Applied Analysis and Optimization
Volume 7, Number 2, 2023, 189–208

Yokohama Publishers
ISSN 2189-1664 Online Journal

FUNCTIONAL MEIR-KEELER MAPS IN ORDERED METRIC SPACES

MIHAI TURINICI

ABSTRACT. A fixed point result involving functional Meir-Keeler maps is established over (transitive) ordered metric spaces. Further, it is shown that some related statements, including the ones due to Choudhury and Kundu [Demonstr. Math., 46 (2013), 327-334] or Du and Rassias [Int. J. Nonlin. Anal. Appl., 11 (2020), 55-66] are ultimately reducible to such techniques.

1. Introduction

Let X be a nonempty set. Call the subset Y of X, *almost singleton* (in short: *asingleton*), provided $[y_1, y_2 \in Y$ implies $y_1 = y_2]$; and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Take a metric $d : X \times X \to R_+ := [0, \infty[$ over X, as well as a selfmap $T \in \mathcal{F}(X)$. [Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ stands for the class of all functions from A to B; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$]. Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. Concerning the existence and uniqueness of such points, a basic result (referred to as: *Banach fixed point theorem*; in short: (B-fpt)) may be stated as follows. Call the selfmap T, $(d;\lambda)$-*contractive* (where $\lambda \geq 0$), if

(con) $d(Tx, Ty) \leq \lambda d(x, y)$, for all $x, y \in X$.

Theorem 1.1. *Suppose that T is $(d;\lambda)$-contractive, for some $\lambda \in [0, 1[$. In addition, let X be d-complete. Then,* $\mathrm{Fix}(T)$ *is a singleton* $\{z\}$, *and* $\lim_n T^n x = z$, *for each* $x \in X$.

This result, obtained in 1922 by Banach [3], found a multitude of applications in operator equations theory; so, it was the subject of many extensions. The most general ones have the (set) *implicit* form

(i-s) $(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \in \mathcal{M}$, for all $x, y \in X$;

where $\mathcal{M} \subseteq R_+^6$ is a (nonempty) subset. A basic particular case of the general contractive property above is

(i-s-2) $(d(Tx, Ty), d(x, y)) \in \mathcal{M}$, for all $x, y \in X$;

where $\mathcal{M} \subseteq R_+^2$ is a (nonempty) subset. The classical example in this particular direction is the one due to Meir and Keeler [23]. Further refinements of the method were proposed by Ćirić [8] and Matkowski [22]; see also Jachymski [17]. In particular, when $\mathcal{M}$ is the zero-section of a certain function $F : R_+^6 \to R$; i.e.,

2020 *Mathematics Subject Classification.* 47H10 (Primary); 54H25 (Secondary).
Key words and phrases. Metric space, fixed point, Picard operator, functional Meir-Keeler property, singular asymptotic map.

$\mathcal{M} = \{(t_1, ..., t_6) \in R_+^6; F(t_1, ..., t_6) \leq 0\}$,

the implicit contractive condition above has the functional form:

(i-f) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \leq 0$, for all $x, y \in X$.

In this setting, certain technical aspects have been considered by Leader [20] and Turinici [37]. Finally, when the function F appearing here admits the explicit form

$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - G(t_2, t_3, t_4, t_5, t_6)$,
$(t_1, t_2, t_3, t_4, t_5, t_6) \in R_+^6$,

(where $G : R_+^5 \to R_+$ is a function), one gets the explicit functional version of this (functional) contraction

(e-f) $d(Tx, Ty) \leq G(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y))$,
for all $x, y \in X$.

For this case, some outstanding results have been established in Boyd and Wong [6], Reich [29], Matkowski [21], and Piticari [28, Ch II]. Further aspects have been discussed in the survey paper by Rhoades [30]; see also Collaco and E Silva [10].

Concerning the recent developments in this area, we must mention the class of implicit functional contractions having as model the ones introduced in 1969 by Krasnoselskii and Stetsenko [19], re-discovered in 2001 by Rhoades [31], and refined in 2008 by Dutta and Choudhury [13]; see also Wardowski [43]. The fixed point or coincidence point results obtained with the aid of such contractions were appreciated as interesting enough to be used in the treatment of various operator equations involving univalued and multivalued maps; see, in this direction, the 2020 survey paper by Karapinar et al [18]. On the other hand, certain efforts have been made towards a structural extension of them, under the *convergence* setting in Petruşel and Rus [27]; see, in this direction, Batra and Vashistha [4].

Having these precise, it is our objective in the following to show that most of these fixed point statements are obtainable by means of a unitary Meir-Keeler procedure. For simplicity reasons, the standard metrical case will be considered. Further aspects, involving the *pseudometric* setting developed in the 2016 paper by Turinici [40], will be delineated elsewhere.

2. Dependent choice principle

Throughout this exposition, the axiomatic system in use is Zermelo-Fraenkel's (abbreviated: (ZF)), as described by Cohen [9, Ch 2]. The notations and basic facts to be considered in this system are more or less standard. Some important ones are discussed below.

