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EXISTENCE AND STABILITY OF APPROXIMATE FIXED POINTS OF NONEXPANSIVE SET-VALUED MAPPINGS

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Abstract. In our recent research we studied a space of single-valued nonexpansive mappings, acting on a closed convex subset of a hyperbolic metric space, which is equipped with the topology of uniform convergence on bounded sets and showed the existence of an open and everywhere dense subset in this space such that every its element possesses an approximate fixed point, which is stable under small perturbations. In the present paper we extend these results for a space of set-valued nonexpansive mappings.

1. Introduction

During more than fifty-five years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 5, 11, 13, 14, 17–20, 22–24, 26–29, 32, 33] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [2, 4, 6–10, 12, 15, 16, 21, 30–33].

In [34] we studied a space of single-valued nonexpansive mappings, acting on a closed convex subset of a hyperbolic metric space, which is equipped with the topology of uniform convergence on bounded sets. It was shown that there exists an open and everywhere dense subset in this space such that every its element possesses an approximate fixed point, which is stable under small perturbations. In the present paper we extend these results for a space of set-valued nonexpansive mappings.

As a matter of fact, it turns out that our results also hold for nonexpansive self-mappings of closed and convex sets in complete hyperbolic spaces, an important class of metric spaces the definition of which we recall in the next section.

2. Hyperbolic spaces

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding will be called a

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metric line. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping will be called a metric segment.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point will be denoted by $(1 - t)x \oplus ty$. We will say that X, or more precisely (X, ρ, M), is a hyperbolic space if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. A set $K \subset X$ is called ρ-convex if $[x, y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and in particular of the Hilbert ball can be found, for example, in [25, 26].

3. Set-valued mappings

Let (X, ρ, M) be a complete hyperbolic space and let K be a nonempty closed ρ-convex subset of X.

For each $x \in K$ and each $r > 0$ set

$$B(x, r) = \{y \in K : \rho(x, y) \le r\}.$$

For each $x \in X$ and each nonempty set $D \subset X$ set

$$\rho(x, D) = \inf\{\rho(x, y) : y \in D\}.$$

Denote by $S(K)$ the collection of all nonempty closed bounded subsets of K.

For each $A, B \in S(K)$ set

$$H(A, B) = \max\{\sup\{\rho(x, B) : x \in A\}, \sup\{\rho(x, A) : x \in B\}\}.$$

Clearly, $(S(K), H)$ is a metric space. It is known that the metric space $(S(K), H)$ is complete.

Denote by $\mathcal{M}$ the set of all mappings $A : K \to S(K)$ such that

$$H(A(x), A(y)) \le \rho(x, y) \text{ for all } x, y \in K. \tag{3.1}$$

Fix $\theta \in K$. We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\mathcal{U}(n) = \{(A, B) \in \mathcal{M} \times \mathcal{M} :$$

$$H(A(x), B(x)) \le n^{-1} \text{ for all } x \in B(\theta, n)\}, \tag{3.2}$$

where n is a natural number. Clearly, the uniform space $\mathcal{M}$ is metrizable and complete.

For each $D \subset X$ denote by $\mathrm{cl}(D)$ the closure of D.

For each nonempty set $D \subset X$ and each $A \in \mathcal{M}$ set

$$A(D) = \cup\{A(x) : x \in D\}.$$

The following theorem is our first main result.

Theorem 3.1. *Let $\bar{r} > 0$, $\bar{M} > 0$ and $\epsilon \in (0,1)$. Then there exists an open everywhere dense subset $\mathcal{F} \subset \mathcal{M}$ such that for each $B \in \mathcal{F}$ there exist a natural number n_B, $\delta_B \in (0, \bar{r})$ and an open neighborhood $\mathcal{U}$ of B in $\mathcal{M}$ such that the following assertion holds.*

