Applied Analysis and Optimization
Volume 7, Number 3, 2023, 223–239

Yokohama Publishers
ISSN 2189-1664 Online Journal

ON THE ALGEBRAIC AND ORDINAL STRUCTURES OF SET RELATIONS IN SEMI-VECTOR SPACE

YOUSUKE ARAYA

Abstract. Convexity is one of the most important concepts in convex analysis and optimization theory. A convex set is usually defined in a subset of vector space X, and a special case of a convex set is a convex cone. It is easy to confirm that both definitions require a multiplication by a non-negative real number, indicating that it is appropriate to define convexity on more general spaces.

In this paper, first the we introduce the concept of semi-vector space, which is a generalization of vector space. Next, we investigate properties of semi-vector space followed by introduction of several types of set relations proposed by Kuroiwa-Tanaka-Ha and Jahn-Ha. Moreover, we investigate algebraic and ordinal structures of the above set relations and introduce several types of nonlinear scalarizing technique for sets, generalizations of Gerstewitz's scalarizing functions for the vector-valued case. Furthermore, we present characterization theorems of set relations based on the nonlinear scalarizing technique. Lastly, we introduce weighted set relations proposed by Chen-Köbis-Köbis-Yao and investigate its algebraic and ordinal structures.

1. Introduction

Let Y be a topological vector space ordered by a closed convex cone $C \subset Y$. Let X be a nonempty set and $F : X \to 2^Y$ a set-valued map with domain X ($F(x) \neq \emptyset$ for each $x \in X$). A set-valued optimization problem is formalized as follows:

$$(\mathrm{P}) \begin{cases} \text{Minimize} \quad F(x) \\ \text{Subject to} \quad x \in X \end{cases}$$

There are two types of criteria of solutions for the set-valued optimization problem; the vectorial criterion and set optimization criterion. The first criterion consists of finding efficient points of the set $F(X) = \cup_{x \in X} F(x)$ and is called set-valued vector optimization problem. On the other hand, Kuroiwa-Tanaka-Ha [28] started developing a new approach to set-valued optimization using the six types of set relations. After that, Jahn-Ha [20] introduced new order relations in set optimization in 2011. The second criterion is based on comparison among values of F, that is, whole images $F(x)$ and seems to be more natural for the set-valued optimization problem. The second type criterion is employed in this study and called set optimization problem.

Chen-Köbis-Köbis-Yao [7] summarized the purpose of investigating the set optimization problem in a practical point of view. For a number of problems in industry, economy and science, it is expected to optimize different conflicting goals (objective

2020 *Mathematics Subject Classification.* 06F20, 46A40, 49J53, 90C29.

Key words and phrases. Semi-vector space, ordinal structure, quotient space, nonlinear scalarization, weighted set relations.

functions) at the same time. Such problem structures lead to vector optimization. Moreover, most complex multi-objective problems are contaminated with uncertain data. For instance, in traffic optimization, uncertain weather conditions, construction works, or traffic jams may influence the computed optimal solutions of a train schedule or shortest path problems. Combining these concepts leads to an application of set optimization problem.

In the last decade, there were some important progresses in set optimization problem. (a) In 2010s, several investigations were performed for nonlinear scalarizing technique for sets, natural generalizations of Gerstewitz's scalarizing functions for vector. Inspired by past studies [15, 17], the authors [2] investigated properties of nonlinear scalarizing functions for sets (duality property, order preserving property, sublinearity and so on). Next, we attempted to derive characterization theorems of set relations yet, mistakes have been found. Gutierrez-Jimenez-Miglierina-Molho [14] and Köbis-Köbis [23] presented full characterization theorems of set relations. In recent years, as presented in [3], the author presented revised version of [2] and introduced existence theorems of cone saddle-points in the framework of set optimization problem. (b) In 2014, Ide-Köbis-Kuroiwa-Schöbel-Tammer [18] revealed strong connections between the set optimization problem and uncertain multi-objective optimization problem. Moreover, they clarified that finding robust solutions to uncertain multi-objective optimization problem can be interpreted as an application to set optimization problem. (c) In 2017, Chen-Köbis-Köbis-Yao [7] introduced new set relations in set optimization, which includes two known set order relations (l-type and u-type) as special cases. They asserted that the advantage of this new set relation is that drawbacks that may occur when just one of the known set order relation are balanced out. Moreover, they presented an existence result for minimal elements with respect to the new order relation and proposed a new numerical method for obtaining approximations of minimal elements. In this paper, we will show new properties of the above new set relation. (d) In 2015, Bao-Mordukhovich-Soubeyran [6] also mentioned that the set optimization problem has strong connections not only with economics but also with behavioral sciences, proposing minimal element theorems, variational principles and variable preferences. In recent years, Hamel-Löhne [16] showed the following famous examples discussed by Kreps [26]. Suppose that you want to go to a restaurant for dinner, and all what counts for you is the quality of the meals. Suppose further that there are two restaurants in your town, and by looking at their menus A and B on your smartphone you realize that for each meal on A there is one on B which you like better (or at least as much as the one on A). In which restaurant would you reserve a dinner table? Hamel-Löhne [16] clarified that the relation between the two menus may be expressed as u-type set relation proposed by Kuroiwa-Tanaka-Ha [28]. They also states that mathematics have contributed the least to the theory of set relations and its applications for a long time, and a number of interesting new questions as well as applications have already been under discussion or will appear in the near future.

Convexity is one of the most important concepts in convex analysis and optimization theory. A convex set is usually defined in a subset of vector space X, and a special case of a convex set is a convex cone. It is easy to confirm that both definitions require a multiplication by a non-negative real number and therefore, is appropriate to define convexity on more general spaces. The new concept, which is called semi-vector space, and its similar concepts were already considered and proposed by a number of researchers, [8, 11, 21, 30, 31, 35, 37, 38] and their references therein. It is confirmed that the family of nonempty convex subsets of vector space Y is an example of semi-vectors space. Therefore, it is appropriate to discuss set optimization problem in the framework of semi-vector space. However, to our knowledge, there have been only few papers on algebraic and ordinal structures of set relations in the framework of semi-vector space so far. Therefore, we aim to clarify algebraic and ordinal structures of the set optimization problem.

The organization of this paper is as follows. First, we recall properties of ordered topological vector space followed by introduction of the concept of semi-vector space, natural generalizations of vector space. Next, we investigate some properties of semi-vector space. The last section is the main results. We introduce several types of set relations proposed by Kuroiwa-Tanaka-Ha [28] and Jahn-Ha [20] and then investigate algebraic and ordinal structures of the above set relations. Moreover, we introduce several types of nonlinear scalarizing technique for sets [2] that is generalizations of Gerstewitz's scalarizing functions for the vector-valued case [9, 10, 12] and present characterization theorems of set relations based on nonlinear scalarizing technique. Furthermore, we introduce weighted set relations proposed by Chen-Köbis-Köbis-Yao [7] and investigate their algebraic and ordinal structures.

2. Preliminaries

Let Y be an ordered topological vector space and 0_Y be an origin of Y. For a set $A \subset Y$, int$A$ and cl$A$ denote the topological interior and the topological closure, respectively. A nonempty set A is called solid if int$A \neq \emptyset$.

We denote $\mathcal{V}$ by the family of nonempty subsets of Y and conv$(\mathcal{V})$ the family of nonempty convex subsets of Y. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

(OP): $V_1 + V_2 := \{v_1 + v_2 \,|v_1 \in V_1, v_2 \in V_2\}$, $\alpha V := \{\alpha v \,|v \in V\}$.