(A) Let X be a nonempty set. By a *relation* over X, we mean any (nonempty) part $\mathcal{R} \subseteq X \times X$; then, $(X, \mathcal{R})$ will be referred to as a *relational structure*. For simplicity, we sometimes write $(x, y) \in \mathcal{R}$ as $x\mathcal{R}y$. Note that $\mathcal{R}$ may be regarded as a mapping between X and $\exp[X]$ (=the class of all subsets in X). In fact, denote

$X(x, \mathcal{R}) = \{y \in X; x\mathcal{R}y\}$ (the *section* of $\mathcal{R}$ through x), $x \in X$;

then, the desired mapping representation is $[\mathcal{R}(x) = X(x, \mathcal{R});\ x \in X]$.

A basic example of relational structure is to be constructed as below. Let $N = \{0, 1, ...\}$ be the set of *natural* numbers, endowed with the usual addition and (partial) order; note that

$(N, \leq)$ is *well ordered*:
any (nonempty) subset of N has a first element.

For each $r \in N$, the section $N(r, >)$ is referred to as the *initial interval* (in N) induced by r. Any set P with $P \sim N$ (in the sense: there exists a bijection from P to N) will be referred to as *effectively denumerable*. In addition, given some natural number $n \geq 1$, any (nonempty) set Q with $Q \sim N(n, >)$ will be said to be *n-finite*; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable* iff it is either effectively denumerable or finite.

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x : N \to X$, where $N = \{0, 1, ...\}$ is the set of *natural* numbers. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$(i(n); n \geq 0)$ is strictly ascending (hence, $i(n) \to \infty$ as $n \to \infty$)

will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Note that, under such a convention, the relation "subsequence of" is *transitive*, in the sense

(z_n)=subsequence of (y_n) and (y_n)=subsequence of (x_n)
imply (z_n)=subsequence of (x_n).

(B) Remember that, an outstanding part of (ZF) is the *Axiom of Choice* (abbreviated: (AC)); which, in a convenient manner, may be written as

(AC) For each couple (J, X) of nonempty sets and each function
$F : J \to \exp(X)$, there exists a (selective) function $f : J \to X$
(characterized as: $f(\nu) \in F(\nu)$, for each $\nu \in J$);

here, $\exp(X)$ stands for the class of all nonempty elements in $\exp[X]$. Sometimes, when the index set J is denumerable, the existence of such a selective function may be determined by using a weaker form of the Axiom of Choice (AC), called: *Dependent Choice* principle (in short: (DC)). Call the relation $\mathcal{R}$ over X, *proper* when

$(X(x, \mathcal{R}) =)\mathcal{R}(x)$ is nonempty, for each $x \in X$.

Then, $\mathcal{R}$ is to be viewed as a mapping between X and $\exp(X)$; and the couple $(X, \mathcal{R})$ will be referred to as a *proper relational structure*. Further, given (the starting) $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is *$(a; \mathcal{R})$-iterative*, provided

$x_0 = a$, and $[x_n \mathcal{R} x_{n+1}$ (i.e.: $x_{n+1} \in \mathcal{R}(x_n)$), for all $n]$.

Proposition 2.1. *Let the relational structure $(X, \mathcal{R})$ be proper. Then, for each $a \in X$ there is at least one $(a, \mathcal{R})$-iterative sequence in X.*

This principle – proposed, independently, by Bernays [5] and Tarski [36] – is deductible from (AC), but not conversely; cf. Wolk [45]. Moreover, by the developments in Moskhovakis [25, Ch 8], and Schechter [35, Ch 6], the *reduced system*

(ZF-AC+DC) it comprehensive enough so as to cover the *usual* mathematics; see also Moore [24, Appendix 2].

A basic consequence of (DC) is the so-called *Denumerable Axiom of Choice* [in short: (AC(N))].

Proposition 2.2. *Let $F : N \to \exp(X)$ be a function. Then, for each $a \in F(0)$ there exists a function $f : N \to X$ with $f(0) = a$ and $f(n) \in F(n)$, for all $n \in N$.*

Proof. Denote $Q = N \times X$; and let us introduce the (proper) relation $\mathcal{R}$ over it:

$$\mathcal{R}(n, x) = \{n + 1\} \times F(n + 1), \ \ n \geq 0, \, x \in X.$$

Then, an application of (DC) to the proper relational structure $(Q, \mathcal{R})$ yields the desired conclusion; we do not give details. □

As a consequence of the above facts,

(DC) $\Longrightarrow$ (AC(N)) in the *strongly reduced system* (ZF-AC);
or, equivalently:
(AC(N)) is deductible in the *reduced system* (ZF-AC+DC).

The reciprocal of this inclusion is not true; see Moskhovakis [25, Ch 8, Sect 8.25] for details.

