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EXTENSION OF THE PROJECTED GRADIENT AND ARMIJO'S RULE CONCEPTS FOR SOLVING CONVEX NONLINEAR MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this work, we propose a method for solving multiobjective optimization problems with boundary constraints. Its algorithm combines the projected gradient with a modified version of the Armijo rule to find the best solutions. Under the assumption that the objective functions are differentiable and convex, the convergence to a Pareto optimal point is established. Furthermore, we have successfully solved test problems with which we calculated some performance indicators related to the convergence and distribution of obtained solutions. All these results have allowed us to highlight the effectiveness and efficiency of the method.

1. Introduction

The multiobjective optimization concept involves the simultaneous optimization of several objectives. In general, these mathematical problems are said to be poorly posed because there is no single solution that optimizes all objective functions simultaneously. Therefore, some criteria are being proposed to facilitate the choice of these many solutions, hence the parturition of the Pareto optimality concept. Multiobjective optimization problems are widely used in many fields of science, such as physics [12], economy [32], logistic and transports [3, 5], education [31], etc.

In the literature, many solving techniques have been developed to solve these types of problems. We can cite heuristic and metaheuristic methods based on genetic and evolutionary strategies [6–9, 23, 24] and iterative methods [13, 16, 18, 19, 21, 25]. Initially, iterative methods were proposed for single objective optimization and were characterized by their good convergence properties. Later, they were extended for multiobjective optimization by the resolution of a single-objective optimization problem obtained with the use of a weighted sum function. For example, the steepest descent method is one of the methods that was extended, but it doesn't use scalar parameters to transform the problem. Later, this improving or expanding process was extended to other iterative methods such as Newton's methods [4, 16], quasi-Newton's methods [1,25], conjugate gradient [21,26] and projected gradient methods [2, 13, 18, 19, 22]

Over the past few years, the projected gradient method for multiobjective optimization has been extensively studied. We refer the reader to [13, 14, 19, 22] for

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some interesting theoretical and practical studies of this method. In [13], the authors have presented a variational formula that defines Pareto's critical points as a solution to the min-max problem ($\min f_p(v) + \frac{1}{2}\|v\|^2$ where $f_p(v) = \max_{j=\overline{1,q}} \left(\nabla f_j(x)^T v\right)$). It should also be stressed that the min-max solution is automatically a descent direction. Here, the length of the descent step is determined by Armijo rule, and that makes the values of the objective function decrease at each iteration. In [10, 11], an equivalent formulation for finding the descent direction is defined, which consists of solving the following problem:

$$\arg\min_{v \in \overline{V}} \|v\| \tag{1.1}$$

where
$\overline{V} = \left\{v \in \mathbb{R}^n : v = \sum_{j=1}^{q} \alpha_j \nabla f_j(x_0), \alpha_j \in]0,1[, \quad \forall j = \overline{1,q}, \quad \sum_{j=1}^{q} \alpha_j = 1\right\}$. This formulation has the same role as in the min-max formulation [13] and can be used whether the gradient vectors are linearly independent or not [11]. In [15, 20, 29], an extension of the Armijo rule was elaborated and contains the classic Armijo rule as a particular case.

Among these extensions of the Armijo rule, we have a modified version which is used for single-objective optimization [15, 29]. Contrary to the initial version, the modified version makes it easier to research the length of the descent step and update the parameter L_k at each iteration. However, it is rare to find works in the literature on the use of this modified version for the resolution of multiobjective optimization problems. Furthermore, the projected gradient method has been used with the initial version for single and multiobjective optimization, and good results have been obtained. That is why we have proposed in this paper that combining projected gradient and modified Armijo rule will be a perfect method in terms of convergence and complexity. The aims of this paper are:

- first, offer a good method for approximating the Pareto front of convex multiobjective optimization problems with boundary constraints;
- then, extend the Armijo rule to the multiple objective case;
- and finally, highlight the theoretical and numerical performance of the modified version of the Armijo rule.

The proposed method is an aggregation method that uses the weighted sum function to solve a multiobjective optimization problem. These weights $\alpha_j, j = \overline{1,q}$ are the solutions of the quadratic program given by equation 1.1. The corresponding vector $v \in \overline{V}$ is the direction in which all objective functions are heading. Under the assumption that objective functions are differentiable and convex, convergence to a Pareto optimal point is established. In addition, we have also presented numerical experiments on some convex multiobjective test problems, showing the practical advantages of our approach.

The remainder of this work is organized as follows. In Section 2, we will introduce the preliminary, which is composed of the basic concepts of multiobjective optimization, the projected gradient method, and the modified Armijo rule. In Section 3, we will talk about the main results of this work, which are theoretical convergence, results from numerical simulations, and a performance study. The conclusion is in the last section.

