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SET-VALUED FIXED POINT AND APPROXIMATING FIXED POINT THEOREMS IN METRIC VECTOR SPACES

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In honor of Professor Dr. Adrian Petrusel on the occasion of his 60-th birthday.

Abstract. In this paper, we prove a fixed point theorem for set-valued mappings in metric vector spaces, in which the considered mappings are not necessarily continuous. We consider the concept of approximating fixed point for set-valued mappings, which is a generalization of fixed point. Then, we prove an approximating fixed point theorems, in which the considered mappings satisfy a certain type of convexity. The proofs involve Fan-KKM Theorem. Some examples are provided for illustrating these theorems.

1. Introduction and preliminaries

One of the first fixed point theorems stipulates that every continuous mapping from a closed ball of a Euclidean space into itself has at least one fixed point. Since any non-degenerate convex compact subset of a Euclidean space is homeomorphic to some closed unit ball, the above result holds for any continuous self-mapping on a compact convex subset of an Euclidean space $\mathbb{R}^n$. The above result was proved for the case $n = 3$ by L. E. J. Brouwer in 1909, while the general case was proved, using the homotopy invariance of the topological degree of continuous mappings, by Brouwer in 1910 (the result was published in 1912, see [2]). Independently, Jacques Hadamard also proved the general case in 1910, using Kronecker indices. Hadamard published the proof of the theorem in the appendix of a book by J. Tannery. Since then fixed point theory has been rapidly developed and has become a major branch in nonlinear analysis (see [8, 18-23]). Kakutani's fixed point theorem [7] is an extension of Brouwer's fixed point theorem from single-valued mappings to set-valued mappings. It has been widely applied in set-valued analysis and has been extended to very broad fields (see [5]).

During the last 80 years, set-valued fixed point theorems and related set-valued analysis have been extensively applied in finance theory [24-25], economics theory [1, 16-17, $24-25, 27$], game theory [5, 15], optimization theory [6, 9-14], social sciences [16-17], variational inequalities [24-27], and so forth. The significate of the applications highlights the importance of set-valued fixed point theorems in both pure mathematics and applied mathematics, which gives the motivation of this paper.

The Fan-KKM Theorem has played an important role in nonlinear analysis. It has been used to prove the existence of some fixed point problems; the solvability of

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some optimization problems and variational inequalities (see $[3-4,7,17,24-25]$). In this paper, we apply the Fan-KKM Theorem to prove a fixed point theorem for set-valued mapping in metric vector spaces, in which the considered mappings do not necessarily hold any type of continuity. To extend the concept of fixed point, we introduce the definition of approximating fixed point for set-valued mappings. Then, we use the Fan-KKM Theorem again to prove an approximating fixed point theorem, in which the considered mappings satisfy a certain type of convexity. However, any type of continuity is not necessary. These theorems are extensions to the set-valued case of some theorems given in [15] for single-valued mappings. We provide some examples to demonstrate these theorems.

Since the Fan-KKM Theorem has been extended and has been generalized to very broad underlying spaces and it has many different versions, for easy reference, we briefly review the definition of KKM mappings and the version of the Fan-KKM Theorem used in this paper (see $[3,19,24-25,27]$)

Let C be a nonempty convex subset of a vector space X. A set-valued mapping $G : C \to 2^X \backslash \{\varnothing\}$ is called a KKM mapping if, for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of C, we have

$$\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{1\le i\le n} G(x_i),$$

where $\operatorname{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Fan-KKM Theorem. Let C be a nonempty closed convex subset of a Hausdorff topological vector space X and let $G : C \to 2^X \backslash \{\emptyset\}$ be a KKM mapping with closed values. If there exists a point $x^* \in C$ such that $G(x^*)$ is a compact subset of C, then

$$\bigcap_{x\in C} G(x) \neq \emptyset.$$

2. A fixed point theorem of set-valued mappings

2.1. **The first theorem and its corollaries.** Let (X, d) be a metric vector space. For any nonempty subset A of X and any point $x \in X$, we denote

$$d(A, x) = \inf\{d(a, x) : a \in A\}. \tag{2.1}$$

$d(A, x)$ is called the distance between x and A with respect to the metric d on X.

