k.$$

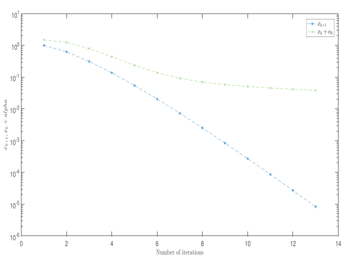
Finally, the sequence $\{x_k\}$ generated by Algorithm 3.1 for this example is

$$\begin{cases} C_k = \{x \in C : |z_k - z| \leq |x_k - z|\}, \\ x_{k+1} = P_{C_k}x_k. \end{cases}$$

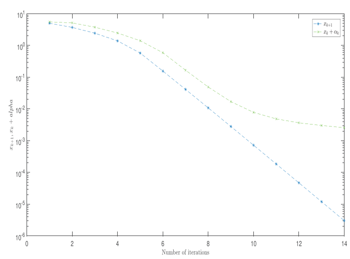
We take $\alpha_k = \frac{1}{2k}$ and stop the algorithm when $x_k < 10^{-5}$. For this example, by (3.8) we have

$$x_{k+1} \leq x_k + 8\alpha_k^2, \ \forall k \in \mathbb{N}. \tag{5.1}$$

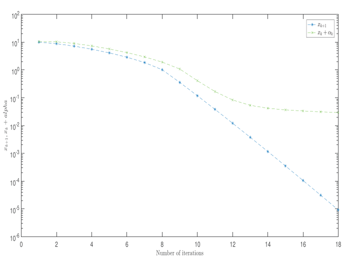
Fig 1 gives the computed results on $\{x_{k+1}\}$ and $\{x_k + 8\alpha_k^2\}$ by Algorithm 3.1 with the different initial point x_1 and the sequence $\{\beta_k\}$. From the curves in Fig 1 we can see the convergence of $\{x_k\}$ and the relation (5.1).



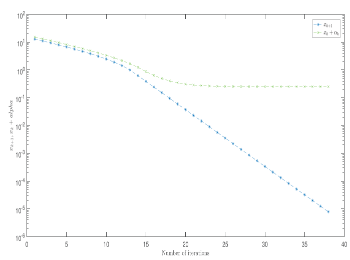
(A) $x_1 = 1, \beta_k = \frac{1}{1+k}$



(B) $x_1 = 5, \beta_k = \frac{1}{k^2}$



(C) $x_1 = 10, \beta_k = \frac{k+1}{k^2+1}$



(D) $x1 = 15, \beta_k = 0.5$

FIGURE 1. Computed results by Algorithm 3.1

6. CONCLUSION

In this paper, we have proposed new projection algorithm for solving an equilibrium problem over fixed point set of a quasi-nonexpansive mapping in real Hilbert spaces. We proved the weak convergence of the proposed algorithm. A numerical experiment is given to illustrate the effectiveness of our algorithm.

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Y. Shi
Department of Mathematics and Physics, North China Electric Power University, Baoding 071003, China
E-mail address: yitongshincepu@163.com

Z. Zhu
Department of Mathematics and Physics, North China Electric Power University, Baoding 071003, China
E-mail address: ziqizhuncepu@163.com

Y. Zhang
Department of Mathematics and Physics, North China Electric Power University, Baoding 071003, China
E-mail address: yueyaozhangncepu@163.com