2. Preliminary

2.1. Notations and basic concepts.

Let us introduce some necessary notations and recall a couple of results. We will denote $\mathcal{K} \subset \mathbb{R}^n$ a closed, convex and pointed subset (i.e, $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$) cone, $\mathcal{K}^*$ as the positive polar cone of $\mathcal{K}$, that is,

$$\mathcal{K}^* := \big\{ y \in \mathcal{K} : \langle y, x \rangle \geq 0, \ \forall x \in \mathcal{K} \big\}.$$

Let $\Omega \subseteq \mathbb{R}^n$ be a closed and convex set. Multiobjective optimization problems, minimization cases, are formulated mathematically by :

$$\min_{x \in \Omega} f(x) = (f_1(x), f_2(x), \dots, f_q(x))^T \tag{2.1}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}, \ j = 1, 2, \dots, q$, is an objective function and $\Omega = \big\{ x \in \mathbb{R}^n : l_i \leq x_i \leq u_i, \ i = 1, \dots, n \big\}$ is a feasible region. $x = (x_1, x_2, \dots, x_n)$ is a decision vector and $\mathcal{Y} = f(\Omega)$ is the objective space. For elements $x, y \in \mathbb{R}^n$, we present the vector inequalities as:

$$\begin{array}{lcll} x = y & \Longleftrightarrow & x_i = y_i, & \forall i = 1, 2, \dots, n, \\ x \geqq y & \Longleftrightarrow & x_i \geqslant y_i, & \forall i = 1, 2, \dots, n, \\ x \geqslant y & \Longleftrightarrow & x_i \geqslant y_i, & \text{and } x \neq y \\ x > y & \Longleftrightarrow & x_i > y_i, & \forall i = 1, 2, \dots, n, \end{array}$$

Definition 2.1. A point $x^* \in \Omega$ is called Pareto optimal solution of problem (2.1) if there is no other point $x \in \Omega$ such that

$$f(x) \leq f(x^*) \text{ and } f(x) \neq f(x^*).$$

In this case, $f(x^*) = \big(f_1(x^*), f_2(x^*), \dots, f_q(x^*)\big)$ is said a non-dominated point. That allows us to put $\mathcal{P}_s = \{x^* \in \Omega : f(x^*)\}$ as non-dominated Pareto optimal solutions set and $\mathcal{P}_f = \{f(x) : x \in \mathcal{P}_s\}$ Pareto front.

Definition 2.2. A point $x^* \in \Omega$ is called weakly Pareto optimal solution of problem (2.1) if there is no $x \in \Omega$ for which

$$f(x) < f(x^*).$$

Noting weakly Pareto optimal solutions by $\mathcal{P}_s$ we have $\mathcal{P}_s \subseteq \overline{\mathcal{P}}_s$.

The weighted sum is a technique that transforms multiobjective optimization problem (2.1) into a single objective optimization problem. It proceeds by an assigning some preference weights to each objective function. Then, if $\alpha_j \in]0, 1[, \ j = 1, \dots, q$ are preference weights we have $\displaystyle\sum_{j=1}^{q} \alpha_j = 1$. That allows us to transform the problem (2.1) as follows :

$$(P_\alpha) \qquad \min_{x\in\Omega} \ \psi(x) = \sum_{j=1}^{q} \alpha_j f_j(x)$$

The theoretical results of the optimality of the solutions of problem (P_α) are presented in the work of R. T. Marler et al. [27]. When the functions f_j are continuous, differentiable, convex and locally lipschitzian, the following proposition gives a necessary and sufficient condition for Pareto optimality.

Proposition 2.3 (See [10])**.** *Let $x^* \in \Omega$ be a Pareto optimal solution of the problem (2.1). Then, there exists some non-negative scalars $\alpha_1, \alpha_2, \ldots, \alpha_q \in]0,1[$ such as $\sum_{j=1}^{q} \alpha_j = 1$ and $\sum_{j=1}^{q} \alpha_j \nabla f_j(x^*) = 0$.*

Definition 2.4 (See [11])**.** Let $x^* \in \Omega$, $\forall j = 1, 2, \ldots, q$, f_j, be smooth function, and $\nabla f_j(x^*)$ its gradient at the point x^*. x^* is said Pareto-stationary if there exists a convex combination of gradients $\nabla f_j(x^*)$ which is equal to zero. In other words

$$\exists \alpha_j \in]0,1[, \ j = 1, 2, \ldots, q \text{ with } \sum_{j=1}^{q} \alpha_j = 1 \text{ such as } \sum_{j=1}^{q} \alpha_j \nabla f_j(x^*) = 0.$$

In the rest of this work, we will consider the following quadratic problem for each $x \in \mathbb{R}^n$ [28] :

$$(2.2) \qquad \begin{array}{l} \min\limits_{\alpha} \left\| \sum\limits_{j=1}^{q} \alpha_j \nabla f_j(x^*) \right\|_2^2 \\ \text{s.t:} \begin{cases} \alpha_j \in]0,1[; \quad \forall j = 1, 2, \ldots, q \\ \sum\limits_{j=1}^{q} \alpha_j = 1. \end{cases} \end{array}$$

Now by defining the function $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto \sum_{j=1}^{q} \alpha_j^* \nabla f_j(x)$, where $\alpha^* = (\alpha_j^*)$ is a global minimum of problem (2.2). Note that if x^* satisfies Proposition 2.3 then, $\vartheta(x) = 0$; otherwise $-\vartheta(x)$ is a descent direction to all objective functions.

We present by this lemma the quasi-Fejér convergence.

Lemma 2.5 (See [22], proposition 1)**.** *Let $T \subset \mathbb{R}^n$ be a non-empty set and $\{a_k\}_k \subset \mathbb{R}^n$ a sequence. Let us assume that, there exists $\{\epsilon_k\}$, a summable positive sequence, such as $\|a_{k+1} - z\|^2 \leq \|a_k - z\|^2 + \epsilon_k$, $\forall z \in T$, $\forall k$. Then,*

(i) *$\{a_k\}_k$ is bounded;*
(ii) *the sequence $\{a_k\}_k$ converge to a^*, with a^* an accumulation point of $\{a_k\}_k$ belonging to T.*

The following propositions are extracted from Lemma 1.1 and 1.2 of [33].