Let $C_i \in \mathcal{U}$ for all integers $i \geq 0$, n_1 be a natural number and a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfy

$$\rho(x_0, \theta) \leq \bar{M},$$

$$x_{i+1} \in C_i(x_i),\ \rho(x_i, x_{i+1}) \leq \rho(x_i, C_i(x_i)) + \bar{r}$$

for all integers $i \geq 0$ and

$$\rho(x_i, x_{i+1}) \leq \rho(x_i, C_i(x_i)) + \delta_B \text{ for all integers } i \geq n_1.$$

Then

$$\rho(x_i, x_{i+1}) \leq \epsilon \text{ for all integers } i \geq n_1 + n_B.$$

Moreover, if an integer $j \geq n_1$ satisfies $\rho(x_j, x_{j+1}) \leq \epsilon$, then

$$\rho(x_i, x_{i+1}) \leq \epsilon \text{ for all integers } i \geq j.$$

Corollary 3.2. *Let $\bar{r} > 0$, $\bar{M} > 0$ and $\epsilon \in (0,1)$ and let an open everywhere dense subset $\mathcal{F} \subset \mathcal{M}$ be as guaranteed by Theorem 3.1. Assume that $B \in \mathcal{F}$ and an open neighborhood $\mathcal{U}$ of B in $\mathcal{M}$ is as as guaranteed by Theorem 3.1.*

Let $C_i \in \mathcal{U}$ for all integers $i \geq 0$ and a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ be such that

$$\rho(x_0, \theta) \leq \bar{M},$$

$$x_{i+1} \in C_i(x_i),\ \rho(x_i, x_{i+1}) \leq \rho(x_i, C_i(x_i)) + \bar{r}$$

for all integers $i \geq 0$ and

$$\lim_{i \to \infty} (\rho(x_i, x_{i+1}) - \rho(x_i, C_i(x_i))) = 0.$$

Then

$$\rho(x_i, x_{i+1}) \leq \epsilon \text{ for all sufficiently large natural numbers } i.$$

The following theorem is our second main result.

Theorem 3.3. *Let $\bar{M} > 0$ and $\epsilon \in (0,1)$. Then there exists an open everywhere dense subset $\mathcal{F} \subset \mathcal{M}$ such that for each $B \in \mathcal{F}$ there exist $x_B \in K$, a natural number n_B and an open neighborhood $\mathcal{U}$ of B in $\mathcal{M}$ such that the following assertion holds.*

Let $C_i \in \mathcal{U}$, $i = 0, 1, \ldots$ and $x_0 \in K$ satisfy

$$\rho(x_0, \theta) \leq \bar{M}.$$

Then there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that

$$x_{i+1} \in C_i(x_i) \text{ for all integers } i \geq 0$$

and

$$\rho(x_i, x_B) \leq \epsilon \text{ for all integers } i \geq n_B.$$

Theorem 3.3 implies the following result.

Theorem 3.4. *There exists a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that for each $B \in \mathcal{F}$, each $M > 0$ and each $\epsilon > 0$ there exist $x_B \in K$, a natural number n_B and an open neighborhood $\mathcal{U}$ of B in $\mathcal{M}$ such that the following assertion holds.*

Let $C_i \in \mathcal{U}$, $i = 0, 1, \ldots$ and $x_0 \in K$ satisfy

$$\rho(x_0, \theta) \leq M.$$

Then there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that

$$x_{i+1} \in C_i(x_i) \text{ for all integers } i \geq 0$$

and

$$\rho(x_i, x_B) \leq \epsilon \text{ for all integers } i \geq n_B.$$

4. Proofs of Theorems 3.1 and 3.3

We prove Theorems 3.1 and 3.3 simultaneously. In the case of Theorem 3.3, set $\bar{r} = 2^{-1}$. We begin with the construction of the set $\mathcal{F}$.