3. Statement of the problem

Let (X, d) be a metric space; and $(\leq)$ be a *quasi-order* (that is: *transitive* relation) over X; then, $(X, d, \leq)$ will be referred to as a *quasi-ordered metric space*. Call the subset Y of X, $(\leq)$-*asingleton* if $[y_1, y_2 \in Y, \ y_1 \leq y_2]$ imply $y_1 = y_2$; and $(\leq)$-*singleton* if, in addition, Y is nonempty. Further, let $T \in \mathcal{F}(X)$ be a selfmap subjected to

(s-pro) T is *semi-progressive*
($X(T, \leq) := \{x \in X; x \leq Tx\}$ is nonempty)
(incr) T is *increasing* ($x \leq y$ implies $Tx \leq Ty$).

We are interested in establishing sufficient conditions for the determination of elements in Fix(T). The basic directions for getting these fixed points are described in our list below, comparable with the one proposed in the 2014 paper by Turinici [38]:

opic-0) We say that T is *fix-$(\leq)$-asingleton*, when Fix(T) is an $(\leq)$-asingleton; and *fix-$(\leq)$-singleton* when Fix(T) is a $(\leq)$-singleton

opic-1) We say that T is a *semi Picard operator* (modulo $(d, \leq)$) if, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is d-asymptotic: $\lim_n d(T^n x, T^{n+1} x) = 0$

opic-2) We say that T is a *Picard operator* (modulo $(d, \leq)$) when, for each $x \in X(T, \leq)$, the iterative sequence $(T^n x; n \geq 0)$ is d-Cauchy: $d(T^n x, T^m x) \to 0$ as $n, m \to \infty$, $n \leq m$

opic-3) We say that T is a *strong Picard operator* (modulo $(d, \leq)$) if, for each $x \in X(T, \leq)$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent and $T^\omega x := \lim_n (T^n x) \in$ Fix(T)

opic-4) We say that T is a *Bellman Picard operator* (modulo $(d, \leq)$) when, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is d-convergent, $T^\omega x := \lim_n (T^n x)$ belongs to Fix(T), and $T^n x \leq T^\omega x$, $\forall n$.

In particular, when $(\leq) = X \times X$ (the *trivial quasi-order* on X) these conventions reduce to the ones in Rus [33, Ch 2, Sect 2.2]; because, in this case, $X(T, \leq) = X$.

Returning to the general setting, the sufficient (regularity) conditions attached to these properties are being founded on *ascending orbital full* (in short: (a-o-f)) concepts. Call the sequence (z_n) in X,

(d1) *ascending*, provided $z_i \leq z_j$ whenever $i < j$.
(d2) *orbital*, if $(z_n = T^n x; n \geq 0)$, for some $x \in X$;
(d3) *full*, when $n \mapsto z_n$ is injective ($i \neq j$ implies $x_i \neq x_j$);

the intersection of these notions yields the precise one.

reg-1) Call X, *(a-o-f,d)-complete* provided (for each (a-o-f)-sequence) d-Cauchy $\Longrightarrow$ d-convergent

reg-2) We say that T is *(a-o-f,d)-continuous*, if ((z_n)=(a-o-f)-sequence and $z_n \xrightarrow{d} z$) imply $Tz_n \xrightarrow{d} Tz$

reg-3) Call the quasi-order $(\leq)$, *(a-o-f,d)-selfclosed* when ((z_n)=(a-o-f)-sequence and $z_n \xrightarrow{d} z$) imply $(z_n \leq z, \forall n)$.

As a basic completion of these, we have to introduce the contractive type conditions to be used. Denote $(P_0(x, y) = d(Tx, Ty); x, y \in X)$; and let $P : X \times X \to R_+$ be a map. We say that T is *Meir-Keeler* $(d, \leq; P)$*-contractive*, if

(mk-1) $x \leq y$, $P(x, y) > 0$ imply $P_0(x, y) < P(x, y)$;
referred to as:
T is *strictly nonexpansive* (modulo $(d, \leq; P)$)
(mk-2) $\forall \varepsilon > 0$, $\exists \delta > 0$: $(x \leq y,\ \varepsilon < P(x, y) < \varepsilon + \delta) \Longrightarrow P_0(x, y) \leq \varepsilon$;
referred to as:
T has the *Meir-Keeler property* (modulo $(d, \leq; P)$).

Note that, by the former of these, the Meir-Keeler property writes

(mk-2a) $\forall \varepsilon > 0$, $\exists \delta > 0$, such that:
$(x \leq y,\ 0 < P(x, y) < \varepsilon + \delta) \Longrightarrow P_0(x, y) \leq \varepsilon$;
referred to as:
T has the *complete Meir-Keeler property* (modulo $(d, \leq; P)$).

The following asymptotic version of Meir-Keeler contractive property is sometimes useful in applications, as we will see. Call the selfmap T, *asymptotic Meir-Keeler* $(d, \leq; P)$*-contractive* if

(a-mk-1) $x \leq y$, $P(x, y) > 0$ imply $P_0(x, y) < P(x, y)$;
referred to as:
T is *strictly nonexpansive* (modulo $(d, \leq; P)$)
(a-mk-2) there are no sequences (u_n), (v_n) in X and no elements $\varepsilon > 0$ with $(u_n \leq v_n, \forall n)$, $P_0(u_n, v_n) \to \varepsilon+$, $P(u_n, v_n) \to \varepsilon+$;
referred to as:
T has the *asymptotic Meir-Keeler property* (modulo $(d, \leq; P)$).