Proposition 2.6 (See [33]). *Let $\mathcal{C}$ be a non-empty and convex set. $\forall x, y \in \mathcal{K}$ and $\forall z \in \mathcal{C}$, the following propositions are satisfied :*

(i) $\langle x - P_{\mathcal{C}}(x), z - P_{\mathcal{C}}(x)\rangle \leq 0$;
(ii) $\langle z - y, z - P_{\mathcal{C}}(y)\rangle \geq \|z - P_{\mathcal{C}}(y)\|$.

2.2. **Projected gradient and modified Armijo's rule.**
Let us recall that the projected gradient method is an iterative, which consists to determine a feasible descent direction. That is defined by : $\delta_k^\alpha = P_\Omega\left(x_k - \xi_k \nabla_\alpha \psi(x_k)\right) - x_k$ $\forall k = 1, 2, \ldots$, where P_Ω is a orthogonal projector operator defined from $\mathbb{R}^n$ to Ω. The iterative step of this method is given by :

$$\delta_k^\alpha = \arg\min_{\delta^\alpha \in \Omega_k} \frac{1}{2}\|\delta^\alpha\|_2^2 + \xi_k \nabla_\alpha \psi(x_k)^T \delta^\alpha, \forall k = 1, 2, \ldots\ ,$$

$$x_{k+1} = x_k + \beta_k \delta_k^\alpha, \forall k = 1, 2, \ldots, \tag{2.3}$$

where $\Omega_k = \{u - x_k : u \in \Omega\}$, $\{\xi_k\}_k \in \mathbb{R}_{++}^*$ (where $\mathbb{R}_{++}^*$ is the set of large positive reals) and $\beta_k \in (0, 1]$. Like that

$$\delta_k^\alpha = \arg\min_{\delta^\alpha \in \Omega_k} \frac{1}{2}\|\delta^\alpha\|_2^2 + \xi_k \nabla_\alpha \psi(x_k)^T \delta^\alpha \tag{2.4}$$

$$= \arg\min_{\delta^\alpha \in \Omega_k} \frac{1}{2}\|\delta^\alpha\|_2^2 + \xi_k \nabla_\alpha \psi(x_k)^T \delta^\alpha + \frac{1}{2}\xi_k^2 \|\nabla_\alpha \psi(x_k)\|^2$$

$$= \arg\min_{\delta^\alpha \in \Omega_k} \frac{1}{2}\left\|\delta^\alpha + \xi_k \nabla_\alpha \psi(x_k)^T\right\|_2^2$$

$$= P_{\Omega_k}(-\xi_k \nabla_\alpha \psi(x_k)). \tag{2.5}$$

For $\theta_k^\alpha = \delta_k^\alpha + x_k$, we have $\theta_k^\alpha = P_{\Omega_k}(-\xi_k \nabla_\alpha \psi(x_k)) + x_k$ and equation (2.5) becomes $\theta_k^\alpha = P_\Omega(x_k - \xi_k \nabla_\alpha \psi(x_k))$ and equation (2.3) becomes $x_{k+1} = x_k + \beta_k\left(\theta_k^\alpha - x_k\right)$ where β_k is the step length which can be determined by inexact linear research.

Now, by positing $\omega^\alpha : \mathbb{R}^n \to \mathbb{R}$, the function indicating the optimal value of problem (2.4) and $\delta^\alpha : \mathbb{R}^n \to \mathbb{R}^n$, the one indicating the optimal solution, we have a characterization of the stationarity using the terms $\omega^\alpha(.)$ and $\delta^\alpha(.)$ is given by the following proposition.

Proposition 2.7 (See [13, 17]). *For all $x \in \Omega$ we have:*

(a) $\omega^\alpha(x) \leq 0$;
(b) *The following conditions are equivalents :*
  (i) *x is not a critical point,*
  (ii) $\omega^\alpha(x) < 0$;
  (iii) $\delta^\alpha(x) \neq 0$;
(c) *the function $\omega^\alpha : \Omega \to \mathbb{R}$ is continuous.*

The inexact linear research method that we use in this work is a modify version of Armijo's rule. The original version is to define a β_k, the largest $\beta \in \left\{S_k, \lambda S_k, \lambda^2 S_k, \ldots\right\}$ such as $f_j(x_k + \beta(\theta_k^\alpha - x_k)) \leq f_j(x_k) + \sigma\beta\left\langle \nabla_\alpha \psi(x_k), \theta_k^\alpha - x_k\right\rangle$ where $S_k = -\dfrac{\nabla\psi(x_k)(\theta_k^\alpha - x_k)}{L\left\|\theta_k^\alpha - x_k\right\|^2}$, $\lambda \in (0, 1)$, $\sigma \in (0, \frac{1}{2})$, $L > 0$.

The modified version of Armijo's rule defined by Shi and al. [30] allows to determine the step length more easily than the original version. This new version can be

presented as follows [30]:
We determine the scalars S_k, λ, L_k, μ, σ, where

$$\beta_k^\alpha = \max\left\{S_k, \lambda S_k, \lambda^2 S_k, \ldots\right\} \tag{2.6}$$

and

$$S_k = -\frac{\nabla\psi(x_k)(\theta_k^\alpha - x_k)}{L_k \left\|\theta_k^\alpha - x_k\right\|^2}, \;\; \lambda \in (0,1), \;\; \sigma \in (0, \frac{1}{2}), \;\; L_k > 0, \;\; \mu \in [0, 2[\,,$$

such as

$$f_j(x_k+\beta(\theta_k^\alpha - x_k)) \leq f_j(x_k) + \sigma\beta\left[\langle\nabla_\alpha\psi(x_k), \theta_k^\alpha - x_k\rangle + \frac{1}{2}\beta\mu L_k \left\|\theta_k^\alpha - x_k\right\|_2^2\right]. \tag{2.7}$$