Let $A \in \mathcal{M}$ and $\gamma \in (0, 1)$. Define

$$A_\gamma(x) = \mathrm{cl}(\{(1-\gamma)z \oplus \gamma\theta : \ z \in A(x)\}), \ x \in K. \tag{4.1}$$

We show that $A_\gamma \in \mathcal{M}$ and for each $x, y \in K$,

$$H(A_\gamma(x), A_\gamma(y)) \leq (1-\gamma)\rho(x, y). \tag{4.2}$$

Let $x, y \in K$ and $z \in A_\gamma(x)$. In order to prove (4.2) it is sufficient to show that

$$\rho(z, A_\gamma(y)) \leq (1-\gamma)\rho(x, y).$$

We may assume without loss of generality that

$$z \in \{(1-\gamma)\xi \oplus \gamma\theta : \ \xi \in A(x)\}.$$

In other words there exists

$$\xi \in A(x)$$

such that

$$z = (1-\gamma)\xi \oplus \gamma\theta.$$

Then

$$\begin{aligned}
\rho(z, A_\gamma(y)) &= \rho((1-\gamma)\xi \oplus \gamma\theta, A_\gamma(y)) \\
&= \inf\{\rho((1-\gamma)\xi \oplus \gamma\theta, (1-\gamma)\eta \oplus \gamma\theta) : \ \eta \in A(y)\} \\
&\leq (1-\gamma)\inf\{\rho(\xi, \eta) : \ \eta \in A(y)\} = (1-\gamma)\rho(\xi, A(y)) \\
&\leq (1-\gamma)H(A(x), H(y)) \leq (1-\gamma)\rho(x, y).
\end{aligned}$$

Therefore (4.2) holds for each $x, y \in K$.

Choose

$$M_0(A, \gamma) > (\rho(\theta, A(\theta)) + 1)\gamma^{-1}.$$

We show that

$$A_\gamma(B(\theta, M_0(A, \gamma))) \subset B(\theta, M_0(A, \gamma) - 1). \tag{4.3}$$

Assume that

$$z \in B(\theta, M_0(A,\gamma)) \text{ and } \xi \in A_\gamma(z).$$

In order to prove (4.3) it is sufficient to show that

$$\xi \in B(\theta, M_0(A,\gamma) - 1).$$

Clearly,

$$\xi \in \mathrm{cl}(\{(1-\gamma)y \oplus \gamma\theta : \ y \in A(z)\}).$$

We may assume without loss of generality that there exists

$$y \in A(z))$$

such that

$$\xi = (1-\gamma)y \oplus \gamma\theta.$$

It follows from the equations above that

$$\begin{aligned}
\rho(\xi,\theta) &= \rho((1-\gamma)y \oplus \gamma\theta, \theta) \le (1-\gamma)\rho(y,\theta) \\
&\le (1-\gamma)\rho(\theta, A(z)) \\
&\le (1-\gamma)(\rho(\theta, A(\theta)) + H(A(\theta), A(z))) \\
&\le \rho(\theta, A(\theta)) + (1-\gamma)\rho(\theta, z) \\
&\le \rho(\theta, A(\theta)) + (1-\gamma)M_0(A,\gamma) \\
&\le M_0(A,\gamma) - \gamma M_0(A,\gamma) + \rho(\theta, A(\theta)) \\
&\le M_0(A,\gamma) - 1.
\end{aligned}$$

In view of (4.2), (4.3) and the Nadler fixed point theorem [20], there exists

$$x(A,\gamma) \in B(\theta, M_0(A,\gamma)) \tag{4.4}$$

such that

$$x(A,\gamma) \in A_\gamma(x(A,\gamma)). \tag{4.5}$$

It follows from (4.3)-(4.5) that

$$x(A,\gamma) \in B(\theta, M_0(A,\gamma) - 1). \tag{4.6}$$

Choose a positive number $M(A,\gamma)$ such that

$$\begin{aligned}
&M(A,\gamma) > 2M_0(A,\gamma) + 2\bar{M} + 2 \\
&+(\sup\{\rho(\xi_1,\xi_2) : \ \xi_1, \xi_2 \in A_\gamma(x(A,\gamma))\} + 2)\gamma^{-1}.
\end{aligned} \tag{4.7}$$