Let C be a nonempty closed and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping. If $x \in F(x)$, for some $x \in C$, then x is called a fixed point of F.

Theorem 2.1. *Let (X, d) be a metric vector space and let C be a nonempty closed and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(A) $\cup\{F(x) : x \in C\} = C$;

(B$_1$) *For any finite subset* $\{x_1, x_2, \ldots, x_n\} \subseteq C$ *and for any* $u = \sum_{i=1}^n \alpha_i x_i$, *in which* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are positive with* $\sum_{i=1}^n \alpha_i = 1$, *we have*

$$\max\{d(F(x_j), u) - d(x_j, u) : j = 1, 2, \ldots, n\} \geq 0;$$

(C_1) *There is* $x^* \in C$ *such that the following subset of* C *is compact*

$$\{y \in C : d(x^*, y) \leq d(F(x^*), y)\}.$$

Then F *has a fixed point.*

Proof. We define a set-valued mapping $G : C \to 2^C$ by

$$G(x) = \{y \in C : d(x, y) \leq d(F(x), y)\}, \text{ for every } x \in C. \tag{2.2}$$

Since $x \in G(x)$, it follows that $G(x)$ is nonempty, for every $x \in C$. For every fixed $x \in C, F(x)$ is a fixed subset in C. We want to show that $G(x)$ is a nonempty closed subset of C. To this end, let w be an arbitrary given cluster point of $G(x)$. For any $\epsilon > 0$, there is a $y' \in G(x)$ such that $d(w, y') < \epsilon$. By (2.1) and (2.2), for any $a \in F(x)$, we have

$$\begin{aligned} d(x, w) &\leq d\left(x, y'\right) + d\left(w, y'\right) \\ &\leq d\left(F(x), y'\right) + d\left(w, y'\right) \\ &\leq d\left(a, y'\right) + d\left(w, y'\right) \\ &\leq d(a, w) + d\left(w, y'\right) + d\left(w, y'\right) \\ &< d(a, w) + 2\epsilon, \text{ for any } a \in F(x). \end{aligned}$$

It implies that $d(x, w) \leq d(F(x), \mathrm{w})$, which follows that $w \in G(x)$. Hence $G(x)$ is closed.

Next, we show that G is a KKM mapping. To this end, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u be a convex combination of $x_1, x_2, \ldots, x_n$. We can suppose that there are positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $u = \sum_{i=1}^n \alpha_i x_i$. By condition (B$_1$) in this theorem, we have

$$\max\{d(F(x_j), u) - d(x_j, u) : j = 1, 2, \ldots, n\} \geq 0$$

It implies that there must be an integer k with $1 \leq k \leq n$ such that

$$d(x_k, u) \leq d(F(x_k), u).$$

That is,

$$\sum_{i=1}^n \alpha_i x_i = u \in G(x_k) \subseteq \bigcup_{1 \leq j \leq n} G(x_j).$$

This implies that $G : C \to 2^C$ is a KKM mapping with nonempty closed values in C. By condition (C$_1$) and by using Fan-KKM Theorem, we obtain $\cap_{x \in C} G(x) \neq \emptyset$. Then, taking any $y_0 \in \cap_{x \in C} G(x)$, we have

$$d(x, y_0) \leq d(F(x), y_0), \text{ for every } x \in C. \tag{2.3}$$

By condition (A) in this theorem, for an arbitrarily $y_0 \in \cap_{x \in C} G(x) \subseteq C$ satisfying (2.3), there is $x_0 \in C$ such that $F(x_0) \ni y_0$. Substituting x_0 for x in (2.3) gets

$$d(x_0, y_0) \leq d(F(x_0), y_0) = 0.$$

This implies that $x_0 = y_0 \in F(x_0)$; and therefore, x_0 is a fixed point of F, which proves this theorem.

□

Corollary 2.2. *Let (X,d) be a metric vector space and let C be a nonempty compact and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping. If F satisfies conditions (A) and* (B_1) *in Theorem 2.1, then F has a fixed point.*

2.2. **Examples to demonstrate Theorem 2.1.** In the following examples, we consider the most simple metric vector space $(X,d) = (\mathbb{R}, |\cdot|)$. Let C be a closed interval of $\mathbb{R}$. Let F be a set-valued mapping on C. For any nonempty subset A in $\mathbb{R}$ and $x \in \mathbb{R}$, we say

$$x \le A \quad \text{if and only if } x \le a, \text{ for every } a \in A.$$

We can similarly define $x \ge A, x < A$ and $x > A$.