We notice that in the modified version of Armijo's rule, the parameter L_k is updated at each iteration. This parameter may be approached by te Lipschitz constant M^* of the objective function gradient $\nabla_\alpha\psi(x_k)$. By posing $\nu_{k-1} = x_k - x_{k-1}$ and $\omega_{k-1} = \nabla_\alpha\psi(x_k) - \nabla_\alpha\psi(x_{k-1})$, we can obtain an estimation of parameter L_k is the following :

$$L_k = \max\left\{\frac{\|\omega_{k-i}\|}{\|\nu_{k-i}\|}, \;\; i = 1, 2, 3, \ldots, M\right\}, \tag{2.8}$$

where M is a fixed integer.

In the following proposition, we show that the new line search is well difined.

Proposition 2.8. *Let $f_j : \Omega \to \mathbb{R}$ be continuously differentiable function. Let us suppose that the gradient function $\nabla f_j, j \in \{1, 2, \ldots, q\}$ is Lipschitz continuous. Let $L_k > 0$ be an approximative value of the Lipschitz constant. If $\delta_k^\alpha = \theta_k^\alpha - x_k$ is a descent direction of f_j for all $j \in \{1, 2, \ldots, q\}$ at x_k, then there is an $\beta_k > 0$ in the set $\left\{S_k, \lambda S_k, \lambda^2 S_k, \ldots\right\}$ such that the following inequality holds:*

$$f_j(x_k + \beta\delta_k^\alpha) \leq f_j(x_k) + \sigma\beta\left(\nabla_\alpha\psi(x_k)^T\delta_k^\alpha + \frac{1}{2}\beta\mu L_k \left\|\delta_k^\alpha\right\|_2^2\right).$$

where $\lambda \in (0,1)$, $\sigma \in (0, \frac{1}{2})$, $L > 0$ are given constant scalars.

Proof *In fact, we only need to prove that a step length β_k is obtained in a finite number of steps.*
Then, if this is not true, for all sufficiently large positive integers m, we have:

$$f_j(x_k + \lambda^m S_k\delta_k^\alpha) > f_j(x_k) + \sigma\lambda^m S_k\left(\nabla_\alpha\psi(x_k)^T\delta_k^\alpha + \frac{1}{2}\lambda^m S_k\mu L_k \left\|\delta_k^\alpha\right\|_2^2\right).$$

By the mean-theorem, there is a $w_k \in (0,1)$ such that

$$\lambda^m S_k\nabla_\alpha\psi(x_k + w_k\lambda^m S_k\delta_k^\alpha)^T\delta_k^\alpha > \sigma\lambda^m S_k\left(\nabla_\alpha\psi(x_k)^T\delta_k^\alpha + \frac{1}{2}\lambda^m S_k\mu L_k \left\|\delta_k^\alpha\right\|_2^2\right).$$

Thus,

$$\nabla_\alpha\psi(x_k + w_k\lambda^m S_k\delta_k^\alpha)^T\delta_k^\alpha - \nabla_\alpha\psi(x_k)^T\delta_k^\alpha > \sigma\Big(\nabla_\alpha\psi(x_k)^T\delta_k^\alpha - \nabla_\alpha\psi(x_k)^T\delta_k^\alpha$$

$$+\frac{1}{2}\lambda^m S_k \mu L_k \left\|\delta_k^\alpha\right\|_2^2\Big).$$

$$\Rightarrow \Big(\nabla_\alpha\psi(x_k + w_k\lambda^m S_k\delta_k^\alpha) - \nabla_\alpha\psi(x_k)\Big)^T \delta_k^\alpha > (1-\sigma)\nabla_\alpha\psi(x_k)^T\delta_k^\alpha + \frac{1}{2}\lambda^m S_k\mu\sigma L_k \left\|\delta_k^\alpha\right\|_2^2.$$

As, $m \to +\infty$, it is obtained that

$$(1-\sigma)\nabla_\alpha\psi(x_k)^T\delta_k^\alpha \leq 0.$$

From $\sigma \in (0, \frac{1}{2})$, it follows that $\nabla_\alpha\psi(x_k)^T\delta_k^\alpha \geq 0$. This contradicts the fact that δ_k^α is a descent direction. □

3. Main result

In this section, we present the algorithm of our method, the theoretical results proving its convergence and numerical simulation results. We also present a discussion on the study of digital performance through performance indices such as convergence and distribution.

3.1. **Method.**

In the following subsections, we will consider $\psi(x_k) = \sum_{j=1}^{q} \alpha_j \nabla f_j(x^*)$ the function whose values α_j are obtained by solving problem (2.2). We define $\nabla_\alpha\psi(x_k)$ as the gradient of the function $\psi(x_k)$ for a given vector α.

There are five main steps for our method. These steps are presented as follows :

Algorithm 1 Armijo's rule with Projected Gradient (APG)

Data : $\sigma \in \left(0, \frac{1}{2}\right)$, $\lambda \in (0,1)$, $\mu \in [0,2[$, $L_0 > 0$, $x_0 \in \Omega$, $0 \leq \epsilon^* \lll 1$, $\xi = 1$,
k=1

Step 1: define the α_j values by solving equation (2.2);
Step 2: define the descent direction δ_k^α which satisfies equation (2.5);

if $\|x_k - P_\Omega\left[x_k - \nabla_\alpha\psi(x_k)\right]\| \leq \epsilon^*$ stop
else go to **Step 3**;
Step 3: estimate the parameter L_k given by equation (2.8) and go to **Step 4**;
Step 4: find the descent step β_k by using equation (2.6) and go to **Step 5**;
Step 5: put $x_{k+1} = (1-\beta_k)\, x_k + \beta_k\theta_k^\alpha$;

$k = k+1$ and go to **Step 1**.