By (4.6) and (4.7),

$$\begin{aligned}
B(\theta, \bar{M}) &\subset B(x(A,\gamma), \bar{M} + M_0(A,\gamma) - 1) \\
&\subset B(x(A,\gamma), M(A,\gamma) - 1).
\end{aligned} \tag{4.8}$$

We show that

$$A_\gamma(B(x(A,\gamma), M(A,\gamma))) \subset B(x(A,\gamma), M(A,\gamma) - 1). \tag{4.9}$$

Let

$$z \in A_\gamma(B(x(A,\gamma), M(A,\gamma))). \tag{4.10}$$

In view of (4.10) there exists

$$y \in B(x(A,\gamma), M(A,\gamma)) \tag{4.11}$$

such that

$$z \in A_\gamma(y). \tag{4.12}$$

By (4.2) and (4.12),

$$\begin{aligned} \rho(z, A_\gamma(x(A,\gamma))) &\le H(A_\gamma(y), A_\gamma(x(A,\gamma))) \\ &\le (1-\gamma)\rho(y, x(A,\gamma)). \end{aligned} \tag{4.13}$$

It follows from (4.5), (4.7), (4.11) and (4.13) that

$$\begin{aligned} &\rho(z, x(A,\gamma)) \\ &\le \rho(z, A_\gamma(x(A,\gamma))) + \sup\{\rho(\xi_1,\xi_2) :\ \xi_1,\xi_2 \in A_\gamma(x(A,\gamma))\} \\ &\le (1-\gamma)\rho(y, x(A,\gamma)) + \sup\{\rho(\xi_1,\xi_2) :\ \xi_1,\xi_2 \in A_\gamma(x(A,\gamma))\} \\ &\le (1-\gamma)M(A,\gamma) + \sup\{\rho(\xi_1,\xi_2) :\ \xi_1,\xi_2 \in A_\gamma(x(A,\gamma))\} \\ &= M(A,\gamma) - \gamma M(A,\gamma) + \sup\{\rho(\xi_1,\xi_2) :\ \xi_1,\xi_2 \in A_\gamma(x(A,\gamma))\} \\ &< M(A,\gamma) - 1. \end{aligned}$$

Thus (4.9) holds.

Choose $\delta(A,\gamma) \in (0,\bar{r})$ such that

$$\delta(A,\gamma) < 12^{-1}\gamma\epsilon. \tag{4.14}$$

There exists an open neighborhood $\mathcal{U}(A,\gamma)$ of A_γ in $\mathcal{M}$ such that

$$\begin{aligned} \mathcal{U}(A,\gamma) \subset \{C \in \mathcal{M} :\ & H(C(z), A_\gamma(z)) \le \delta(A,\gamma) \\ & \text{for all } z \in B(x(A,\gamma), M(A,\gamma))\}. \end{aligned} \tag{4.15}$$

By (4.1), for each $z \in K$,

$$\begin{aligned} H(A(z), A_\gamma(z)) &= H(A(z), \mathrm{cl}(\{(1-\gamma)\xi \oplus \gamma\theta :\ \xi \in A(z)\})) \\ &\le \gamma \sup\{\rho(\theta,\xi) :\ \xi \in A(z)\}. \end{aligned} \tag{4.16}$$

For each $z \in K$ and each $\xi \in A(z)$,

$$\begin{aligned} \rho(\theta,\xi) &\le \rho(\xi, A(\theta)) + \sup\{\rho(\eta_1,\eta_2) :\ \eta_1,\eta_2 \in A(\theta)\} \\ &\quad + \sup\{\rho(\theta,\eta) :\ \eta \in A(\theta)\} \\ &\le \sup\{\rho(\eta_1,\eta_2) :\ \eta_1,\eta_2 \in A(\theta)\} \\ &\quad + \sup\{\rho(\theta,\eta) :\ \eta \in A(\theta)\} + H(A(z), A(\theta)) \\ &\le \sup\{\rho(\theta,\eta) :\ \eta \in A(\theta)\} + \rho(z,\theta) \\ &\quad + \sup\{\rho(\eta_1,\eta_2) :\ \eta_1,\eta_2 \in A(\theta)\} \\ &\le \rho(z,\theta) + 3\sup\{\rho(\theta,\eta) :\ \eta \in A(\theta)\}. \end{aligned} \tag{4.17}$$