Example 2.3. Let $C = [0,\infty)$ and let F be a set-valued on C with nonempty values satisfying
(i) $F(0) \ge 10$;
(ii) $0 \le F(x) \le x$, for $0 < x \le 5$ and $F(0,5] = [0,5]$;
(iii) $F(x) \ge x$, for $5 < x < \infty$ and $F(5,\infty) = (5,\infty)$.
Then
(I) F satisfies all conditions $(\mathrm{A}, \mathrm{B}_1, \mathrm{C}_1)$ in Theorem 2.1;
(II) F has at least one fixed point, $x = 5$.

Proof. Conditions (i-iii) in this example show that F satisfies condition (A) in Theorem 2.1. We next show that F satisfies conditions (B_1). For any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u be an arbitrary convex combination of $x_1, x_2, \ldots, x_n$ with $u = \sum_{i=1}^n \alpha_i x_i$, for some positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$. Suppose $0 \le x_1 < x_2 < \ldots < x_n < \infty$. It implies that $0 \le x_1 < u < x_n < \infty$. Then, the proof of (B_1) is divided to three cases:

Case 1. $0 < x_1 < u \le 5$. By $F(x_1) \le x_1$, we have

$$0 < u - x_1 \le u - a, \text{ for all } a \in F(x_1).$$

This implies that $d(x_1, u) \le d(F(x_1), u)$.

Case 2. $0 = x_1 < u \le 5$.

$$0 < u - 0 = u \le a - u, \text{ for all } a \in F(0).$$

This implies that $d(0,u) \le d(F(0),u)$.

Case 3. $5 < u < x_n < \infty$. By $F(x_n) \ge x_n$, we have

$$0 < x_n - u \le a - u, \text{ for all } a \in F(x_n).$$

This implies that $d(x_n, u) \le d(F(x_n), u)$.

It follows that F satisfies conditions (B_1). Then, we show that F satisfies condition (C_1) in Theorem 2.1. Take any point x^* with $x^* > 5$. Since $F(x^*) \ge x^* > 0$, we have

$$d(F(x^*), 0) = \inf F(x^*) \ge x^* > 0$$

It follows that

$$\{y \in C : |x^* - y| \leq d(F(x^*), y)\} = \left[0, \frac{x^* + d(F(x^*), 0)}{2}\right]$$

It is a compact subset in C. So F satisfies condition (C_1). $\square$

More specifically, we have the following example, which is a special case of Example 2.3.

Example 2.4. Let $C = [0, \infty)$. Define three continuous functions ξ, φ and ψ on $(0, \infty)$ as follows:

$$\xi(x) = \begin{cases} \frac{1}{4}x, & \text{for } x \in (0,4] \\ 4x - 15, & \text{for } x \in (4,6] \\ 2x - 3, & \text{for } x \in (6,\infty) \end{cases}$$

$$\varphi(x) = \begin{cases} \frac{1}{2}x, & \text{for } x \in (0,4]; \\ 3x - 10, & \text{for } x \in (4,6] \\ \frac{3}{2}x - 1, & \text{for } x \in (6,\infty) \end{cases}$$

and

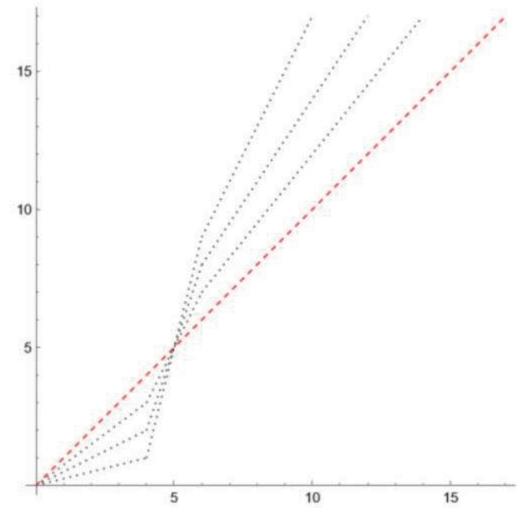
$$\psi(x) = \begin{cases} \frac{3}{4}x, & \text{for } x \in (0,4] \\ 2x - 5, & \text{for } x \in (4,6] \\ \frac{5}{4}x - \frac{1}{2}, & \text{for } x \in (6,\infty) \end{cases}$$