In practice, in **Step** 1 the α_j's are defined from the initial point by solving equation (2.2). In **Step** 2, at each iteration, if the stopping condition is not verified, we have $\|x_k - P_\Omega\left[x_k - \nabla_\alpha\psi(x_k)\right]\| > \epsilon^*$. As $x_k \in \Omega$, that involves $\nabla_\alpha\psi(x_k) \neq 0$, hence the point x_k is not Pareto-stationary. Then, move to **Step** 3. At this step, we estimate the parameter L_k given by equation (2.8). The **Step 4** is dedicated to find the descent step length. In the **Step 5**, for each new $x_{k+1} = x_k + \beta_k \delta_k^{\alpha}$ computed, we go back to **Step** 1 to find new values α_j; which will allow us to define a new descent direction.

3.2. **Theoretical results.**
As presented in the introduction, we assume that the objective functions are convex we establish a proposition and two theorems to prove the convergence.

Proposition 3.1. *The sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by the Algorithm 1 is feasible, and $\{f_j(x_k)\}_{k\in\mathbb{N}}$ is monotonically decreasing.*

Proof By induction. The initial iterate x_0 belongs to Ω by the initialization of the method. Assuming that $x_k \in \Omega$, since $\beta_k \in (0, 1]$ and $\delta_k^{\alpha} \in \Omega_k = \Omega - x_k$, since Ω is convex and $x_{k+1} = x_k + \beta_k \delta_k^{\alpha}$, we conclude that x_{k+1} belongs to Ω.

Let $K_\beta := \{k : \beta_k \leq S_k\}$. If $k \in K_\beta$ then, $\forall j = 1, 2, \ldots, q$,

$$
\begin{aligned}
f_j(x_{k+1}) - f_j(x_k) &\leq \sigma\beta_k \left[\nabla_\alpha \psi(x_k)^T(\theta_k - x_k) + \frac{1}{2}\beta_k \mu L_k \|\theta_k - x_k\|_2^2\right] \\
&\leq \sigma S_k \left[\nabla_\alpha \psi(x_k)^T(\theta_k - x_k) + \frac{1}{2} S_k \mu L_k \|\theta_k - x_k\|_2^2\right] \\
&\leq -\sigma(1 - \frac{1}{2}\mu) L_k^{-1} \left(\frac{\nabla_\alpha \psi(x_k)^T(\theta_k - x_k)}{\|\theta_k - x_k\|}\right)^2. \qquad (3.1)
\end{aligned}
$$

Since $\sigma \in \left(0, \frac{1}{2}\right)$, $\mu \in (0, 2)$, and by definition, L_k is positive. Let us set $\mathcal{E} = \sigma(1 - \frac{1}{2}\mu) L_k^{-1} \geq 0$. From the equation (3.1), we have :

$$
f_j(x_{k+1}) - f_j(x_k) \leq -\mathcal{E}\left(\frac{\nabla_\alpha \psi(x_k)^T(\theta_k - x_k)}{\|\theta_k - x_k\|}\right)^2, \quad j = 1, 2, \ldots, q.
$$

Hence the result.

Theorem 3.2. *Let $f_j \colon \mathbb{R}^n \to \mathbb{R}$, for all $j = 1, 2, \ldots, q$ be continuously differentiable functions in Ω, $\sigma \in (0, 1)$ and $\{x_k\}$ a sequence generated by Algorithm 1 with $x_{k+1} = x_k + \beta_k(\theta_k^{\alpha} - x_k)$. Let us assume that $\mathcal{T} = \{x \in \Omega; f(x) \leq f(x_0)\}$ is bounded. Then the sequence $\{x_k\}$ generated by Algorithm 1 converge to a critical point x^* of the problem (P_α).*

Proof From the Proposition 3.1, $\{f_j(x_k)\}_{k\in\mathbb{N}}$ is monotonically decreasing, that means $f_j(x_k) > f_j(x_{k+1})$, $\forall j = 1, \ldots, q$. By using Armijo's rule defined in the equation (2.7), as $\{f_j(x_k)\}_{k\in\mathbb{N}}$ is bounded and the gradients of $f_j(x_k)$, $j = 1, \ldots, q$ are Lipschitzian. Let us set $K_\beta := \{k : \beta_k \leq S_k\}$. If $k \in K_\beta$, we obtain the following inequalities :

$$
f_j(x_{k+1}) - f_j(x_k) \leq \sigma\beta_k \left[\nabla_\alpha \psi(x_k)^T(\theta_k - x_k) + \frac{1}{2}\beta_k \mu L_k \|\theta_k - x_k\|_2^2\right]
$$

$$\leq \sigma\beta_k \left[\nabla_\alpha \psi(x_k)^T(\theta_k - x_k) + \frac{1}{2}S_k\mu L_k \|\theta_k - x_k\|_2^2\right]$$

$$\leq \sigma\beta_k(1 - \frac{1}{2}\mu)\nabla_\alpha \psi(x_k)^T(\theta_k - x_k). \tag{3.2}$$