We denote cl$(\mathcal{V})$ by the family of nonempty closed subsets of Y. The sum of two sets $V_1, V_2 \in$ cl$(\mathcal{V})$ and the product of $\alpha \in \mathbb{R}$ and $V \in$ cl$(\mathcal{V})$ are defined by

(cl-OP)**:** $V_1 + V_2 := \mathrm{cl}\{v_1 + v_2 \,|v_1 \in V_1, v_2 \in V_2\}$, $\alpha V := \{\alpha v \,|v \in V\}$.

see also Godini [11]. We denote int$(\mathcal{V})$ by the family of nonempty open subsets of Y. The sum of two sets $V_1, V_2 \in$ int$(\mathcal{V})$ and the product of $\alpha \in \mathbb{R}$ and $V \in$ int$(\mathcal{V})$ are defined by

(int-OP)**:** $V_1 + V_2 := \mathrm{int}\{v_1 + v_2 \,|v_1 \in V_1, v_2 \in V_2\}$, $\alpha V := \{\alpha v \,|v \in V\}$.

The following proposition are fundamental properties of addition and scalar multiplication for sets in vector space, which is presented by fundamental methods of set theory.

Proposition 2.1 ([25,41])**.** *For $A, B, D \in \mathcal{V}$, $\lambda, \mu \in \mathbb{R}$ and $\lambda_1, \mu_1 \geq 0$, the following relations hold.*

(i) $A + B = B + A$;
(ii) $(A + B) + D = A + (B + D)$;

(iii) $A + \{0\} = A$;
(iv) $0_Y \in A + (-A)$.
(v) *There does not always exists* $\hat{A} \in \mathcal{V}$ *such that* $A + \hat{A} = \{0_Y\}$.
(vi) $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$.
(vii) $(\lambda_1 + \mu_1) \cdot A \subset \lambda_1 \cdot A + \mu_1 \cdot A$.
If A *is convex, then* $(\lambda_1 + \mu_1) \cdot A = \lambda_1 \cdot A + \mu_1 \cdot A$.
(viii) $\lambda \cdot (\mu \cdot A) = (\lambda\mu) \cdot A$.
(ix) $1 \cdot A = A$.
(x) $0 \cdot A = 0_Y$.

Lemma 2.2 ([27]). *For* $C \subset Z$ *a closed convex cone and* $A, B, V \in \mathcal{V}$*, the following relations hold:*
(i) $C + C = C$;
(ii) $C + \mathrm{int}C = \mathrm{int}C$;
(iii) $\mathrm{int}A + \mathrm{int}B \subset \mathrm{int}(A + B)$;
(iv) $\mathrm{cl}A + \mathrm{cl}B \subset \mathrm{cl}(A + B)$;
(v) $\mathrm{cl}(V + C) + C = \mathrm{cl}(V + C)$.

Proof. The last property follows from properties (i) and (iv). □

3. Semi-vector space

In the previous section, it was confirmed in Proposition 2.1 that it is consistent to define convexity on more general spaces. The natural framework for convexity seems to be a semi-vector space, proposed by Löhne [31], rather than a vector space.

Definition 3.1 (semi-vector space: see also [31, 37]). A nonempty set Z equipped with an addition $+ : Z \times Z \to Z$ and a multiplication $\odot : \mathbb{R}_+ \times Z \to Z$ is said to be a semi-vector space with the natural element $\theta \in Z$ if for all $z, z_1, z_2 \in Z$ and $\alpha, \beta \geq 0$ the following axioms are satisfied:
(SV1) $(z_1 + z_2) + z = z_1 + (z_2 + z)$,
(SV2) $z + \theta = z$,
(SV3) $z_1 + z_2 = z_2 + z_1$,
(SV4) $\alpha \odot (\beta \odot z) = (\alpha\beta) \odot z$,
(SV5) $1 \odot z = z$,
(SV6) $0 \odot z = \theta$,
(SV7) $\alpha \odot (z_1 + z_2) = \alpha \odot z_1 + \alpha \odot z_2$,
(SV8) $\alpha \odot z + \beta \cdot z = (\alpha + \beta) \odot z$
When Z is a topological space, it will be called a topological semi-vector space if $+$ and $\odot$ are continuous.

Axiom (SV8) states that every singleton sets are convex. Prakash-Sertel [37] called Axiom (SV8) "pointwise convex". See also the following example [31].

Example 3.2 ([31]). It is remarked that some singleton sets can be nonconvex. Indeed, let $Z = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$) and consider the element $A := \{0, 1\} \in Z$. Then we have that $\frac{1}{2}A + \frac{1}{2}A = \{0, \frac{1}{2}, 1\} \neq A$.

Proposition 3.3 ([31, 37]). *For every semi-vector space, the following statements are equivalent:*

(i) *Every singleton sets is convex,*
(ii) (SV8) *holds.*

The pioneering idea of semi-vector space has been found by Rådström [40], who pointed out properties of addition and scalar multiplication for sets in vector space (Proposition 2.1). Moreover, he presented question if additive semigroup can be embedded in a group and if multiplication with scalars can be extended to this group so that for positive scalars the new multiplication coincides with the original one on the semigroup. Furthermore, he gave an embedding theorem in normed vector space.

The concept of semi-vector space and its similar concepts have already been proposed by a number of researchers, such as, Gähler-Gähler [8](semi vector space) for problems of fuzzy analysis, Pap [35] for problems of measure theory, Godini [11](almost linear space) for approximation theory, Prakash-Sertel [37,38](semivector space) for topological fixed point problems. Janyška-Modugno-Vitolo [21] proposed the concept of positive spaces and their rational powers and showed that how these concepts can be used as scale spaces in a broad class of physical theories.

Example 3.4 ([11,30,37])**.** We give examples of semi-vector space.
(1) Every vector space V.
(2) $\mathbb{R}_+ := \{\lambda \in \mathbb{R} \,|\, \lambda \geq 0\}$ and its one-point compactification $\mathbb{R}_+ \cup \{\infty\}$.
(3) Every convex cone $C \subset Z$ of a semi-vector space with $0_Z \in C$.
(4) $(\mathrm{conv}(\mathcal{V}), +, \cdot)$ with operation (OP).
(5) $(\mathrm{cl}(\mathrm{conv}(\mathcal{V})), +, \cdot)$ with operation (cl-OP).
(6) $(\mathrm{int}(\mathrm{conv}(\mathcal{V})), +, \cdot)$ with operation (int-OP).

Proposition 3.5 ([30])**.** *A semi-vector space having a nontrivial conical element (that is, an element $z \in Z$ with $z = \alpha \cdot z$ for all $\alpha > 0$) cannot be embedded into a vector space.*

Of course, a partially ordered vector space is a special case of a semi-vector space. However, it is remarkable that most of set relations treated in the set optimization problem do not satisfy antisymmetric properties. See Proposition 4.2 in the next section. Thus, we introduce the following concept.

Definition 3.6 (preordered semi-vector space)**.** Let $(Z, +, \cdot)$ be a semi-vector space and let $\leq$ be a preorder on the set Z. $(Z, +, \cdot, \leq)$ is called a preordered semi-vector space if $\leq$ satisfies conditions (O1) and (O2):

(O1) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in E$,
(O2) $x \leq y$ implies $\alpha \cdot x \leq \alpha \cdot y$ for every $x, y \in E$ and $\alpha \geq 0$.

In a similar way as [31], let us define convex functions in the general setting of semi-vector space.