Here, given the sequence $(r_n; n \geq 0)$ in R and the point $r \in R$, we denoted

$r_n \to r+$, if $r_n \to r$ and $(r_n > r$, for all $n)$.

The relationships with the standard Meir-Keeler condition are described by

Theorem 3.1. *We have, in (ZF-AC+DC),*

T is Meir-Keeler $(d, \le; P)$-contractive iff
T is asymptotic Meir-Keeler $(d, \le; P)$-contractive.

Proof. i): Suppose that T is Meir-Keeler $(d, \le; P)$-contractive; but [in contradiction with the conclusion] T is not endowed with the asymptotic Meir-Keeler property (modulo $(d, \le; P)$); i.e.:

there are sequences (u_n), (v_n) in X and elements $\varepsilon > 0$ with
$(u_n \le v_n, \forall n)$, $P_0(u_n, v_n) \to \varepsilon+$, $P(u_n, v_n) \to \varepsilon+$.

Given this $\varepsilon > 0$, let $\delta > 0$ be the number associated to it, by the Meir-Keeler $(d, \le; P)$-contractive property. From the convergence relations above, there exists some rank $n(\delta)$ with

(rela) $(\forall n \ge n(\delta))$: $u_n \le v_n$, and
$\varepsilon < P_0(u_n, v_n) < \varepsilon + \delta$, $\varepsilon < P(u_n, v_n) < \varepsilon + \delta$.

But, according to the first and third part of (rela), we must have

$P_0(u_n, v_n) \le \varepsilon$, for all $n \ge n(\delta)$.

This, however, contradicts the second part of (rela). Hence, our working assumption cannot be accepted; and the assertion follows.

ii): Suppose that T is asymptotic Meir-Keeler $(d, \le; P)$-contractive; but [in contradiction with the conclusion] T is not endowed with the Meir-Keeler property (modulo $(d, \le; P)$); i.e. (for some $\varepsilon > 0$)

$A(\delta) := \{(u, v) \in X \times X; u \le v, \varepsilon < P(u, v) < \varepsilon + \delta,\ P_0(u, v) > \varepsilon\}$
is nonempty, for each $\delta > 0$.

Taking a zero converging sequence $(\delta_n; n \ge 0)$ in R_+^0, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a couple of sequences $(x_n; n \ge 0)$ and $(y_n; n \ge 0)$ in X, so as

$(\forall n)$: $(x_n, y_n) \in A(\delta_n)$; i.e. (by the strict nonexpansive condition)
$x_n \le y_n$, $\varepsilon < P_0(x_n, y_n) < P(x_n, y_n) < \varepsilon + \delta_n$.

As a direct consequence of this relation,

$(x_n \le y_n, \forall n)$, and $(P_0(x_n, y_n) \to \varepsilon+,\ P(x_n, y_n) \to \varepsilon+)$, as $n \to \infty$.

This contradicts the asymptotic Meir-Keeler property (modulo $(d, \le; P)$) of T. Hence, the Meir-Keeler property (modulo $(d, \le; P)$) of T follows. □

In the following, two basic examples of such contractions are constructed.

(Ex-I) Given the mapping $P : X \times X \to R_+$ and the function $\varphi \in \mathcal{F}(R_+^0, R)$, let us say that T is *$(d, \le; P; \varphi)$-contractive*, if

(phi-con) $P_0(x, y) \le \varphi(P(x, y))$, $\forall x, y \in X$, $x \le y$, $P(x, y) > 0$.

The functions φ to be considered may described as below. Let $\mathcal{F}(re)(R_+^0, R)$ stand for the subclass of all $\varphi \in \mathcal{F}(R_+^0, R)$, with

φ is *regressive*: $\varphi(t) < t$, for each $t > 0$.

Call $\varphi \in \mathcal{F}(re)(R_+^0, R)$, *Meir-Keeler admissible* if

(mk-adm) $\forall \gamma > 0$, $\exists \beta > 0$, $(\forall t)$: $\gamma < t < \gamma + \beta \Longrightarrow \varphi(t) \le \gamma$.

A related concept may be introduced as below. Call $\varphi \in \mathcal{F}(re)(R_+^0, R)$, *Matkowski admissible* if

(M-adm) for each sequence $(t_n; n \geq 0)$ in R^0_+ with $(t_{n+1} \leq \varphi(t_n), \forall n)$, we have $\lim_n t_n = 0$.

In particular, letting $\mathcal{F}(re, in)(R^0_+, R)$ stand for the class of all increasing functions in $\mathcal{F}(re)(R^0_+, R)$, we have (cf. Matkowski [21])

$\varphi \in \mathcal{F}(re, in)(R^0_+, R)$ is *Matkowski admissible* iff for each $t > 0$, $\lim_n \varphi^n(t) = 0$, as long as $(\varphi^n(t); n \geq 0)$ exists.