Following Proposition 2.6, we have :

$$\langle \nabla_\alpha \psi(x_k), x_k - \theta_k \rangle \geq \|x_k - \theta_k\|. \tag{3.3}$$

By combining equations (3.2) and (3.3), we obtain the following inequality :

$$\sigma\beta_k(1 - \frac{1}{2}\mu)\|\theta_k - x_k\| \leq f_j(x_k) - f_j(x_{k+1});$$

$$\sum_{k'=0}^{k} \sigma\beta_{k'}(1 - \frac{1}{2}\mu)\|\theta_{k'} - x_{k'}\| \leq f_j(x_0) - f_j(x_{k+1}).$$

As the sequence $\{f_j(x_k)\}$ is bounded and according to Armijo's rule the sequence $\{f_j(x_k)\}$ is monotonically decreasing, then $f_j(x_k)$ converge to $f_j(x^*)$ when $k \to +\infty$ and x^* is an accumulation point of $\{x_k\}$. In this condition we obtain the following inequality :

$$\sum_{k=0}^{+\infty} \sigma\beta_k(1 - \frac{1}{2}\mu)\|\theta_k - x_k\| \leq f_j(x_0) - f_j(x^*) < +\infty, \tag{3.4}$$

because $\sigma\beta_k \|\theta_k - x_k\|$ is finite [13]. So, we obtain $\beta_k \|\theta_k - x_k\| \to 0$ when $k \to +\infty$. From the instruction of the algorithm, the descent step β_k is bounded. If $\beta_k \to +\infty$ then, from equation (3.4), we have $+\infty \leq f_j(x_0) - f_j(x^*) < +\infty$, that means $\mathcal{T}$ is not bounded, which is contradictory. If $\beta_k \in (0, 1]$, $\forall k$, then $\{\theta_k - x_k\}$ converge to 0 when k tends to infinity. Let us note that $\mathcal{T}$ is bounded at least an accumulated point. Let $\{y_1^*, y_2^*, \ldots, y_m^*\}$ be the set of accumulation points of $\{x_k\}$. By using the Proposition 4 from [13], we have the function δ_k continuous on Ω, because y_i^* is an accumulation point for each $i \in \{1, 2, \ldots, m\}$, so $\delta_k(y_i^*)$ is a critical point of f_i for each $i \in \{1, 2, \ldots, m\}$. □

Theorem 3.3. *Let $f_j : \mathbb{R}^n \to \mathbb{R}$, for all $j = 1, 2, \ldots, q$ be convex, continuous, differentiable and locally Lipschitzian functions in Ω. Let $\{x_k\}$ be a sequence generated by Algorithm 1 with $x_{k+1} = x_k + \beta_k(\theta_k^\alpha - x_k)$. Then, the accumulation point of $\{x_k\}$ is a weakly Pareto optimal solution of the problem (2.1).*

Proof Let us consider x^* an accumulation point of the sequence $\{x_k\}$. We have show in the theorem (3.2) that the sequence $\{x_k\}$ generated by the Algorithm 1 converge to a critical point x^* for all f_j, $\forall j \in \{1, 2, \ldots, q\}$. From **Step 5** of the Algorithm 1, we have :

$$x_{k+1} = x_k + \beta_k(\theta_k - x_k).$$

Therefore, we have

$$\|x_{k+1} - x^*\|^2 = \|x_k - x^* + \beta_k(\theta_k - x_k)\|^2$$

$$= \|x_k - x^*\|^2 + \beta_k^2 \|\theta_k - x_k\|^2 - 2\beta_k \langle \theta_k - x_k, x^* - x_k \rangle. \tag{3.5}$$

In addition

$$\langle \theta_k - x_k, x^* - x_k \rangle = \langle \theta_k - x_k + \nabla_\alpha \psi(x_k) - \nabla_\alpha \psi(x_k), x^* - x_k \rangle$$

$$
\begin{aligned}
&= -\left\langle \nabla_\alpha \psi(x_k), x^* - x_k \right\rangle + \left\langle \theta_k - x_k + \nabla_\alpha \psi(x_k), x^* - x_k \right\rangle \\
&= \left\langle \nabla_\alpha \psi(x_k), x_k - x^* \right\rangle - \left\langle \theta_k - x_k + \nabla_\alpha \psi(x_k), x_k - x^* \right\rangle \\
&= \left\langle \nabla_\alpha \psi(x_k), x_k - x^* \right\rangle - \left\langle \theta_k - x_k + \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle \\
&- \left\langle \theta_k - (x_k - \nabla_\alpha \psi(x_k)), \theta_k - x^* \right\rangle \\
&\geq \left\langle \nabla_\alpha \psi(x_k), x_k - x^* \right\rangle - \left\langle \theta_k - x_k + \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle .
\end{aligned}
$$

From the convexity of f_j, we have $\nabla_\alpha \psi(x_k).(x_k - x^*) \geq \psi(x_k) - \psi(x^*)$, and

$$
\begin{aligned}
\left\langle \theta_k - x_k, x^* - x_k \right\rangle &\geq \psi(x_k) - \psi(x^*) - \left\langle \theta_k - x_k + \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle \\
&\geq -\left\langle \theta_k - x_k + \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle \\
&\geq \|\theta_k - x_k\|^2 - \left\langle \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle ,
\end{aligned}
$$

that involve that

$$
-2\beta_k \left\langle \theta_k - x_k, x^* - x_k \right\rangle \leq -2\beta_k \|\theta_k - x_k\|^2 + 2\beta_k \left\langle \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle . \tag{3.6}
$$