Definition 3.7. Let W be a semi-vector space and let Z be a preordered semi-vector space. A function $f : W \to Z$ is said to be convex if for all $\lambda \in [0, 1]$ and $w_1, w_2 \in W$, one has

$$f(\lambda \cdot w_1 + (1 - \lambda) \cdot w_2) \leq \lambda \cdot f(w_1) + (1 - \lambda) \cdot f(w_2).$$

4. Algebraic and ordinal structures of set relations

In this section, let $C \subset Y$ be a solid closed convex cone, that is, $\mathrm{int}C \neq \emptyset$, $\mathrm{cl}C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$. For $a, b \in Y$ and a solid convex cone $C \subset Y$, define what is called vector ordeing as follows

$$a \leq_C b \quad \text{by} \quad b - a \in C \qquad\qquad a \leq_{\mathrm{int}C} b \quad \text{by} \quad b - a \in \mathrm{int}C.$$

For a given real vector space E, there is a canonical one-to-one correspondence between the collection of order relations with properties (O1) and (O2) as described above and the collection of pointed convex cone, see for detail, [6,12,19,32,39]. However, the situation is more complicated for the set optimization problems. In this section, we will investigate the difference between the vector optimization problem and the set optimization problem in terms of algebraic and ordinal structures.

4.1. **l-type, u-type and $l\&u$-type set relations.** First, we recall several types of binary relationships on $\mathcal{V}$ by using a solid convex cone $C \subset Y$.

Definition 4.1 (Kuroiwa-Tanaka-Ha [28], Jahn-Ha [20]). For A, $B \in \mathcal{V}$ and a solid closed convex cone $C \subset Y$, we define

$$(\textbf{lower type})\ A \leq^l_C B \quad \text{by} \quad B \subset A + C \quad (A \leq^l_{\mathrm{int}C} B \quad \text{by} \quad B \subset A + \mathrm{int}C),$$

$$(\textbf{upper type})\ A \leq^u_C B \quad \text{by} \quad A \subset B - C \quad (A \leq^u_{\mathrm{int}C} B \quad \text{by} \quad A \subset B - \mathrm{int}C),$$

$$(\textbf{lower and upper type}) \quad A \leq^{l\&u}_C B \quad \text{by} \quad B \subset A + C \text{ and } A \subset B - C$$
$$(A \leq^{l\&u}_{\mathrm{int}C} B \quad \text{by} \quad B \subset A + \mathrm{int}C \text{ and } A \subset B - \mathrm{int}C).$$

Proposition 4.2 ([3,4]). *For A, B, $D \in \mathcal{V}$, $a, b \in Y$ and $\alpha \geq 0$, the following statements hold.*

(i) *$A \leq^l_C B$ implies $A + D \leq^l_C B + D$ and*
$A \leq^u_C B$ implies $A + D \leq^u_C B + D$.
(ii) *$A \leq^l_C B$ implies $\alpha A \leq^l_C \alpha B$ and $A \leq^u_C B$ implies $\alpha A \leq^u_C \alpha B$.*
(iii) *$\leq^l_C$ and $\leq^u_C$ are reflexive and transitive.*
(iv) *$A \leq^{l\&u}_C B$ implies $A \leq^l_C B$ and $A \leq^{l\&u}_C B$ implies $A \leq^u_C B$.*
(v) *$A \leq^l_C B$ and $A \leq^u_C B$ are not comparable, that is, $A \leq^l_C B$ does not imply $A \leq^u_C B$ and $A \leq^u_C B$ does not imply $A \leq^l_C B$.*
(vi) *$A \leq^u_C b$ implies $A \leq^l_C b$ and $a \leq^l_C B$ implies $a \leq^u_C B$.*

We see by (i) and (ii) of Proposition 4.2 that $(\mathcal{V}, +, \cdot, \leq^l_C)$, $(\mathcal{V}, +, \cdot, \leq^u_C)$ and $(\mathcal{V}, +, \cdot, \leq^{l\&u}_C)$ are preordered semi-vector spaces.

Example 4.3 (see also [20]). Let $a_1, a_2, b_1, b_2 \in Y$ be arbitrarily given with $a_1 \leq_C a_2$ and $b_1 \leq_C b_2$. We consider the following order intervals

$$A = [a_1, a_2] := \{y \in Y \,|a_1 \leq_C y \leq_C a_2\},$$
$$B = [b_1, b_2] := \{y \in Y \,|b_1 \leq_C y \leq_C b_2\}.$$

By the definition of $\leq^l_C$, $\leq^u_C$ and $\leq^{l\&u}_C$, we have

$$[a_1, a_2] \leq^l_C [b_1, b_2] \iff a_1 \leq_C b_1,$$
$$[a_1, a_2] \leq^u_C [b_1, b_2] \iff a_2 \leq_C b_2,$$
$$[a_1, a_2] \leq^{l\&u}_C [b_1, b_2] \iff a_1 \leq_C b_1 \quad \text{and} \quad a_2 \leq_C b_2.$$

Proposition 4.4 (Cancelation law: see also [37,38,40]). *For A, $B \in \mathcal{V}$ and $C \subset Y$ a closed convex cone, the following statements hold.*

(i) *If $B \in \mathcal{V}$ is bounded, then we have that $B \leq^l_C B + A$ implies $0_Y \leq^l_C A$.*
(ii) *If $B \in \mathcal{V}$ is bounded, then we have that $B + A \leq^u_C B$ implies $A \leq^u_C 0_Y$.*
(iii) *If $B \in \mathcal{V}$ is compact, then we have that $B \leq^l_{\mathrm{int}C} B+A$ implies $0_Y \leq^l_{\mathrm{int}C} A$.*
(iv) *If $B \in \mathcal{V}$ is compact, then we have that $B+A \leq^u_{\mathrm{int}C} B$ implies $A \leq^u_{\mathrm{int}C} 0_Y$.*

Proof. These are immediate consequences of (i) and (ii) of Proposition 2.2 in [38]. □

Inspired by Godini [11] and Maeda [33], we define new types of set relations on $\mathrm{cl}(\mathcal{V})$ and $\mathrm{int}(\mathcal{V})$ by using a solid convex cone $C \subset Y$.

Definition 4.5. For A, $B \in \mathrm{cl}(\mathcal{V})$ and a solid closed convex cone $C \subset Y$, we define the following set relations with operation (cl-OP):

(**l-closure**): $A \leqq^l_C B$ by $B \subset A + C$,
(**u-closure**): $A \leqq^u_C B$ by $A \subset B - C$,
(**$l\&u$-closure**): $A \leqq^{l\&u}_C B$ by $B \subset A + C$ and $A \subset B - C$.

Definition 4.6. For A, $B \in \mathrm{int}(\mathcal{V})$ and a solid closed convex cone $C \subset Y$, we define the following set relations with operation (int-OP):

(**l-interior**): $A <^l_{\mathrm{int}C} B$ by $B \subset A + \mathrm{int}C$,
(**u-interior**): $A <^u_{\mathrm{int}C} B$ by $A \subset B - \mathrm{int}C$,
(**$l\&u$-interior**): $A <^{l\&u}_{\mathrm{int}C} B$ by $B \subset A + \mathrm{int}C$ and $A \subset B - \mathrm{int}C$.

In a similar way as Proposition 4.2, we obtain several properties of new set relations.