Here, as usual, φ^n stands for the n-th iterate of φ, for each $n \in N$.

Concerning the relationships between these admissible concepts, we have

Theorem 3.2. *For each $\varphi \in \mathcal{F}(re)(R^0_+, R)$, we have*

φ is Meir-Keeler admissible iff φ is Matkowski admissible.

For a complete proof of this we refer to the paper by Turinici [41]. Some partial aspects of the problem can be found in Jachymski [16].

To get concrete examples of such functions, the constructions below are in effect. Let $\varphi \in \mathcal{F}(re)(R^0_+, R)$ be given. Denote, for each $s \in R^0_+$

$\Lambda^+\varphi(s) = \inf_{\varepsilon>0} \Phi^*(s+)(\varepsilon)$; where $\Phi^*(s+)(\varepsilon) = \sup \varphi]s, s+\varepsilon[$;

this will be referred to as: *right superior limit* of φ at s. From the regressive property of φ, we have

$-\infty \leq \Lambda^+\varphi(s) \leq s, \quad \forall s \in R^0_+$;

but the alternative of the extremal terms being attained cannot be avoided. The following consequence of this will be useful.

Theorem 3.3. *Let $\varphi \in \mathcal{F}(re)(R^0_+, R)$ and $s \in R^0_+$ be arbitrary fixed. Then,*

(32-a) $\limsup_n(\varphi(t_n)) \leq \Lambda^+\varphi(s)$, for each sequence (t_n) in R^0_+ with $t_n \to s+$

(32-b) there exists a sequence (r_n) in R^0_+ with $r_n \to s+$ and $\varphi(r_n) \to \Lambda^+\varphi(s)$.

Proof. Denote, for simplicity,

$\alpha = \Lambda^+\varphi(s)$; hence, $\alpha = \inf_{\varepsilon>0} \Phi^*(s+)(\varepsilon)$, and $-\infty \leq \alpha \leq s$.

i) Given $\varepsilon > 0$, there exists a rank $p(\varepsilon) \geq 0$ such that $s < t_n < s+\varepsilon$, for all $n \geq p(\varepsilon)$; hence

$\limsup_n(\varphi(t_n)) \leq \sup\{\varphi(t_n); n \geq p(\varepsilon)\} \leq \Phi^*(s+)(\varepsilon)$.

Passing to infimum over $\varepsilon > 0$, yields (see above)

$\limsup_n(\varphi(t_n)) \leq \inf_{\varepsilon>0} \Phi^*(s+)(\varepsilon) = \alpha$;

and the claim follows.

ii): Define $(\beta_n := \Phi^*(s+)(2^{-n-1}s); n \geq 0)$; clearly,

(p1) $(\forall n)$: $-\infty < \beta_n \leq s + 2^{-n-1}s$, if we note that $-\infty < \varphi(t) < t < s + 2^{-n-1}s$, for all $t \in]s, s+2^{-n-1}s[$;

(p2) (β_n) is descending, $(\beta_n \geq \alpha, \ \forall n)$, $\inf_n \beta_n = \alpha$; hence $\lim_n \beta_n = \alpha$.

By these properties, there may be directly constructed a sequence $(\gamma_n; n \geq 0)$ in R, with

$\gamma_n < \beta_n, \forall n; \lim_n \gamma_n = \lim_n \beta_n = \alpha$.

(For example, $(\gamma_n = \beta_n - 3^{-n}; n \geq 0)$ has such a property; so, (DC) is not used here). Define the map $n \mapsto E(n)$ from N to $\exp(R_+^0)$, as

$$E(n) = \{t \in]s, s + 2^{-n-1}s[; \beta_n \geq \varphi(t) > \gamma_n\},\ n \in N;$$

clearly, this construction is meaningful, by the supremum definition. From the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $(r_n; n \geq 0)$ in R_+^0 may be obtained so as

$(\forall n)$: $r_n \in E(n)$; that is (by the above definition):
$s < r_n < s + 2^{-n-1}s$ and $\beta_n \geq \varphi(r_n) > \gamma_n$.

This yields ($r_n \to s+$ and $\varphi(r_n) \to \alpha$); wherefrom, we are done. □

In the following, some particular examples of Meir-Keeler (or, equivalently: Matkowski) admissible functions will be given.

I-1) Call $\varphi \in \mathcal{F}(re)(R_+^0, R)$, *Boyd-Wong admissible* if

(bw-adm) $\Lambda^+\varphi(s) < s$, for all $s > 0$.

(This convention is related to the developments in Boyd and Wong [6]). In particular, $\varphi \in \mathcal{F}(re)(R_+^0, R)$ is Boyd-Wong admissible provided it is upper semicontinuous at the right on R_+^0:

$\Lambda^+\varphi(s) \leq \varphi(s)$, $\forall s \in R_+^0$.