ξ, ϕ, and ψ satisfy the following conditions:
(i) $0 < \xi(x) < \varphi(x) < \psi(x) < x$, for $0 < x < 5$
(ii) $\xi(x) > \varphi(x) > \psi(x) > x$, for $5 < x < \infty$.
Then, we define $F : C \to 2^C$ by

$$F(x) = \begin{cases} [10, 16], & \text{for } x = 0; \\ [\xi(x), \varphi(x)], & \text{for } 0 < x \leq 5 \text{ and } x \text{ is rational;} \\ [\varphi(x), \psi(x)], & \text{for } 0 < x \leq 5 \text{ and } x \text{ is irrational;} \\ [\varphi(x), \xi(x)], & \text{for } 5 < x < \infty \text{ and } x \text{ is rational;} \\ [\psi(x), \varphi(x)], & \text{for } 5 < x < \infty \text{ and } x \text{ is irrational.} \end{cases}$$

Then, we have
(I) F satisfies all conditions $(\mathrm{A}, \mathrm{B}_1, \mathrm{C}_1)$ in Theorem 2.1;
(II) F has one fixed point, $x = 5$.

Proof. As we mentioned before this example, Example 2.4 is a special case of Example 2.3. $\square$



2.3. **Some counter examples regarding to Theorem 2.1.** In this subsection, we give three counter examples to respectively show that every condition in Theorem 2.1 is necessary for the considered mapping to have a fixed point.

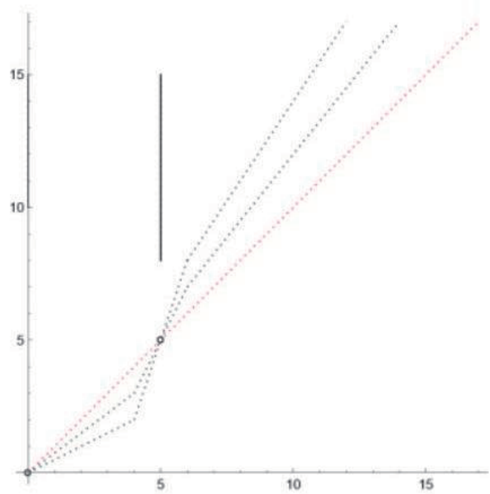
Example 2.5. Let $C = [0, \infty)$ and let $F : C \to 2^C$ be the set-valued function given in Example 2.3. Based on F, we define a set-valued function $H : C \to 2^C$ by

$$H(x) = \begin{cases} F(x), & \text{for } x \in [0, \infty)\backslash\{0, 5\}; \\ [10, 15], & \text{for } x = 0; \\ [8, 15], & \text{for } x = 5. \end{cases}$$

Then

(I) H satisfies condition $(\mathrm{B}_1, \mathrm{C}_1)$ but not (A) in Theorem 2.1;

(II) H has no fixed point.



Example 2.6. Let $C = [0, 10]$. Let $F : C \to 2^C$ be a set-valued mapping satisfying the following conditions:

$$F(x) = \begin{cases} [4,10], & \text{for } x = 0; \\ > x, & \text{for } x \in (0,5); \\ < x, & \text{for } x \in [5,10); \\ [0,6], & \text{for } x = 10. \end{cases}$$

Then

(I) F satisfies condition $(\mathrm{A}, \mathrm{C}_1)$ but not (B_1) in Theorem 2.1;

(II) F has no fixed point.