Proposition 4.7. *For A, B, $D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold.*

(i) *$A \leqq^l_C B$ implies $A + D \leqq^l_C B + D$ and $A \leqq^u_C B$ implies $A + D \leqq^u_C B + D$.*
(ii) *$A \leqq^l_C B$ implies $\alpha A \leqq^l_C \alpha B$ and $A \leqq^u_C B$ implies $\alpha A \leqq^u_C \alpha B$.*
(iii) *$\leqq^l_C$ and $\leqq^u_C$ are reflexive and transitive.*
(iv) *$A \leqq^{l\&u}_C B$ implies $A \leqq^l_C B$ and $A \leqq^{l\&u}_C B$ implies $A \leqq^u_C B$.*

Proposition 4.8. *For A, B, $D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold.*

(i) *$A <^l_{\mathrm{int}C} B$ implies $A+D \leq^l_{\mathrm{int}C} B+D$ and $A <^u_{\mathrm{int}C} B$ implies $A+D <^u_{\mathrm{int}C} B + D$.*
(ii) *$A <^l_{\mathrm{int}C} B$ implies $\alpha A <^l_{\mathrm{int}C} \alpha B$ and $A <^u_{\mathrm{int}C} B$ implies $\alpha A <^u_{\mathrm{int}C} \alpha B$.*
(iii) *$<^l_{\mathrm{int}C}$ and $<^u_{\mathrm{int}C}$ are reflexive and transitive.*
(iv) *$A <^{l\&u}_{\mathrm{int}C} B$ implies $A <^l_{\mathrm{int}C} B$ and $A <^{l\&u}_{\mathrm{int}C} B$ implies $A <^u_{\mathrm{int}C} B$.*

Definition 4.9 ([17]). It is said that $A \in \mathcal{V}$ is C-proper $\big[(-C)$-proper$\big]$ if

$$A + C \neq Y \qquad [A - C \neq Y].$$

The symbol $\mathcal{V}_C$ denotes the family of C-proper subsets of Y, $\mathcal{V}_{-C}$ denotes the family of $(-C)$-proper subsets of Y and $\mathcal{V}_{\pm C}$ denotes the family of C-proper and $(-C)$-proper subsets of Y, respectively.

Definition 4.10 ([32])**.** It is said that $A \in \mathcal{V}$ is

(i) C-closed $\big[(-C)$-closed$\big]$ if $A + C$ $[A - C]$ is a closed set,

(ii) C-bounded $\big[(-C)$-bounded$\big]$ if for each neighborhood U of zero in Y there is some positive number $t > 0$ such that
$$A \subset tU + C \quad [A \subset tU - C],$$

(iii) C-compact $\big[(-C)$-compact$\big]$ if any cover of A the form
$$\{U_\alpha + C |\ U_\alpha \text{ are open}\} \quad [\{U_\alpha - C |\ U_\alpha \text{ are open}\}]$$
admits a finite subcover.

(iv) C-convex $\big[(-C)$-convex$\big]$ if $A + C$ $[A - C]$ is a convex set.

Every C-compact set is C-closed and C-bounded.

The symbol $\mathrm{cl}(\mathcal{V}_C)$ denotes the family of C-proper and C-closed subsets of Y, $\mathrm{cl}(\mathcal{V}_{-C})$ denots the family of $(-C)$-proper and $(-C)$-closed subsets of Y, $\mathrm{cl}(\mathcal{V}_{\pm C})$ denotes the family of C-proper, $(-C)$-proper, C-closed and $(-C)$-closed subsets of Y, $\mathrm{conv}(\mathcal{V}_C)$ denotes the family of C-convex subsets of Y, $\mathrm{conv}(\mathcal{V}_{-C})$ denots the family of $(-C)$-convex subsets of Y and $\mathrm{conv}(\mathcal{V}_{\pm C})$ denotes the family of C-convex and $(-C)$-convex subsets of Y, respectively.

Introducing the equivalence relations
$$A \simeq_l B \Longleftrightarrow A \leq^l_C B \quad \text{and} \quad B \leq^l_C A,$$
$$A \simeq_u B \Longleftrightarrow A \leq^u_C B \quad \text{and} \quad B \leq^u_C A,$$
$$A \simeq_{l\&u} B \Longleftrightarrow A \leq^{l\&u}_C B \quad \text{and} \quad B \leq^{l\&u}_C A,$$
we can generate the set of equivalence classes which are denoted by $[\cdot]^l$, $[\cdot]^u$ and $[\cdot]^{l\&u}$, respectively. The followings are easily confirmed.
$$A \in [B]^l \Longleftrightarrow A + C = B + C,$$
$$A \in [B]^u \Longleftrightarrow A - C = B - C,$$
$$A \in [B]^{l\&u} \Longleftrightarrow A + C = B + C \quad \text{and} \quad A - C = B - C.$$
Similarly, we define the following new equivalence relations
$$A \cong_l B \Longleftrightarrow A \leqq^l_C B \quad \text{and} \quad B \leqq^l_C A,$$
$$A \cong_u B \Longleftrightarrow A \leqq^u_C B \quad \text{and} \quad B \leqq^u_C A,$$
$$A \cong_{l\&u} B \Longleftrightarrow A \leqq^{l\&u}_C B \quad \text{and} \quad B \leqq^{l\&u}_C A,$$
$$A \sim_l B \Longleftrightarrow A <^l_{\mathrm{int}C} B \quad \text{and} \quad B <^l_{\mathrm{int}C} A,$$
$$A \sim_u B \Longleftrightarrow A <^u_{\mathrm{int}C} B \quad \text{and} \quad B <^u_{\mathrm{int}C} A,$$
$$A \sim_{l\&u} B \Longleftrightarrow A <^{l\&u}_{\mathrm{int}C} B \quad \text{and} \quad B <^{l\&u}_{\mathrm{int}C} A,$$
we can generate a partial ordering on the set of equivalence classes which are denoted by $\mathrm{cl}[\cdot]^l$, $\mathrm{cl}[\cdot]^u$, $\mathrm{cl}[\cdot]^{l\&u}$, $\mathrm{int}[\cdot]^l$, $\mathrm{int}[\cdot]^u$ and $\mathrm{int}[\cdot]^{l\&u}$, respectively. We can easily see that for $A, B \in \mathrm{cl}(\mathcal{V})$
$$A \in \mathrm{cl}[B]^l \Longleftrightarrow A + C = B + C,$$
$$A \in \mathrm{cl}[B]^u \Longleftrightarrow A - C = B - C,$$
$$A \in \mathrm{cl}[B]^{l\&u} \Longleftrightarrow A + C = B + C \quad \text{and} \quad A - C = B - C,$$

and for $A, B \in \mathrm{int}(\mathcal{V})$

$$A \in \mathrm{int}[B]^l \Longleftrightarrow A + \mathrm{int}C = B + \mathrm{int}C,$$
$$A \in \mathrm{int}[B]^u \Longleftrightarrow A - \mathrm{int}C = B - \mathrm{int}C,$$
$$A \in \mathrm{int}[B]^{l\&u} \Longleftrightarrow A + \mathrm{int}C = B + \mathrm{int}C \quad \text{and} \quad A - \mathrm{int}C = B - \mathrm{int}C.$$

By the definitions of the equivalent classes described above, we obtain the following relationships.

Proposition 4.11. *The following statements hold.*

(i) $[-A]^l = [A]^u$
(ii) $[-A]^u = [A]^l$
(iii) $[-A]^{l\&u} = [A]^{l\&u}$
(iv) *If there is some* $b \in Y$ *such that* $A \in [b]^l (A \in [b]^u)$ *and* $-A \in [-b]^l (-A \in [-b]^u)$*, then we have that* $A - A \in [0_Y]^l (A - A \in [0_Y]^u)$.
(v) *If there is some* $b \in Y$ *such that* $A \in [b]^{l\&u}$*, then we have that* $A - A \in [0_Y]^{l\&u}$.