By using the equations (3.5) and (3.6) we obtain the following inequality :

$$
\begin{aligned}
&\|x_k - x^*\|^2 + \beta_k^2 \|\theta_k - x_k\|^2 - 2\beta_k \left\langle \theta_k - x_k, x^* - x_k \right\rangle \leq \\
&\qquad \|x_k - x^*\|^2 + \beta_k^2 \|\theta_k - x_k\|^2 - 2\beta_k \|\theta_k - x_k\|^2 + 2\beta_k \left\langle \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle ,
\end{aligned}
$$

that gives,

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + \beta_k^2 \|\theta_k - x_k\|^2 - 2\beta_k \|\theta_k - x_k\|^2 + 2\beta_k \left\langle \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle .
$$

As $\beta_k \in (0, 1] \Rightarrow \beta_k^2 - 2\beta_k \leq -\beta_k$, thus we have:

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \beta_k \|\theta_k - x_k\|^2 + 2\beta_k \left\langle \nabla_\alpha \psi(x_k), x_k - \theta_k \right\rangle . \tag{3.7}
$$

Let us remain that $\forall j \in \{1, 2, \ldots, q\}$, $f_j(x_k)$ is bounded and $\nabla f_j(x_k)$ are Lipschitzian. Let us set that $K_\beta := \{k \in \mathbb{N} : \beta_k \leq S_k\}$. Using the Armijo's rule defined in equation (2.7), if $k \in K_\beta$, then we obtain the following inequalities :

$$
\begin{aligned}
f_j(x_{k+1}) - f_j(x_k) &\leq \sigma\beta_k \left[\nabla_\alpha \psi(x_k)^T (\theta_k - x_k) + \frac{1}{2}\beta_k \mu L_k \|\theta_k - x_k\|_2^2 \right] \\
&\leq \sigma\beta_k \left[\nabla_\alpha \psi(x_k)^T (\theta_k - x_k) + \frac{1}{2} S_k \mu L_k \|\theta_k - x_k\|_2^2 \right] \\
&\leq -\sigma\beta_k (1 - \frac{1}{2}\mu) \nabla_\alpha \psi(x_k)^T (x_k - \theta_k),
\end{aligned}
$$

$$
\Rightarrow \left\langle \nabla_\alpha \psi(x_k), \theta_k - x_k \right\rangle \geq \frac{1}{\sigma\beta_k (1 - \frac{1}{2}\mu)} \left(f_j(x_{k+1}) - f_j(x_k) \right) . \tag{3.8}
$$

By combining the equations (3.7) and (3.8), we obtain equation (3.9) as follows :

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \beta_k \|\theta_k - x_k\|^2 + \frac{2}{\sigma(1 - \frac{1}{2}\mu)} \left(f_j(x_k) - f_j(x_{k+1}) \right) .
$$

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + \frac{2}{\sigma(1 - \frac{1}{2}\mu)} \left(f_j(x_k) - f_j(x_{k+1}) \right) . \tag{3.9}
$$

by setting $\epsilon_k = \dfrac{2}{\sigma(1-\frac{1}{2}\mu)}\left(f_j(x_k) - f_j(x_{k+1})\right)$, equation (3.9) becomes

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + \epsilon_k. \tag{3.10}$$

And we have:

$$\sum_{k'=0}^{k} \epsilon_{k'} = \sum_{k'=0}^{k} \frac{2}{\sigma(1-\frac{1}{2}\mu)}\left(f_j(x_{k'}) - f_j(x_{k'+1})\right),$$

when $k \to +\infty$

$$\sum_{k=0}^{+\infty} \epsilon_k \leq \frac{2}{\sigma(1-\frac{1}{2}\mu)}\left(f_j(x_0) - f_j(x^*)\right) < +\infty. \tag{3.11}$$

Regarding to the equation (3.10) and equation (3.11), the sequence $\{x_k\}$ generated by the Algorithm 1 verify the conditions of Lemma 2.5 then, it is quasi-Fejér convergent to the solutions set of initial problem. Moreover, according to Theorem 3.2, the sequence $\{x_k\}$ converges weakly to a Pareto-stationary solution of problem (2.1). □

3.3. **Numerical Results.**
To implement the algorithm, we have set: $\xi = 1$ at each iteration; $L_0 = 1$; $\mu = 1.5$; $\epsilon^* = 10^{-12}$; $\sigma = 0.38$; $\lambda = 0.87$; and the initial point x_0 for each problem is generated randomly. To generate the Pareto front, we run the algorithm for all test problems in a multi-start fashion. The test problems we used for the numerical test are all convex problems [6–9] and are recorded in the Table 1. The Figures 1, 2, 3, 4, 5 show the Pareto front obtained for the 05 test problems with a Multi-Start execution of 100 runs. Like any multi-objective algorithm, the study of the performance of the solutions obtained is in itself a bi-objective problem. This study focuses, on the one hand, on the minimization of the distance between the solutions obtained and the solutions belonging to the true Pareto front (analytical front) and, on the other hand, on the minimization of the distance between two consecutive solutions obtained. There are several metrics in the literature to measure the performance of the solutions obtained by an algorithm, in this work we will use those proposed by Deb (2002) [7]. The first parameter measures the convergence of obtained solutions to Pareto front, and the second parameter measures the distribution of obtained solutions on Pareto front. Let us set by γ the convergence parameter and Δ the distribution parameter.

$$\gamma = \frac{\sqrt{\sum_{i=1}^{N} d_i^2}}{N}$$

and

$$\Delta = \frac{d_f + d_l + \sum_{i=1}^{N-1} \left|d_i - \overline{d}\right|}{d_f + d_l + (N-1)\overline{d}}$$

where N is the number of solutions given by the Algorithm 1; d_f and d_l are respectively the euclidean distances between the upper extreme solutions and the lower extreme solutions given by the Algorithm 1; the d_i represents the euclidean distances between two consecutive solutions obtained and $\overline{d}$ the arithmetic mean of d_i.