Proof. Since C is compact and $F(0) \cup F(10) = C$, it follows that F satisfies conditions $(\mathrm{A}, \mathrm{C}_1)$ in Theorem 2.1. We only prove that F does not satisfy condition (B_1) in Theorem 2.1. Take $n = 2$ with $x_1 = 0, x_2 = 10$ and $u = 5$. Then u is a convex combination of $\{x_1, x_2\}$, which satisfies

$$u - x_1 = x_2 - u = 5,$$

and

$$d\left(F\left(x_1\right), u\right) = d\left(F\left(x_2\right), u\right) = 0.$$

It follows that

$$\max\left\{d\left(F\left(x_j\right), u\right) - d\left(x_j, u\right) : j = 1, 2\right\} = -5$$

This shows that F does not satisfy condition (B_1) in Theorem 2.1. It is clear that F has no fixed point. □

Example 2.7. Let $C = (-\infty, \infty)$ and define $F : C \to 2^C$ by

$$F(x) = [x-2, x-1], \text{ for all } x \in C.$$

Then

(I) F satisfies conditions $(\mathrm{A}, \mathrm{B}_1)$ but not (C_1) in Theorem 2.1;

(II) F has no fixed point.

Proof. Similar to the proof of Example 2.2, we can show that F satisfies conditions $(\mathrm{A}, \mathrm{B}_1)$ in Theorem 2.1. For any $x \in (-\infty, \infty)$, from $F(x) = [x-2, x-1]$, we have

$$\begin{aligned} &\{y \in C : |x-y| \leq d(F(x), y)\} \\ =&\{y \in C : |x-y| \leq |x-1-y|\} \\ =&\left[x - \frac{1}{2}, \infty\right). \end{aligned}$$

This implies that, for any $x \in (-\infty, \infty)$, the set $\{y \in C : |x-y| \leq d(F(x), y)\}$ is not compact, which proves that F does not satisfy condition (C_1) in Theorem 2.1. □

3. Approximating fixed point and an approximating fixed point theorem of set-valued mappings in metric vector spaces

3.1. **The second theorem and its corollaries.**

Definition 3.1. Let C be a nonempty subset of a metric space (X, d). Let $F : C \to 2^C$ be a set-valued mapping. If $d(F(x), x) = 0$, for some $x \in C$, then x is called an *approximating fixed point* of F.

Approximating fixed points have the following properties.

1. Every fixed point of F is an approximating fixed point of F;
2. If a set-valued mapping $F : C \to 2^C$ has nonempty closed values, then, x is an approximating fixed point of F if and only if x is a fixed point of F.

Theorem 3.2. *Let (X, d) be a metric vector space and let C be a nonempty closed and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping. Suppose that F satisfies the following conditions:*
(A) $\cup\{F(x) : x \in C\} = C$
(B_2) *For any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$ and for any $u = \sum_{i=1}^n \alpha_i x_i$, in which $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive with $\sum_{i=1}^n \alpha_i = 1$, we have*

$$\max\{d(F(x_j), u) - d(F(x_j), x_j) : j = 1, 2, \ldots, n\} \geq 0$$

(C_2) There is $x^* \in C$ such that the following subset of C is compact

$$\{y \in C : d(F(x^*), x^*) \leq d(F(x^*), y)\}$$

Then F *has an approximating fixed point.*

Proof. The proof of this theorem is similar to the proofs of Theorem 2.1 in this paper and Theorem 3 in [15]. We define a set-valued mapping $G : C \to 2^C$ by

$$G(x) = \{y \in C : d(F(x), x) \leq d(F(x), y)\}, \text{ for every } x \in C$$

It is clear that $x \in G(x)$, for every $x \in C$. Since, for any fixed $x \in C$, $d(F(x), \cdot) - d(F(x), x)$ is a continuous function on C (on X) (the proof is similar to the proof of Theorem 2.1), it implies that, for every $x \in C, G(x)$ is a nonempty closed subset of C.