Lemma 4.12. *We define a quotient space* $(\mathcal{V}/\simeq_l)$ *and its operations as follows*

($l+$) $[A]^l + [B]^l := [A + B]^l$ *for all* $A, B \in \mathcal{V}$,
($l\odot$) $\alpha \odot [A]^l := [\alpha \odot A]^l$ *for all* $A \in \mathcal{V}, \alpha \geq 0$.

(the definitions of u-type and l&u-type operations are similar to the following ones.) Then the above operations are well-defined.

Proof. By the definition of the equivalent class $[\cdot]^l$ and C being a convex cone, we have

(a) $A \simeq_l \hat{A}$ and $B \simeq_l \hat{B}$ implies $A + B \simeq_l \hat{A} + \hat{B}$,
(b) $A \simeq_l \hat{A}$ and $\alpha \geq 0$ implies $\alpha A \simeq_l \alpha \hat{A}$,

and hence we obtain the conclusion. The operations of u-type and $l\&u$-type are similar. □

Theorem 4.13. *The neutral elements are defined as follows*

$$[0_Y]^l := \{A \in \mathcal{V} \,|\, A + C = C\}, \qquad [0_Y]^u := \{A \in \mathcal{V} \,|\, A - C = -C\},$$
$$[0_Y]^{l\&u} := \{A \in \mathcal{V} \,|\, A + C = C \quad \text{and} \quad A - C = -C\}.$$

Then $(\mathrm{conv}(\mathcal{V}_C)/\simeq_l)$, $(\mathrm{conv}(\mathcal{V}_{-C})/\simeq_u)$ *and* $(\mathrm{conv}(\mathcal{V}_{\pm C})/\simeq_{l\&u})$ *are semi-vector space with the operations* ($l+$), ($l\odot$), ($u+$), ($u\odot$), ($l\&u+$), ($l\&u\odot$)*, respectively.*

Proof. It is easy to confirm that $\mathcal{V}$ satisfies the axioms of semi-vector space (SV1)–(SV7). Let us show (SV8). By the definition of the operations, we only show

(l-SV8): $\lambda_1 \odot A + \lambda_2 \odot A \in [(\lambda_1 + \lambda_2) \odot A]^l \qquad$ for $\lambda_1, \lambda_2 \geq 0$.

Since C is a convex cone, we have

$$\begin{aligned} \lambda_1 \odot A + \lambda_2 \odot A + C &= \lambda_1 \odot A + \lambda_2 \odot A + \lambda_1 \odot C + \lambda_2 \odot C \\ &= \lambda_1 \odot (A + C) + \lambda_2 \odot (A + C). \end{aligned}$$

Since $A + C$ is a convex set and C is a convex cone, we have

$$\lambda_1 \odot (A + C) + \lambda_2 \odot (A + C) = (\lambda_1 + \lambda_2) \odot (A + C)$$

$$= (\lambda_1 + \lambda_2) \odot A + (\lambda_1 + \lambda_2) \odot C = (\lambda_1 + \lambda_2) \odot A + C.$$

and hence (l-SV8) holds. The rest of the proof is similar as the above. □

In a similar way as the above theorem, we obtain the following results.

Theorem 4.14. *We define a quotient space* $(\mathrm{cl}(\mathcal{V})/\cong_l)$ *and its operations as follows (the definitions of u-type and $l\&u$-type operations are similar to the following ones.): for* $A, B \in \mathrm{cl}(\mathcal{V})$,

(cl$-l+$): $\mathrm{cl}[A]^l + \mathrm{cl}[B]^l := \mathrm{cl}[A+B]^l$ *for all* $A, B \in \mathcal{V}$,
(cl$-l\odot$): $\alpha \odot \mathrm{cl}[A]^l := \mathrm{cl}[\alpha \odot A]^l$ *for all* $A \in \mathcal{V}, \alpha \geq 0$.

We also define the neutral elements as follows

$$\mathrm{cl}[0_Y]^l := \{A \in \mathrm{cl}(\mathcal{V}) \,|\, A + C = C\}, \quad \mathrm{cl}[0_Y]^u := \{A \in \mathrm{cl}(\mathcal{V}) \,|\, A - C = -C\},$$

$$\mathrm{cl}[0_Y]^{l\&u} := \{A \in \mathrm{cl}(\mathcal{V}) \,|\, A + C = C \quad \text{and} \quad A - C = -C\}.$$

Then $(\mathrm{conv}(\mathrm{cl}(\mathcal{V}_C))/\cong_l)$, $(\mathrm{conv}(\mathrm{cl}(\mathcal{V}_{-C}))/\cong_u)$ *and* $(\mathrm{conv}(\mathrm{cl}(\mathcal{V}_{\pm C}))/\cong_{l\&u})$ *are semi-vector spaces.*

Next, we consider the concept of minimality on preordered semi-vector space $(\mathcal{V}, l+, l\odot, \leq_C^l)$, $(\mathcal{V}, u+, u\odot, \leq_C^u)$ and $(\mathcal{V}, l\&u+, l\&u\odot, \leq_C^{l\&u})$, respectively.

Definition 4.15. Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-minimal element if for any $A \in \mathcal{S}$,

$$A \leq_C^{l[u,l\&u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_C^{l[u,l\&u]} A.$$

Moreover, $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-weak minimal element if for any $A \in \mathcal{S}$,

$$A \leq_{\mathrm{int}C}^{l[u,l\&u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_{\mathrm{int}C}^{l[u,l\&u]} A.$$

The symbol $l[u, l\&u]$-$\mathrm{Min}(\mathcal{S}; C)$ denotes the family of $l[u, l\&u]$-minimal elements of $\mathcal{S}$ and $l[u, l\&u]$-$\mathrm{wMin}(\mathcal{S}; \mathrm{int}C)$ denotes the family of $l[u, l\&u]$-weak minimal elements of $\mathcal{S}$.

It is easily seen that

- $l[u, l\&u]$-$\mathrm{Min}(\mathcal{S}; C) \subset l[u, l\&u]$-$\mathrm{wMin}(\mathcal{S}; \mathrm{int}C) \subset \mathcal{S}$.

Next, we consider the concept of the set-valued convex map on semi-vector spaces. The following concepts plays an important role to show duality theorems in the set optimization problem (see [4]).

Definition 4.16. Let K be a convex set in a semi-vector space X. A set-valued map $F : X \to \mathcal{V}$ is said to be $l[u, l\&u]$-C-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, we have

$$F(\lambda x_1 + (1 - \lambda) x_2) \leq_C^{l[u,l\&u]} \lambda F(x_1) + (1 - \lambda) F(x_2).$$

4.2. **Characterizations of set relations and equivalent classes via nonlinear scalarization technique.** In 1983, Gerstewitz [9] introduced a nonlinear scalarizing function in vector optimization problem. Agreeing $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$, for $k^0 \in C \setminus (-C)$ we define $\varphi_{C,k^0} : Y \to (-\infty, \infty]$

$$\varphi_{C,k^0}(y) = \inf\{t \in \mathbb{R} \,|\, y \leq_C tk^0\} = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 - C\}.$$