By (4.16) and (4.17), for each $z \in K$,

$$H(A(z), A_\gamma(z)) \le \gamma(\rho(z,\theta) + 3\sup\{\rho(\theta,\eta) :\ \eta \in A(\theta)\}).$$

By the relation above, for each neighborhood V of A in $\mathcal{M}$ there exists $\gamma_V \in (0,1)$ such that for each $\gamma \in (0, \gamma_V]$, $A_\gamma \in V$. Therefore

$$\{A_\gamma : \ A \in \mathcal{M}, \ \gamma \in (0,1)\}$$

is an everywhere dense subset of $\mathcal{M}$. Set

$$\mathcal{F} = \cup\{\mathcal{U}(A,\gamma) : \ A \in \mathcal{M}, \ \gamma \in (0,1)\}. \tag{4.18}$$

Clearly, $\mathcal{F}$ is an open everywhere dense subset of $\mathcal{M}$.

Completion of the proof of Theorem 3.1

Assume that

$$B \in \mathcal{F}. \tag{4.19}$$

By (4.18) and (4.19), there exist $A \in \mathcal{M}$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(A,\gamma). \tag{4.20}$$

Set

$$\delta_B = \delta(A,\gamma) \tag{4.21}$$

and choose a natural number n_B such that

$$n_B > 4M(A,\gamma)(\gamma\epsilon)^{-1} + 4. \tag{4.22}$$

Assume that

$$C_i \in \mathcal{U}(A,\gamma), \ i = 0,1,\dots, \tag{4.23}$$

n_1 is a natural number and that a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies

$$\rho(x_0,\theta) \le \bar{M}, \tag{4.24}$$

$$x_{i+1} \in C_i(x_i), \ \rho(x_i,x_{i+1}) \le \rho(x_i, C_i(x_i)) + \bar{r} \tag{4.25}$$

for all integers $i \ge 0$ and

$$\rho(x_i,x_{i+1}) \le \rho(x_i, C_i(x_i)) + \delta_B \text{ for all integers } i \ge n_1. \tag{4.26}$$

In order to complete the proof it is sufficient to show the following:

$$\rho(x_i,x_{i+1}) \le \epsilon \text{ for all integers } i \ge n_1 + n_B;$$

if an integer $j \ge n_1$ satisfies $\rho(x_j,x_{j+1}) \le \epsilon$, then

$$\rho(x_{j+1},x_{j+2}) \le \epsilon.$$

First we show that

$$x_i \in B(x(A,\gamma), M(A,\gamma) - 1/2) \tag{4.27}$$

for all integers $i \ge 0$. In view of (4.8) and (4.24), inclusion (4.27) holds for $i = 0$.

Assume that an integer $j \ge 0$ and (4.27) holds with $i = j$. By (4.9) and (4.27) with $i = j$,

$$A_\gamma(x_j) \subset B(x(A,\gamma), M(A,\gamma) - 1). \tag{4.28}$$

In view of (4.15), (4.23) and (4.27) (with $i = j$),

$$H(C_j(x_j), A_\gamma(x_j)) \le \delta(A,\gamma).$$

Together with (4.25) this implies that

$$\rho(x_{j+1}, A_\gamma(x_j)) \le H(C_j(x_j), A_\gamma(x_j)) \le \delta(A, \gamma).$$

Combined with (4.14) and (4.28) this implies that

$$\rho(x_{j+1}, B(x(A,\gamma), M(A,\gamma) - 1)) \le \delta(A,\gamma) < 1/4$$

and

$$\rho(x_{j+1}, x(A,\gamma)) < M(A,\gamma) - 1/2.$$

Thus (4.27) holds for $i = j+1$. Therefore we have shown by induction that (4.27) holds for all integers $i \ge 0$.