Next, we show that G is a KKM mapping. To this end, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u be a convex combination of $x_1, x_2, \ldots, x_n$. We can suppose that there are positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $u = \sum_{i=1}^n \alpha_i x_i$. By condition ($B_2$) in this theorem, we have

$$\max\{d(F(x_j), u) - d(F(x_j), x_j) : j = 1, 2, \ldots, n\} \geq 0$$

This implies that there must be an integer k with $1 \leq k \leq n$ such that

$$d(F(x_k), x_k) \leq d(F(x_k), u)$$

That is,

$$\sum_{i=1}^{n} \alpha_i x_i = u \in G(x_k) \subseteq \bigcup_{1 \le j \le n} G(x_j)$$

This implies that $G : C \to 2^C$ is a KKM mapping with nonempty closed values in C. By condition (C_2) and by using Fan-KKM Theorem, we obtain $\cap_{x \in C} G(x) \neq \emptyset$. Then, taking any $y_0 \in \cap_{x \in C} G(x)$, we have

$$d(F(x), x) \le d(F(x), y_0), \text{ for every } x \in C \tag{3.1}$$

By condition (A) in this theorem, for an arbitrarily selected $y_0 \in \cap_{x \in C} G(x) \subseteq C$ satisfying (3.1), there is $x_0 \in C$ such that $F(x_0) \ni y_0$. Substituting x_0 for x in (3.1) gives

$$d(F(x_0), x_0) \le d(F(x_0), y_0) = 0$$

This implies that x_0 is an approximating fixed point of F, which proves this theorem. □

Corollary 3.3. *Let (X, d) be a metric vector space and let C be a nonempty compact and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping. If F satisfies conditions* (A) *and* (B_1) *in Theorem 3.2, then F has an approximating fixed point.*

From the properties of approximating fixed points of set-valued mappings, as a consequence of Theorem 3.2, we have the following fixed point theorem for mappings with nonempty closed values in metric vector spaces.

Proposition 3.4. *Let (X, d) be a metric vector space and C be a nonempty closed and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping satisfying all conditions (A, B_2, C_2) in Theorem 3.2. If, in addition, F has nonempty closed values, then F has a fixed point.*

In particular, Corollary 3.3 and Proposition 3.4 induce the following fixed point theorem on nonempty compact and convex subsets of metric vector spaces.

Corollary 3.5. *Let (X, d) be a metric vector space and let C be a nonempty compact and convex subset of X. Let $F : C \to 2^C$ be a set-valued mapping. If F has nonempty closed values and satisfies conditions* (A) *and* (B_1) *in Theorem* 3.2*, then F has a fixed point.*

3.2. **Some examples to demonstrate Corollary 3.3.** In this subsection, we provide some examples to demonstrate Corollary 3.3. In the following subsection, we will give some examples to demonstrate Theorem 3.2.

Example 3.6. Let $C = [0, 10]$. We define $F : C \to 2^C$ by

$$F(x) = \begin{cases} (0, 10], & \text{for } x = 0; \\ [0, x), & \text{for } 0 < x \le 10, \text{ and } x \text{ is rational}; \\ (x, 10], & \text{for } 0 < x \le 10, \text{ and } x \text{ is irrational}. \end{cases}$$

Then, we have

(I) F satisfies conditions $(\mathrm{A}, \mathrm{B}_2)$ in Corollary 3.3;
(II) Every point in $[0, 10]$ is an approximating fixed point of F;
(III) F does not have fixed point.

Proof. Parts (II) and (III) are easy to see. It is clear that F satisfies condition (A) in Corollary 3.3. Notice that

$$d(F(x), x) = 0, \text{ for every } x \in [0, 10]$$

and

$$d(F(x), u) \geq 0, \text{ for any } x, u \in [0, 10] \text{ with } u \neq x$$

These prove that F satisfies conditions (B_2) in Corollary 3.3. □

Example 3.7. Let $C = [0, 10]$. Let $F : C \to 2^C$ be a set-valued mapping. Suppose that F satisfies condition (A) in Corollary 3.3 and the values of F satisfy the following conditions

$$F(x) \leq x, \text{ for } x \in [0, 5] \quad \text{ and } F(x) \geq x, \text{ for } x \in (5, 10]$$

Then
(I) F satisfies conditions $(\mathrm{A}, \mathrm{B}_2)$ in Corollary 3.3;
(II) F has at least two fixed points, $x = 0$ and $x = 10$.