Also, φ_{C,k^0} has a dual form as follows: $\psi_{C,k^0} : Y \to [-\infty, \infty)$,

$$\psi_{C,k^0}(y) = \sup\{t \in \mathbb{R} \,|\, tk^0 \leq_C y\} = \sup\{t \in \mathbb{R} \,|\, y \in tk^0 + C\}.$$

$$\varphi_{C,k^0}(y) = -\psi_{C,k^0}(-y).$$

These functions have wide applications in vector optimization (see also Luc [32], Gerth-Weidner [10], Göpfert-Riahi-Tammer-Zălinescu [12]). After that, we investigated the properties of the following two-variable infimum type $h_{\inf} : Y \times Y \to (-\infty, \infty]$ and supremum type $h_{\sup} : Y \times Y \to [-\infty, \infty)$ of nonlinear scalarizing functions for the vector optimization problem, which are generalizations of the above scalarizing function (see [1]):

$$h_{\inf}(y, a) = \inf\{t \in \mathbb{R} \,|\, y \leq_C tk^0 + a\} = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 + a - C\},$$

$$h_{\sup}(y, a) = \sup\{t \in \mathbb{R} \,|\, tk^0 + a \leq_C y\} = \sup\{t \in \mathbb{R} \,|\, y \in tk^0 + a + C\},$$

($h_{\inf}(y, a) := \varphi_{C,k^0}(y - a)$ for $a, y \in Y$). We can easily show

$$h_{\sup}(y, a) = -h_{\inf}(-y, -a).$$

Next, we introduce the following nonlinear scalarizing functions for sets, which are natural extension of $h_{\inf}$ and $h_{\sup}$. Agreeing $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$, we define $h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf}, h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup} : \mathcal{V} \times \mathcal{V} \to [-\infty, \infty]$ as follows. The functions $h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf}, h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup}$ play the role of utility functions.

$$h^l_{\inf}(V_1, V_2) = \inf\left\{t \in \mathbb{R} \,\middle|\, V_1 \leq^l_C tk^0 + V_2\right\} = \inf\{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset V_1 + C\},$$

$$h^u_{\inf}(V_1, V_2) = \inf\{t \in \mathbb{R} \,|\, V_1 \leq^u_C tk^0 + V_2\} = \inf\{t \in \mathbb{R} \,|\, V_1 \subset tk^0 + V_2 - C\},$$

$$h^{l\&u}_{\inf}(V_1, V_2) = \inf\left\{t \in \mathbb{R} \,\middle|\, V_1 \leq^{l\&u}_C tk^0 + V_2\right\}$$
$$= \inf\{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset V_1 + C \quad \text{and} \quad V_1 \subset tk^0 + V_2 - C\},$$

$$h^l_{\sup}(V_1, V_2) = \sup\left\{t \in \mathbb{R} \,\middle|\, tk^0 + V_2 \leq^l_C V_1\right\} = \sup\{t \in \mathbb{R} \,|\, V_1 \subset tk^0 + V_2 + C\},$$

$$h^u_{\sup}(V_1, V_2) = \sup\{t \in \mathbb{R} \,|\, tk^0 + V_2 \leq^u_C V_1\} = \sup\{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset V_1 - C\},$$

$$h^{l\&u}_{\sup}(V_1, V_2) = \sup\left\{t \in \mathbb{R} \,\middle|\, tk^0 + V_2 \leq^{l\&u}_C V_1\right\}$$
$$= \sup\{t \in \mathbb{R} \,|\, V_1 \subset tk^0 + V_2 + C \quad \text{and} \quad tk^0 + V_2 \subset V_1 - C\}.$$

Proposition 4.17 (see also [2,3])**.** *The following statements hold:*

(i) $h^l_{\sup}(V_1, V_2) = -h^u_{\inf}(-V_1, -V_2)$ *and* $h^u_{\sup}(V_1, V_2) = -h^l_{\inf}(-V_1, -V_2)$;
(ii) $h^{l\&u}_{\sup}(V_1, V_2) = -h^{l\&u}_{\inf}(-V_1, -V_2)$;
(iii) $h^l_{\inf}(V_1, V_2) = h^u_{\inf}(-V_2, -V_1)$ *and* $h^u_{\inf}(V_1, V_2) = h^l_{\inf}(-V_2, -V_1)$;
(iv) $h^l_{\inf}(V_1, V_2) \leq h^{l\&u}_{\inf}(V_1, V_2)$ *and* $h^u_{\inf}(V_1, V_2) \leq h^{l\&u}_{\inf}(V_1, V_2)$;
(v) $h^{l\&u}_{\sup}(V_1, V_2) \leq h^l_{\sup}(V_1, V_2)$ *and* $h^{l\&u}_{\sup}(V_1, V_2) \leq h^u_{\sup}(V_1, V_2)$.

The above results show that l-type and u-type are dual concepts each other.

Definition 4.18. We say that the function $f : \mathcal{V} \to [-\infty, \infty]$ is

(i) $\le_C^l$-increasing if $V_1 \le_C^l V_2$ implies $f(V_1) \le f(V_2)$,

(ii) strictly $\le_{\mathrm{int}C}^l$-increasing if

$V_1 \le_{\mathrm{int}C}^l V_2$ with $V_1 \ne V_2$ implies $f(V_1) < f(V_2)$.

The definitions of $\le_C^u$-increasing, $\le_C^{l\&u}$-increasing, strictly $\le_{\mathrm{int}C}^u$-increasing and strictly $\le_{\mathrm{int}C}^{l\&u}$-increasing are similar to the above ones, respectively.

Theorem 4.19 ([2,3])**.** *The functions $h_{\inf}^l, h_{\inf}^u, h_{\inf}^{l\&u}, h_{\sup}^l, h_{\sup}^u, h_{\sup}^{l\&u}$ have the following properties:*

(i) *$h_{\inf}^l(\cdot, V)$ and $h_{\sup}^l(\cdot, V)$ are $\le_C^l$-increasing for every $V \in \mathcal{V}$;*

(ii) *$h_{\inf}^u(\cdot, V)$ and $h_{\sup}^u(\cdot, V)$ are $\le_C^u$-increasing for every $V \in \mathcal{V}$;*

(iii) *$h_{\inf}^{l\&u}(\cdot, V)$ and $h_{\sup}^{l\&u}(\cdot, V)$ are $\le_C^{l\&u}$-increasing for every $V \in \mathcal{V}$.*

Proposition 4.20 ([5,14])**.** *For $V_1, V_2, V_3, V_4 \in \mathcal{V}$ and $\alpha \ge 0$, the following statements hold:*

(i) *$h_{\inf}^l(V_1, V_2) \le t \iff tk^0 + V_2 \subset \mathrm{cl}(V_1 + C)$;*

(ii) *$h_{\inf}^u(V_1, V_2) \le t \iff V_1 \subset \mathrm{cl}(tk^0 + V_2 - C)$;*

(iii) *$h_{\inf}^l(V_1 + C, V_2 + C) = h_{\inf}^l(V_1, V_2)$;*

(iv) *$h_{\inf}^u(V_1 - C, V_2 - C) = h_{\inf}^u(V_1, V_2)$;*

(v) *If $V_2 \in [V_1]^l$, then we have that $h_{\inf}^l(V_2, V_1) = h_{\inf}^l(V_1, V_2)$;*

(vi) *If $V_2 \in [V_1]^u$, then we have that $h_{\inf}^u(V_2, V_1) = h_{\inf}^u(V_1, V_2)$;*

(vii) *$h_{\inf}^l(V_1 + V_2, V_3 + V_4) \le h_{\inf}^l(V_1, V_3) + h_{\inf}^l(V_2, V_4)$ and*
$h_{\inf}^u(V_1 + V_2, V_3 + V_4) \le h_{\inf}^u(V_1, V_3) + h_{\inf}^u(V_2, V_4)$;

(viii) *$h_{\inf}^l(\alpha V_1, \alpha V_2) = \alpha h_{\inf}^l(V_1, V_2)$ and $h_{\inf}^u(\alpha V_1, \alpha V_2) = \alpha h_{\inf}^u(V_1, V_2)$.*

Next theorems are characterizations of set relations and equivalent classes via $h_{\inf}^l$ and $h_{\inf}^u$. See [3,14,23] and their references therein.