We show that there exists an integer $i \in [n_1, n_1 + n_B]$ such that

$$\rho(x_i, x_{i+1}) \le \epsilon.$$

Assume the contrary. Then

$$\rho(x_i, x_{i+1}) > \epsilon \tag{4.29}$$

for all $i = n_1, \dots, n_1 + n_B$. Assume that an integer i satisfies

$$i \in \{n_1, \dots, n_1 + n_B - 1\}. \tag{4.30}$$

In view of (4.26) and (4.30),

$$\rho(x_{i+1}, x_{i+2}) \le \rho(x_{i+1}, C_{i+1}(x_{i+1})) + \delta_B. \tag{4.31}$$

By (4.15), (4.21), (4.23) and (4.27),

$$H(C_j(x_j), A_\gamma(x_j)) \le \delta(A,\gamma) = \delta_B \tag{4.32}$$

for all integers $j \ge 0$. It follows from (4.2), (4.25) and (4.32) that

$$\begin{aligned} \rho(x_{i+1}, C_{i+1}(x_{i+1})) &\le H(C_i(x_i), C_{i+1}(x_{i+1})) \\ &\le H(C_i(x_i), A_\gamma(x_i)) + H(A_\gamma(x_i), A_\gamma(x_{i+1})) \\ &\quad + H(A_\gamma(x_{i+1}), C_{i+1}(x_{i+1})) \\ &\le 2\delta(A,\gamma) + H(A_\gamma(x_i), A_\gamma(x_{i+1})) \\ &\le 2\delta(A,\gamma) + (1-\gamma)\rho(x_i, x_{i+1}). \end{aligned} \tag{4.33}$$

By (4.21), (4.26), (4.30) and (4.33),

$$\rho(x_{i+1}, x_{i+2}) \le 3\delta(A,\gamma) + (1-\gamma)\rho(x_i, x_{i+1}).$$

Together with (4.14), (4.29) and (4.30) this implies that

$$\begin{aligned} \rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2}) &\ge \gamma\rho(x_i, x_{i+1}) - 3\delta(A,\gamma) \\ &\ge \gamma\epsilon - 3\delta(A,\gamma) \ge \gamma\epsilon/2 \end{aligned}$$

and

$$\rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2}) \ge 2^{-1}\gamma\epsilon.$$

Since the inequality above holds for all integers i satisfying (4.30) it follows from (4.27) that

$$\begin{aligned} 2M(A,\gamma) &\ge \rho(x_{n_1}, x_{n_1+1}) - \rho(x_{n_1+n_B-1}, x_{n_1+n_B}) \\ &= \sum_{i=n_1}^{n_1+n_B-2} (\rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2})) \end{aligned}$$

$$\geq (n_B - 1)\gamma\epsilon/2$$

and

$$n_B \leq 4(\gamma\epsilon)^{-1}M(A,\gamma) + 1.$$

This contradicts (4.22). The contradiction we have reached proves that there exists an integer

$$j \in [n_1, n_1 + n_B] \tag{4.34}$$

such that

$$\rho(x_j, x_{j+1}) \leq \epsilon. \tag{4.35}$$

Assume that an integer j satisfies

$$j \geq n_1,\ \rho(x_j, x_{j+1}) \leq \epsilon. \tag{4.36}$$

It follows from (4.2), (4.14), (4.21), (4.25), (4.26), (4.32) and (4.36) that

$$\begin{aligned}\rho(x_{j+1}, x_{j+2}) &\leq \rho(x_{j+1}, C_{j+1}(x_{j+1})) + \delta_B \\ &\leq \delta_B + H(C_j(x_j), C_{j+1}(x_{j+1})) \\ &\leq \delta_B + H(C_j(x_j), A_\gamma(x_j)) + H(A_\gamma(x_j), A_\gamma(x_{j+1})) \\ &\quad + H(A_\gamma(x_{j+1}), C_{j+1}(x_{j+1})) \\ &\leq 3\delta(A,\gamma) + (1-\gamma)\rho(x_j, x_{j+1}) \\ &\leq 3\delta(A,\gamma) + (1-\gamma)\epsilon \leq \epsilon.\end{aligned}$$