This, e.g., is fulfilled when φ is continuous at the right on R_+^0; for, in such a case,

$\Lambda^+\varphi(s) = \varphi(s)$, for each $s \in R_+^0$.

Proposition 3.4. *Each Boyd-Wong admissible function in $\mathcal{F}(re)(R_+^0, R)$ is Meir-Keeler admissible.*

Proof. Suppose that $\varphi \in \mathcal{F}(re)(R_+^0, R)$ is Boyd-Wong admissible, and fix $\gamma > 0$; hence $\Lambda^+\varphi(\gamma) < \gamma$. By definition, there exists $\beta = \beta(\gamma) > 0$ with the property [$\gamma < t < \gamma + \beta$ implies $\varphi(t) < \gamma$]; proving that the function φ is Meir-Keeler admissible. □

I-2) Given $\varphi \in \mathcal{F}(re)(R_+^0, R)$, we call it *Geraghty admissible* [15], if

$(t_n; n \geq 0)$= sequence in R_+^0 and $\varphi(t_n)/t_n \to 1$ imply $t_n \to 0$.

Proposition 3.5. *Each Geraghty admissible function is Boyd-Wong admissible; hence, Meir-Keeler admissible.*

Proof. Let $\varphi \in \mathcal{F}(re)(R_+^0, R)$ be Geraghty admissible; and suppose by contradiction that φ is not Boyd-Wong admissible. From a previous observation, there exists some $s \in R_+^0$ with $\Lambda^+\varphi(s) = s$. Combining with a preceding auxiliary fact, there exists a sequence $(r_n; n \geq 0)$ in R_+^0 with

$r_n \to s+$ and $\varphi(r_n) \to s$; whence $\varphi(r_n)/r_n \to 1$;

telling us that φ is not Geraghty admissible. The obtained contradiction proves our claim. □

Returning to the general case, the following auxiliary fact establishes the necessary connections between these functional contractions and the the Meir-Keeler (or, Matkowski) ones.

Theorem 3.6. *Assume that the selfmap T is $(d, \leq; P; \varphi)$-contractive, where $\varphi \in \mathcal{F}(re)(R_+^0, R)$. Then, T is Meir-Keeler $(d, \leq; P)$-contractive when φ is Meir-Keeler admissible (or, equivalently: Matkowski admissible). In particular, this is retainable whenever φ is Boyd-Wong admissible; hence, all the more, when it is Geraghty admissible.*

Proof. i) Let $x, y \in X$ be such that $x \leq y$ and $P(x, y) > 0$. By the contractive condition [and φ=regressive], one has $P_0(x, y) < P(x, y)$; so that, T is strictly nonexpansive (modulo $(d, \leq; P)$).

ii) Let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number assured by the Meir-Keeler admissible property of φ. Further, let $x, y \in X$ be such that $x \leq y$ and $\varepsilon < P(x, y) < \varepsilon + \delta$. By the contractive condition and Meir-Keeler admissible property,

$$P_0(x, y) \leq \varphi(P(x, y)) \leq \varepsilon;$$

so that, T has the Meir-Keeler property (modulo $(d, \leq; P)$).

ii): Evident, by the above auxiliary facts. □

(Ex-II) Let (ψ, φ) be a couple of functions over $\mathcal{F}(R_+^0, R)$ endowed with

(norm) (ψ, φ) is *normal*:

ψ is increasing and φ is *strictly positive* [$\varphi(t) > 0$, $\forall t > 0$].

(This concept is related with the developments in Rhoades [31]; see also Dutta and Choudhury [13]). The following extra condition will be largely used in various concrete cases:

(r-s-pos) (φ is *right sequentially positive*)

for each sequence (t_n) in R_+^0 and each $\varepsilon > 0$ with $t_n \to \varepsilon+$,

the relation $\lim_n \varphi(t_n) = 0$ is impossible.

Given the mapping $P : X \times X \to R_+$ and the couple (ψ, φ) of functions over $\mathcal{F}(R_+^0, R)$, let us say that T is $(d, \leq; P; (\psi, \varphi))$-*contractive*, provided

$$\psi(P_0(x, y)) \leq \psi(P(x, y)) - \varphi(P(x, y)),$$
$$\forall x, y \in X,\ x \leq y,\ P_0(x, y) > 0,\ P(x, y) > 0.$$

The following auxiliary fact establishes the necessary connection between this contractive concept and the Meir-Keeler one.

Proposition 3.7. *Suppose that T is $(d, \leq; P; (\psi, \varphi))$-contractive, for a normal couple (ψ, φ) over $\mathcal{F}(R_+^0, R)$ with φ=right sequentially positive. Then, in the reduced system (ZF-AC+DC),*

T is asymptotic Meir-Keeler $(d, \leq; P)$-contractive;

or, equivalently: Meir-Keeler $(d, \leq; P)$-contractive.