Proof. Part (II) is easily seen. We only show part (I). For any $x \in C$, we have

$$d(F(x), x) = \begin{cases} x - \sup F(x), & \text{for } 0 \leq x \leq 5 \\ \inf F(x) - x, & \text{for } 5 < x \leq 10 \end{cases}$$

To prove (I), we simply take $n = 2$ and take arbitrary x_1, u, x_2 with $0 \leq x_1 < u < x_2 \leq 10$. By the above equalities, the proof of (B_2) is divided into two cases.

Case 1. $0 \leq x_1 < u \leq 5$. We have

$$\begin{aligned} & d\left(F\left(x_1\right), x_1\right) \\ = & x_1 - \sup F\left(x_1\right) \\ < & u - \sup F\left(x_1\right) \\ = & d\left(F\left(x_1\right), u\right). \end{aligned}$$

Case 2. $5 < u < x_2 \leq 10$. We have

$$\begin{aligned} & d\left(F\left(x_2\right), x_2\right) \\ = & \inf F\left(x_2\right) - x_2 \\ < & \inf F\left(x_2\right) - u \\ = & d\left(F\left(x_2\right), u\right). \end{aligned}$$

These imply that F satisfies condition (B_2) in Corollary 3.3. □

Example 3.8. Let $C = [0, 10]$. Let $F : C \to 2^C$ be a set-valued mapping. Suppose that F satisfies condition (A) in Corollary 3.3 and the values of F satisfy the following conditions

$$F(x) \leq x, \text{ for every } x \in [0, 10] \quad (\text{or } F(x) \geq x, \text{ for every } x \in [0, 10].\ )$$

Then
(I) F satisfies conditions (B_2) in Corollary 3.3;
(II) F has at least two fixed points, $x = 0$ and $x = 10$.

Proof. The proof of this example is similar to the proof of Example 3.7 and it is omitted. □

3.3. **Some examples to demonstrate Theorem 3.2.** In the previous subsection, we give some examples to demonstrate Corollary 3.3, which is a special case of Theorem 3.2. Considering the significant difference between compact subsets and (just) closed subsets, in this subsection, we will construct some examples to demonstrate Theorem 3.2, in which the underlying subsets are not compact.

Example 3.9. Let $C = [0, \infty)$. Define $F : C \to 2^C$ by

$$F(x) = [2x, \infty), \text{ for every } x \geq 0.$$

Then
(I) F satisfies conditions (A, B_2, C_2) in Theorem 3.2;
(II) F has one fixed point, $x = 0$.

Proof. It is clear that F satisfies condition (A). To prove that F satisfies condition (B_2) in Theorem 3.2, we simply take $n = 2$ and take arbitrary x_1, u, x_2 with $0 \leq x_1 < u < x_2 < \infty$. We have

$$d\left(F\left(x_2\right), u\right) = 2x_2 - u > x_2 = d\left(F\left(x_2\right), x_2\right).$$

This proves that F satisfies condition (B_2). Finally, we show that F satisfies condition (C_2). For every $x > 0$, we have

$$\{y \in C : d(F(x), x) \leq d(F(x), y)\} = \{y \in C : x \leq 2x - y\} = [0, x].$$

This is compact, which proves that F satisfies condition (C_2). It is clear that F has a fixed point, $x = 0$. □

Example 3.10. Let $C = [0, \infty)$. Define $F : C \to 2^C$ by

$$F(x) = \begin{cases} [0, 2x], & \text{for } 0 \leq x \leq 1 \\ [x + 2, \infty), & \text{for } x > 1 \end{cases}$$

Then
(I) F satisfies conditions (A, B_2, C_2) in Theorem 3.2,
(II) $\mathcal{F}(F) = [0, 1]$,
where $\mathcal{F}(F)$ denotes the collection of fixed points of F.

Proof. We only prove that F satisfies conditions $(\mathrm{B}_2, \mathrm{C}_2)$ in Theorem 3.2. To prove that F satisfies condition (B_2) in Theorem 3.2, we simply take $n = 2$ and take arbitrary x_1, u, x_2 with $0 \leq x_1 < u < x_2 < \infty$. We have the following two cases:

Case 1. $0 \leq x_1 < u < x_2 \leq 1$. In this case, we have

$$d\left(F\left(x_1\right), x_1\right) = d\left(F\left(x_2\right), u\right) = d\left(F\left(x_2\right), x_2\right) = 0$$