Thus if an integer j satisfies (4.36), then $\rho(x_{j+1}, x_{j+2}) \leq \epsilon$. Combined with (4.34) and (4.35) this completes the proof of Theorem 3.1.

Completion of the proof of Theorem 3.3

Let

$$B \in \mathcal{F}. \tag{4.37}$$

By (4.18) and (4.37), there exist $A \in \mathcal{M}$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(A,\gamma). \tag{4.38}$$

Set

$$x_B = x(A,\gamma) \tag{4.39}$$

and choose a natural number n_B such that

$$n_B > 4 + 2M(A,\gamma)(\gamma\epsilon)^{-1}. \tag{4.40}$$

Assume that

$$C_i \in \mathcal{U}(A,\gamma),\ i = 0, 1, \ldots, \tag{4.41}$$

$$x_0 \in B(\theta, \bar{M}). \tag{4.42}$$

By induction define a sequence $\{x_i\}_{i=0}^\infty \subset K$ such that for each integer $i \geq 0$,

$$x_{i+1} \in C_i(x_i), \tag{4.43}$$

$$\rho(x(A,\gamma), x_{i+1}) \leq \rho(x(A,\gamma), C_i(x_i)) + 4^{-1}\delta(A,\gamma). \tag{4.44}$$

In order to complete the proof it is sufficient to show the following:

$$\rho(x_i, x(A,\gamma)) \le \epsilon \text{ for all integers } i \ge n_B.$$

First we show that

$$\rho(x_i, x(A,\gamma)) \le M(A,\gamma) \tag{4.45}$$

for all integers $i \ge 0$. Note that in view of (4.8) and (4.42), inequality (4.45) holds for $i = 0$.

Assume that an integer $j \ge 0$ and

$$\rho(x_j, x(A,\gamma)) \le M(A,\gamma) \tag{4.46}$$

By (4.2), (4.5) and (4.44),

$$\begin{aligned} \rho(x_{j+1}, x(A,\gamma)) &\le \rho(x(A,\gamma), C_j(x_j)) + 4^{-1}\delta(A,\gamma) \\ &\le \rho(x(A,\gamma), A_\gamma(x_j)) \\ &\quad + H(A_\gamma(x_j), C_j(x_j)) + 4^{-1}\delta(A,\gamma) \\ &\le H(A_\gamma(x(A,\gamma)), A_\gamma(x_j)) \\ &\quad + H(A_\gamma(x_j), C_j(x_j)) + 4^{-1}\delta(A,\gamma) \\ &\le (1-\gamma)\rho(x(A,\gamma), x_j) \\ &\quad + H(A_\gamma(x_j), C_j(x_j)) + 4^{-1}\delta(A,\gamma). \end{aligned} \tag{4.47}$$

In view of (4.15), (4.41) and (4.46),

$$H(A_\gamma(x_j), C_j(x_j)) \le \delta(A,\gamma).$$

Together with (4.47) this implies that

$$\rho(x_{j+1}, x(A,\gamma)) \le (1-\gamma)\rho(x(A,\gamma), x_j) + 2\delta(A,\gamma).$$

Thus we have shown that the following property holds:

(P) if an integer $j \ge 0$ satisfies

$$\rho(x_j, x(A,\gamma)) \le M(A,\gamma),$$

then

$$\rho(x_{j+1}, x(A,\gamma)) \le (1-\gamma)\rho(x(A,\gamma), x_j) + 2\delta(A,\gamma). \tag{4.48}$$

Assume that an integer $j \ge 0$ and that (4.46) holds. Then by property (P), (4.7), (4.14) and (4.46),

$$\begin{aligned} \rho(x_{j+1}, x(A,\gamma)) &\le (1-\gamma)M(A,\gamma) + 2\delta(A,\gamma) \\ &< (1-\gamma)M(A,\gamma) + 1 < M(A,\gamma) \end{aligned}$$

and (4.45) holds with $i = j+1$. Therefore we have shown by induction that (4.45) holds for all integers $i \ge 0$. Together with property (P) this implies that (4.48) holds for all integers $j \ge 0$.