Proof. i) Let $x, y \in X$ be such that $x \leq y$, $P(x, y) > 0$. If $P_0(x, y) = 0$, we are done; so, without loss, assume that $P_0(x, y) > 0$. As $P(x, y) > 0$, we have (along with φ=strictly positive), $\varphi(P(x, y)) > 0$; wherefrom

$$\psi(P_0(x, y)) < \psi(P(x, y)) \text{ (by the contractive condition);}$$

and this, via [ψ=increasing], yields $P_0(x, y) < P(x, y)$. Putting these together, one derives that T is strictly nonexpansive (modulo $(d, \leq; P)$).

ii) We have to establish that T is endowed with the asymptotic Meir-Keeler property (modulo $(d, \leq; P)$). Suppose not:

there are sequences (u_n), (v_n) in X and elements $\varepsilon > 0$
with $(u_n \leq v_n, \forall n)$, $P_0(u_n, v_n) \to \varepsilon+$, $P(u_n, v_n) \to \varepsilon+$.

By the contractive property,

$(\forall n)$: $\psi(P_0(u_n, v_n)) \leq \psi(P(u_n, v_n)) - \varphi(P(u_n, v_n))$;
or, equivalently (along with φ=strictly positive):
$(0 <)\varphi(P(u_n, v_n)) \leq \psi(P(u_n, v_n)) - \psi(P_0(u_n, v_n))$.

Passing to $\limsup$ as $n \to \infty$, gives

$0 \leq \limsup \varphi(P(u_n, v_n)) \leq \psi(\varepsilon + 0) - \psi(\varepsilon + 0) = 0$;
whence, $\lim_n \varphi(P(u_n, v_n)) = 0$;

in contradiction with the choice of φ. Hence, the underlying asymptotic property holds; and we are done. □

Note that, some other examples of such contractions are available. But, for the developments below, this will suffice.

4. Main result

Let $(X, d, \leq)$ be a quasi-ordered metric space; and T be a selfmap of X; supposed to be semi-progressive and increasing. As precise, we have to determine whether $\text{Fix}(T)$ is nonempty; and, if this holds, to establish whether T is fix-asingleton (that is: fix-singleton). The basic directions as well as the sufficient conditions under which this problem is to be solved were already listed. In addition, the contractive Meir-Keeler setting of our problem, expressed in terms of a certain mapping $P \in \mathcal{F}(X \times X, R_+)$, is being settled. It remains now to discuss the specific regularity conditions upon P to be used. Denote for each $x, y \in X$,

$A_1(x, y) = d(x, y)$, $A_2(x, y) = \max\{d(x, Tx), d(y, Ty)\}$,
$L_0(x, y) = \min\{d(x, y), d(Tx, Ty)\}$,
$L_1(x, y) = \min\{d(x, y), d(Tx, Ty), d(x, Tx), d(y, Ty)\}$,
$M(x, y) = \text{diam}\{x, Tx, y, Ty\}$.

(I) The first condition upon P writes

P is *L-positive* (where $L \in \{L_0, L_1\}$):
$L(x, y) > 0$ implies $P(x, y) > 0$.

It has the role of working with our iterative sequences (under $L = L_1$) as well as to assure the fix-asingleton property (under $L = L_0$). Note that

P is L_0-positive implies P is L_1-positive;

but the reciprocal is not in general true.

(II) The second condition upon P is expressed as

P is *orbitally bounded*: $P(x, Tx) \leq A_2(x, Tx)$, for all $x \in X$;

it allows us deducing the d-asymptotic and full properties for the iterative sequences to be considered (see below).

(III) The third condition upon P writes

P is *diametral*: $P(x, y) \leq M(x, y)$, for all $x, y \in X$.

It has the role of deducing (in addition to the Meir-Keeler contractive assumption) the d-Cauchy property for the iterative sequences in question.

(IV) The fourth condition upon P is a couple of orbital asymptotic ones, formulated as

(o-sg-asy) P is *orbitally singular asymptotic*:
for each (a-o-f) sequence (x_n) in X and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n \leq z,\ L_1(x_n, z) > 0,\ \forall n)$, we have $\liminf_n P(x_n, z) < d(z, Tz)$
(s-o-sg-asy) P is *strongly orbitally singular asymptotic*:
for each (a-o-f) sequence (x_n) in X and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n \leq z,\ L_1(x_n, z) > 0,\ \forall n)$, we have $(\exists) \lim_n P(x_n, z) < d(z, Tz)$.

These, essentially, allow us deducing (in addition to the Meir-Keeler contractive assumption) the fixed point property for the limit of underlying iterative sequence.

Under the above preliminaries, we are now in position to state our basic fixed point result in this exposition.