Theorem 4.21 ([3])**.** *Suppose that $C \subset Y$ be a solid closed convex cone and $k^0 \in \mathrm{int}C$.*

(i) *If if $V_1 \in \mathcal{V}_C$ is C-closed and $V_2 \in \mathcal{V}$, then we have*

$$V_2 \subset V_1 + C \iff h_{\inf}^l(V_1, V_2) \le 0.$$

Moreover, if $V_1 \in \mathcal{V}_C$ and $V_2 \in \mathcal{V}$ is C-compact, then we have

$$V_2 \subset V_1 + \mathrm{int}C \iff h_{\inf}^l(V_1, V_2) < 0.$$

(ii) *If $V_1 \in \mathcal{V}$ and $V_2 \in \mathcal{V}_{-C}$ is $(-C)$-closed, then we have*

$$V_1 \subset V_2 - C \iff h_{\inf}^u(V_1, V_2) \le 0.$$

Moreover, if $V_1 \in \mathcal{V}$ is $(-C)$-compact and $V_2 \in \mathcal{V}_{-C}$, then we have

$$V_1 \subset V_2 - \mathrm{int}C \iff h_{\inf}^u(V_1, V_2) < 0.$$

(iii) *If $V_1 \in \mathcal{V}_C$ is C-closed and $V_2 \in \mathcal{V}_{-C}$ is $(-C)$-closed, then we have*

$$V_2 \subset V_1 + C \quad \text{and} \quad V_1 \subset V_2 - C \iff h_{\inf}^{l\&u}(V_1, V_2) \le 0.$$

Moreover, if $V_1 \in \mathcal{V}_C$ *is* $(-C)$*-compact and* $V_2 \in \mathcal{V}_{-C}$ *is* C*-compact, then we have*

$$V_2 \subset V_1 + \mathrm{int}C \quad \text{and} \quad V_1 \subset V_2 - \mathrm{int}C \Longleftrightarrow h^{l\&u}_{\inf}(V_1, V_2) < 0.$$

By the definitions of equivalent classes $[\cdot]^l$, $[\cdot]^u$, $[\cdot]^{l\&u}$ and the above theorems, we obtain the following results.

Corollary 4.22. *Let* $C \subset Y$ *be a solid closed convex cone and* $k^0 \in \mathrm{int}C$.

(i) *If* $V_1 \in \mathcal{V}_C$ *and* $V_2 \in \mathcal{V}_C$ *are* C*-closed, then we have*

$$V_1 \in [V_2]^l \Longleftrightarrow h^l_{\inf}(V_1, V_2) = h^l_{\inf}(V_2, V_1) \leq 0.$$

(ii) *If* $V_1 \in \mathcal{V}_{-C}$ *and* $V_2 \in \mathcal{V}_{-C}$ *are* $(-C)$*-closed, then we have*

$$V_1 \in [V_2]^u \Longleftrightarrow h^u_{\inf}(V_1, V_2) = h^u_{\inf}(V_2, V_1) \leq 0.$$

(iii) *If* $V_1 \in \mathcal{V}_{\pm C}$ *and* $V_2 \in \mathcal{V}_{\pm C}$ *are* C*-closed and* $(-C)$*-closed, then we have*

$$V_1 \in [V_2]^{l\&u} \Longleftrightarrow h^{l\&u}_{\inf}(V_1, V_2) = h^{l\&u}_{\inf}(V_2, V_1) \leq 0.$$

Following the same line of the proof of the above theorems, we obtain the following results.

Corollary 4.23. *Let* $C \subset Y$ *be a solid closed convex cone and* $k^0 \in \mathrm{int}C$.

(i) *If* $V_1 \in \mathrm{cl}(\mathcal{V}_C)$ *and* $V_2 \in \mathrm{cl}(\mathcal{V})$*, then we have*

$$V_1 \leqq^l_C V_2 \Longleftrightarrow h^l_{\inf}(V_1, V_2) \leq 0.$$

(ii) *If* $V_1 \in \mathrm{cl}(\mathcal{V})$ *and* $V_2 \in \mathrm{cl}(\mathcal{V}_{-C})$*, then we have*

$$V_1 \leqq^u_C V_2 \Longleftrightarrow h^u_{\inf}(V_1, V_2) \leq 0.$$

(iii) *If* $V_1 \in \mathrm{cl}(\mathcal{V}_C)$ *and* $V_2 \in \mathrm{cl}(\mathcal{V}_{-C})$*, then we have*

$$V_1 \leqq^{l\&u}_C V_2 \Longleftrightarrow h^{l\&u}_{\inf}(V_1, V_2) \leq 0.$$

Corollary 4.24. *Let* $C \subset Y$ *be a solid closed convex cone and* $k^0 \in \mathrm{int}C$.

(i) *If* $V_1 \in \mathrm{cl}(\mathcal{V}_C)$ *and* $V_2 \in \mathrm{cl}(\mathcal{V}_C)$*, then we have*

$$V_1 \in \mathrm{cl}[V_2]^l \Longleftrightarrow h^l_{\inf}(V_1, V_2) = h^l_{\inf}(V_2, V_1) \leq 0.$$

(ii) *If* $V_1 \in \mathrm{cl}(\mathcal{V}_{-C})$ *and* $V_2 \in \mathrm{cl}(\mathcal{V}_{-C})$*, then we have*

$$V_1 \in \mathrm{cl}[V_2]^u \Longleftrightarrow h^u_{\inf}(V_1, V_2) = h^u_{\inf}(V_2, V_1) \leq 0.$$

(iii) *If* $V_1 \in \mathrm{cl}(\mathcal{V}_{\pm C})$ *and* $V_2 \in \mathrm{cl}(\mathcal{V}_{\pm C})$*, then we have*

$$V_1 \in \mathrm{cl}[V_2]^{l\&u} \Longleftrightarrow h^{l\&u}_{\inf}(V_1, V_2) = h^{l\&u}_{\inf}(V_2, V_1) \leq 0.$$

4.3. **The weighted set order relations.** In this subsection, we introduce new type set relations proposed by Chen-Köbis-Köbis-Yao in 2017 and investigate its algebraic and ordinal structures in semi-vector space.

Definition 4.25 (Chen-Köbis-Köbis-Yao [7])**.** Let $C \subset Y$ be a solid closed convex cone, $\lambda \in [0,1]$ and $k^0 \in \text{int}C$. For A, $B \in \mathcal{V}$, we define

$$A \preceq_{k^0}^{\lambda} B \iff \lambda h_{\inf}^{l}(A,B) + (1-\lambda) h_{\inf}^{u}(A,B) \leq 0,$$

$$A \prec_{k^0}^{\lambda} B \iff \lambda h_{\inf}^{l}(A,B) + (1-\lambda) h_{\inf}^{u}(A,B) < 0.$$

In this subsection, we investigate the set relations $\preceq_{k^0}^{\lambda}$ and $\prec_{k^0}^{\lambda}$ which are continuous research of [7].