Table 1. Multiobjective problems

Codes	Mathematics formulation
$PNL1$	$\begin{cases} \min f_1(x) = x^2 \\ \min f_2(x) = (x-2)^2 \\ -4 \leqslant x \leqslant 4 \end{cases}$
$PNL2$	$\begin{cases} \min f_1(x) = \cosh(x) \\ \min f_2(x) = x^2 - 12x + 35 \\ x \in [0,5] \end{cases}$
$PNL3$	$\begin{cases} \min f_1(x_1,x_2) = x_1 \\ \min f_2(x_1,x_2) = \dfrac{1+x_2}{x_1} \\ 0.1 \leqslant x_1 \leqslant 1 \\ 0 \leqslant x_2 \leqslant 5 \end{cases}$
$PNL4$	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = g(x)\left(1 - \sqrt{\frac{f_1(x)}{g(x)}}\right) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum_{i=2}^{n} x_i \\ x = (x_1, x_2, \ldots, x_{10}) \in [0,15]^{10} \end{cases}$
$PNL5$	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = \frac{g(x)}{x_1} \\ g(x) = 2 - \exp\left\{-\left(\frac{x_2-0.2}{0.004}\right)^2\right\} - 0,8\exp\left\{-\left(\frac{x_2-0.6}{0.4}\right)^2\right\} \\ x = (x_1,x_2) \in [0.1,1] \times [0,1] \end{cases}$

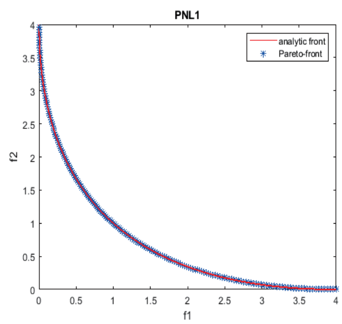

Figure 1. Pareto front of PNL1

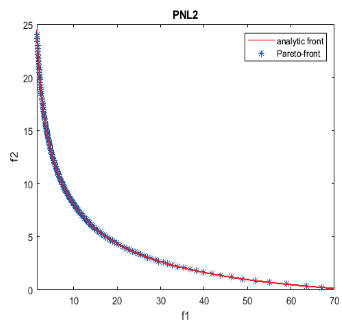

Figure 2. Pareto front of PNL2

The values of the convergence and distribution are presented in the table below.

Over all the 05 test problems, it is observed that the values of the convergence index are close to 0, which explains a good convergence, and that of the distribution index are slightly far from 0, which shows a low distribution of solutions.

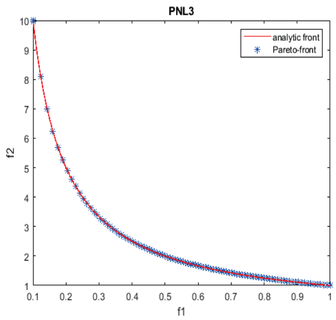

Figure 3. Pareto front of PNL3

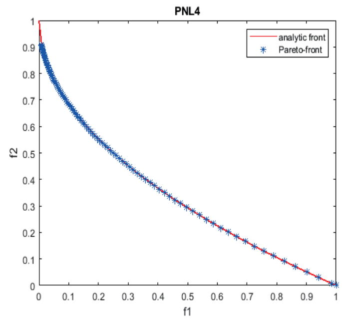

Figure 4. Pareto front of PNL4

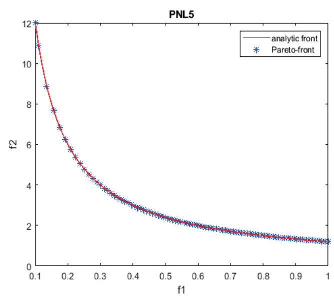

Figure 5. Pareto front of PNL5

TABLE 2. Performance parameters

Problems	PNL1	PNL2	PNL3	PNL4	PNL5
γ	0.0026	0.0074	0.0114	0.0003	0.0022
Δ	0.6666	0.9457	1.1065	0.9996	1.1131

3.4. **Comments.** The numerical solutions in the figures and numerical convergence parameters indicate that the proposed algorithm achieves good convergence of solutions on the analytical front. Except for problem PLN1, the distribution of solutions in the whole Pareto front on test problems is weak. This could be explained by the choice of parameters and the fact that the initial points are generated in a random way.

4. CONCLUSION

In this work, we combined three concepts: the projected gradient, Armijo rule, and the weighted sum in order to solve multiobjective optimization problems. First, we presented the scope of its different concepts before combining them to design a new method for solving convex, nonlinear, multiobjective optimization problems. To demonstrate convergence of methods, a proposition and two theorems were established. Numerical experiments on five test problems were successfully performed. Based on the numerical results, we calculated performance indices related to the convergence and distribution of solutions in relation to the Pareto front. All these theoretical and numerical results demonstrate that the method is a suitable choice for solving convex nonlinear multiobjective optimization problems.

In future research, we will start by improve the distribution performance of our method. Then, we will investigate the solution to constrained multiobjective optimization problems. In this works, a study will be done to compare our solutions to other methods for some test problems from the literature. Finally, a convergence study for non-convex problems will be another research topic.

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