Case 2. $0 \leq x_1 < u < x_2$ and $x_2 > 1$. In this case, we have

$$d\left(F\left(x_2\right), u\right) > d\left(F\left(x_2\right), x_2\right) = 2$$

These prove that F satisfies condition (B_2). Then, we show that F satisfies condition (C_2). For every $x > 1$, we have

$$\{y \in C : d(F(x), x) \leq d(F(x), y)\} = \{y \in C : 2 \leq 2x - y\} = [0, 2x - 2]$$

This is compact, which proves that F satisfies condition (C_2). It is clear that $\mathcal{F}(F) = [0, 1]$. □

3.4. **Some counter examples regarding to Theorem 3.2.** In this subsection, we give three counter examples to respectively show that every condition in Theorem 3.2 is necessary for the considered mapping to have an approximating fixed point.

Example 3.11. Let $C = [0, \infty)$. Define $F : C \to 2^C$ by

$$F(x) = [x + 1, \infty), \text{ for every } x \geq 0$$

Then,

(I) F satisfies conditions $(\mathrm{B}_2, \mathrm{C}_2)$ but not (A) in Theorem 3.2;

(II) F does not have any approximating fixed point.

Proof. It is clear to see that F does not satisfy condition (A) in Theorem 3.2. We prove that F satisfies condition (B_2) in Theorem 3.2. To this end, we simply take $n = 2$ and take arbitrary x_1, u, x_2 with $0 \leq x_1 < u < x_2 < \infty$. We have

$$d\left(F\left(x_2\right), u\right) > d\left(F\left(x_2\right), x_2\right) = 1$$

This proves that F satisfies condition (B_2). Finally, we show that F satisfies condition (C_2). Notice that, for every $x \geq 0$, we have

$$\{y \in C : d(F(x), x) \leq d(F(x), y)\} = \{y \in C : 1 \leq d(F(x), y)\} = [0, x].$$

This is compact, which proves that F satisfies condition (C_2). It is clear that F does not have any approximating fixed point. □

Example 3.12. Let $C = [0, 10]$. Define $F : C \to 2^C$ by

$$F(x) = \begin{cases} [5, 10], & \text{for } x = 0 \\ \left[0, \frac{x}{2}\right], & \text{for } 0 < x \leq 10 \end{cases}$$

Then,

(I) F satisfies conditions $(\mathrm{A}, \mathrm{C}_2)$ but not (B_2) in Theorem 3.2;

(II) F does not have any approximating fixed point.

Proof. We only prove that F does not satisfy condition (B_2) in Theorem 3.2. We simply take $n = 2$ and take arbitrary x_1, u, x_2 with $0 = x_1 < u < x_2 \leq 5$, we have

$$d\left(F\left(x_1\right), x_1\right) = d(F(0), 0) = 5 > 5 - u = d(F(0), u) = d\left(F\left(x_1\right), u\right),$$

and

$$d\left(F\left(x_2\right), x_2\right) = \frac{x_2}{2} > d\left(\left[0, \frac{x_2}{2}\right], u\right) = d\left(F\left(x_2\right), u\right).$$

This proves that F does not satisfy condition (B_2) in Theorem 3.2. It is clear that F does not have any approximating fixed point.

□

Example 3.13. Let $C = (-\infty, \infty)$. Define $F : C \to 2^C$ by

$$F(x) = [x + 1, x + 2], \text{ for all } x \in C.$$

Then,

(I) F satisfies conditions $(\mathrm{A}, \mathrm{B}_2)$ but not (C_2) in Theorem 3.2;

(II) F does not have any approximating fixed point.

Proof. It is clear that F satisfies condition (A) in Theorem 3.2. Similarly to the proof of Example 3.9, we can show that F satisfies condition (B_2). Next we show that F does not satisfy condition (C_2) in Theorem 3.2. For every $x \in C$, we calculate

$$\begin{aligned} \{y \in C : d(F(x), x) \leq d(F(x), y)\} &= \{y \in C : 1 \leq d(F(x), y)\} \\ &= (-\infty, x] \cup [x + 3, \infty) \end{aligned}$$

This proves that F does not satisfy condition (C_2). It is clear that F does not have any approximating fixed point. □

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