We show that there exists an integer $j \in [0, n_B]$ such that

$$\rho(x_j, x(A,\gamma)) \le \epsilon.$$

Assume the contrary. Then for all $j \in \{0, \dots, n_B\}$

$$\rho(x_j, x(A,\gamma)) > \epsilon.$$

Together with (4.14) and (4.48) this implies that for all integers $j \in [0, n_B]$,

$$\begin{aligned}\rho(x_j, x(A,\gamma)) - \rho(x_{j+1}, x(A,\gamma)) &\geq \gamma\rho(x_j, x(A,\gamma)) - 2\delta(A,\gamma)\\ &> \gamma\epsilon - 2\delta(A,\gamma)\\ &> \gamma\epsilon/2.\end{aligned}$$

Together with (4.45) this implies that

$$\begin{aligned}M(A,\gamma) &\geq \rho(x_0, x(A,\gamma)) - \rho(x_{n_B}, x(A,\gamma))\\ &= \sum_{j=0}^{n_B-1} (\rho(x_j, x(A,\gamma)) - \rho(x_{j+1}, x(A,\gamma)))\\ &\geq n_B\gamma\epsilon/2\end{aligned}$$

and

$$n_B \leq 2(\gamma\epsilon)^{-1} M(A,\gamma).$$

This contradicts (4.40). The contradiction we have reached proves that there exists an integer j such that

$$j \in [0, n_B],\ \rho(x_j, x(A,\gamma)) \leq \epsilon. \tag{4.49}$$

Assume that an integer $i \geq 0$ satisfies

$$\rho(x_i, x(A,\gamma)) \leq \epsilon. \tag{4.50}$$

We show that

$$\rho(x_{i+1}, x(A,\gamma)) \leq \epsilon.$$

There are two cases:

$$\rho(x_i, x(A,\gamma)) \leq \epsilon/2; \tag{4.51}$$

$$\rho(x_i, x(A,\gamma)) > \epsilon/2. \tag{4.52}$$

Assume that (4.51) holds. By (4.14), (4.48) and (4.51),

$$\rho(x_{i+1}, x(A,\gamma)) \leq 2\delta(A,\gamma) + \rho(x_i, x(A,\gamma)) \leq 2\delta(A,\gamma) + \epsilon/2 < \epsilon.$$

Assume that (4.52) holds. It follows from (4.14), (4.48), (4.50) and (4.52) that

$$\begin{aligned}\rho(x_{i+1}, x(A,\gamma)) &= \rho(x_i, x(A,\gamma)) - (\rho(x_i, x(A,\gamma)) - \rho(x_{i+1}, x(A,\gamma)))\\ &\leq \epsilon - (\rho(x_i, x(A,\gamma)) - (1-\gamma)\rho(x_i, x(A,\gamma)) - 2\delta(A,\gamma))\\ &= \epsilon - (\gamma\rho(x_i, x(A,\gamma)) - 2\delta(A,\gamma))\\ &\leq \epsilon - \gamma\epsilon/2 + 2\delta(A,\gamma) \leq \epsilon.\end{aligned}$$

Thus in the both cases

$$\rho(x_{i+1}, x(A,\gamma)) \leq \epsilon. \tag{4.53}$$

We have shown that if an integer $i \geq 0$ satisfies (4.50), then (4.53) holds. Together with (4.49) this implies that (4.50) holds for all integers $i \geq n_B$. This completes the proof of Theorem 3.3.

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