Theorem 4.26 ([5,7])**.** *Let $C \subset Y$ be a solid closed convex cone, $\lambda \in [0,1]$ and $k^0 \in \text{int}C$. Then for A, B, $D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold.*

(i) *$A \preceq_{k^0}^{\lambda} B$ implies $A + D \preceq_{k^0}^{\lambda} B + D$.*
(ii) *$A \preceq_{k^0}^{\lambda} B$ implies $\alpha A \preceq_{k^0}^{\lambda} \alpha B$.*
(iii) *$\preceq_{k^0}^{\lambda}$ is reflexive and transitive.*
(iv) *$\prec_{k^0}^{\lambda}$ is transitive.*
(v) *If $A \in \mathcal{V}_C$ is C-closed and $B \in \mathcal{V}_{-C}$ is $(-C)$-closed, then $A \leq_C^{l\&u} B$ implies $A \preceq_{k^0}^{\lambda} B$.*
(vi) *If $A \in \mathcal{V}_C$ is $(-C)$-compact and $B \in \mathcal{V}_{-C}$ is C-compact, then we have that $A <_{\text{int}C}^{l\&u} B$ implies $A \prec_{k^0}^{\lambda} B$.*

We see by Proposition 4.26 that $(\mathcal{V}, +, \cdot, \preceq_{k^0}^{\lambda})$ is a preordered semi-vector space. In a similar way as Section 2, we can define the following new equivalence relation

$$A \simeq_{\lambda,k^0} B \Longleftrightarrow A \preceq_{k^0}^{\lambda} B \quad \text{and} \quad B \preceq_{k^0}^{\lambda} A.$$

Hence, we can generate the set of equivalence class which is denoted by $[\cdot]_{k^0}^{\lambda}$.

Proposition 4.27 ([5])**.** *The following statement holds:*

$$[-A]_{k^0}^{\lambda} = [A]_{k^0}^{\lambda}.$$

Lemma 4.28. *We define a quotient space $(\mathcal{V}/\simeq_{\lambda,k^0})$ and its operations as follows*

(w-+): *$[A]_{k^0}^{\lambda} + [B]_{k^0}^{\lambda} := [A+B]_{k^0}^{\lambda}$ for all $A, B \in \mathcal{V}$,*
(w-$\odot$): *$\alpha \odot [A]_{k^0}^{\lambda} := [\alpha \odot A]_{k^0}^{\lambda}$ for all $A \in \mathcal{V}, \alpha \geq 0$.*

Then the above operations are well-defined.

Proof. Using (vii) and (viii) of Proposition 4.20, we obtain

(a) $A \simeq_{\lambda,k^0} \hat{A}$ and $B \simeq_{\lambda,k^0} \hat{B}$ implies $A + B \simeq_{\lambda,k^0} \hat{A} + \hat{B}$,
(b) $A \simeq_{\lambda,k^0} \hat{A}$ and $\alpha \geq 0$ implies $\alpha A \simeq_{\lambda,k^0} \alpha \hat{A}$.

□

Theorem 4.29. *We define the neutral element as follows:*

$$[0_Y]_{k^0}^{\lambda} := \left\{ D \in \mathcal{V} \;\middle|\; \begin{array}{l} \lambda h_{\inf}^{l}(0_Y, D) + (1-\lambda) h_{\inf}^{u}(0_Y, D) \leq 0 \\ \lambda h_{\inf}^{l}(D, 0_Y) + (1-\lambda) h_{\inf}^{u}(D, 0_Y) \leq 0 \end{array} \right\}$$

Then $(\text{conv}(\mathcal{V}_{\pm C})/\simeq_{\lambda,k^0})$ is a semi-vector space with the operations (w-+) and (w-$\odot$) in Lemma 4.28.

Proof. We can easily confirm that $\mathrm{conv}(\mathcal{V}_{\pm C})$ satisfies the axioms of semi-vector space (SV1)–(SV7). Using (l-SV8) in Theorem 4.13 and (iii), (iv) of Proposition 4.20, (SV8) holds. □

In a similar way as Section 4.1, we can consider the concept of minimality and convex map on preordered semi-vector space $(\mathcal{V}, +, \odot, \preceq^{\lambda}_{k^0})$.

Definition 4.30. ($\preceq^{\lambda}_{k^0}$-minimal and $\prec^{\lambda}_{k^0}$-minimal element [7]) Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $\preceq^{\lambda}_{k^0}$-minimal element if for any $A \in \mathcal{S}$,

$$A \preceq^{\lambda}_{k^0} \bar{A} \quad \text{implies} \quad \bar{A} \preceq^{\lambda}_{k^0} A.$$

Moreover, $\bar{A} \in \mathcal{S}$ is a $\prec^{\lambda}_{k^0}$-minimal element if for any $A \in \mathcal{S}$,

$$A \prec^{\lambda}_{k^0} \bar{A} \quad \text{implies} \quad \bar{A} \prec^{\lambda}_{k^0} A.$$

The symbol $\mathrm{Min}(\mathcal{S}; C, \lambda, k^0)$ denotes the family of $\preceq^{\lambda}_{k^0}$-minimal elements of $\mathcal{S}$ and $\mathrm{wMin}(\mathcal{S}; \mathrm{int}C, \lambda, k^0)$ denotes the family of $\prec^{\lambda}_{k^0}$-minimal elements of $\mathcal{S}$.

It is easily seen that

- $l\&u$-$\mathrm{Min}(\mathcal{S}; C) \subset \mathrm{Min}(\mathcal{S}; C, \lambda, k^0)$ and
- $l\&u$-$\mathrm{wMin}(\mathcal{S}; \mathrm{int}C) \subset \mathrm{wMin}(\mathcal{S}; \mathrm{int}C, \lambda, k^0)$.

For further investigation of minimality of the weighted set relations, see [5].

Definition 4.31 ($\preceq^{\lambda}_{k^0}$-convexity)**.** Let K be a convex set in a semi-vector space X. A set-valued map $F : X \to \mathcal{V}$ is said to be $\preceq^{\lambda}_{k^0}$-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, one has

$$F(\lambda x_1 + (1 - \lambda)x_2) \preceq^{\lambda}_{k^0} \lambda F(x_1) + (1 - \lambda)F(x_2).$$

5. Conclusions

In this paper, we first introduced the concept of preordered semi-vector space which is a natural generalization of ordered vector space. The concept of semi-vector space is convenient for dealing with the set optimization problem and has characteristic property "pointwise convex". Cancelation law of set relations (Proposition 4.4) holds under additional conditions.

We have found by Proposition 4.26 that the weighted set relation $\preceq^{\lambda}_{k^0}$ has the same algebraic and ordinal structures as $\leq^{l}_{C}$, $\leq^{u}_{C}$ and $\leq^{l\&u}_{C}$. In particular, the concept of C-convexity proposed by Luc [32] plays a fundamental role to define an ordinary algebraic operation on $\mathcal{V}$. Moreover, we have found that $\preceq^{\lambda}_{k^0}$ contains $\leq^{l\&u}_{C}$ under some natural conditions by characterization theorems of set relations via nonlinear scalarization technique (Theorem 4.21).

Since set relation $\leq^{l\&u}_{C}$ has wide applications in engineering (see [20, 34]), the author infer that investigations of $\preceq^{\lambda}_{k^0}$ will be an important subject of the set optimization problem. The above facts raise a crucial question for how λ in $\preceq^{\lambda}_{k^0}$ is determined, and this problem has strong relationship with probability and statistics. The concept of convex metric space proposed by Takahashi [42] is also an interesting topic and it will be also important subject in pursuing how this concept in semi-vector space is introduced.

Acknowledgments

We would like to express our gratitude to the referee for his/her useful comments on this paper.

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Manuscript received April 11 2022
revised March 5 2023

Y. Araya
Faculty of Systems Science and Technology, Akita Prefectural University, 84-4 Aza Ebinokuchi Tsuchiya, Yurihonjo City, Akita, 015-0055, Japan
E-mail address: `y-araya@akita-pu.ac.jp`