Theorem 4.1. *Suppose that T is Meir-Keeler $(d, \leq; P)$-contractive, (or, equivalently: asymptotic Meir-Keeler $(d, \leq; P)$-contractive), where the mapping $P : X \times X \to R_+$ is orbitally bounded and diametral. In addition, let X be (a-o-f,d)-complete. Then,*

(41-a) *T is a Picard operator (modulo $(d, \leq)$) when, in addition, P is L_1-positive*

(41-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (41-a), T is (a-o-f,d)-continuous*

(41-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (41-a), $(\leq)$ is (a-o-f,d)-selfclosed and P is orbitally singular asymptotic*

(41-d) *T has the fix-asingleton property when (in addition) the mapping P is L_0-positive.*

Proof. Let us firstly establish the fix-asingleton property. Take some couple $z_1, z_2 \in \mathrm{Fix}(T)$ with $z_1 \leq z_2$; and suppose by contradiction that $z_1 \neq z_1$; hence, $L_0(z_1, z_2) = d(z_1, z_2) > 0$. As P is L_0-positive, we must have $P(z_1, z_2) > 0$; so, that by the strict nonexpansive property of T (modulo $(d, \leq; P)$): $d(z_1, z_2) = d(Tz_1, Tz_2) < P(z_1, z_2)$. On the other hand, as P is diametral, $P(z_1, z_2) \leq M(z_1, z_2) = d(z_1, z_2)$. Since the obtained relations are contradictory, our working assumption cannot be accepted; wherefrom, the assertion follows.

It remains to establish that T is a strong/Bellman Picard operator (modulo $(d, \leq)$). Fix some $x_0 \in X$; and put $(x_n = T^n x_0; n \geq 0)$; it is an ascending and orbital sequence. If $x_n = x_{n+1}$ for some $n \geq 0$, we are done; so, without loss, one may assume that the *non-telescopic* property holds

(n-tele) $x_n \neq x_{n+1}$ (that is, $\rho_n := d(x_n, x_{n+1}) > 0$), $\forall n$.

The argument will be divided into several parts.

Part 1. By the imposed condition,

$(\forall n)$: $L_1(x_n, x_{n+1}) = \min\{\rho_n, \rho_{n+1}\} > 0$; hence $P(x_n, x_{n+1}) > 0$

if we remember that P is L_1-positive. The Meir-Keeler $(d, \leq; P)$-contractive condition applies to (x_n, x_{n+1}), for each n; and yields (by the strict nonexpansive condition upon T and the orbital boundedness of P)

(iter-1) $(\forall n)$: $\rho_{n+1} = P_0(x_n, x_{n+1}) < P(x_n, x_{n+1}) \leq$
$A_2(x_n, x_{n+1}) = \max\{\rho_n, \rho_{n+1}\}$;

From the strict inequality between the extremal members of this relation, we get the evaluations

(iter-2) $(\forall n)$: $\rho_{n+1} < \rho_n$; wherefrom, $\rho_{n+1} < P(x_n, x_{n+1}) \leq \rho_n$.

Two consequences of this fact are retainable.

Conseq 1. By the first half of (iter-2), (ρ_n) is strictly descending. We claim that, in this case,

(x_n) is full: $i \neq j$ implies $x_i \neq x_j$ (whence, $d(x_i, x_j) > 0$).

In fact, suppose by contradiction that there exist $i, j \in N$ with $i < j$, $x_i = x_j$. Then, by definition, $x_{i+1} = x_{j+1}$; so that $\rho_i = \rho_j$; in contradiction with $\rho_i > \rho_j$; and the assertion follows.

Conseq 2. By the same strict descending property of $(\rho_n; n \geq 0)$, we have that $\rho := \lim_n \rho_n$ exists in R_+; and $[\rho_n > \rho, \forall n]$. Assume that $\rho > 0$; and let $\sigma > 0$ be the number given by the Meir-Keeler property (modulo $(d, \leq; P)$) of T. From the convergence relation above, there exists a rank $n(\sigma)$ such that

$(\forall n \geq n(\sigma))$: $\rho < \rho_n < \rho + \sigma$; hence, by (iter-1),
$\rho < \rho_{n+1} < P(x_n, x_{n+1}) \leq \rho_n < \rho + \sigma$.

This, by the quoted condition, yields (for the same n),

$(\rho <)\ \rho_{n+1} = P_0(x_n, x_{n+1}) \leq \rho$;

a contradiction. Hence, $\rho = 0$; so that, necessarily, $(x_n; n \geq 0)$ is a d-asymptotic sequence.

Part 2. Summing up, (x_n) is orbital, full, and d-asymptotic. We claim that, under the precise conditions, (x_n) is d-Cauchy. Let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number associated by the Meir-Keeler property; without loss, one may assume that $\delta < \varepsilon$. From the obtained d-asymptotic property, there exists a rank $n(\delta) \geq 0$ such that

(d-asy) $d(x_n, x_{n+1}) < \delta/4$ (hence, $d(x_n, x_{n+2}) < \delta/2$), $\forall n \geq n(\delta)$.

We claim, via ordinary induction, that for each index $i \geq 1$, the property below holds

(d-C;i) $d(x_n, x_{n+i}) < \varepsilon + \delta/2$, $\forall n \geq n(\delta)$;

wherefrom, the d-Cauchy property of $(x_n; n \geq 0)$ is clear. The case $i \in \{1, 2\}$ is evident, by (d-asy). Assume that, for a certain $k \geq 2$, (d-C;i) holds for all $i \in \{1, ..., k\}$; we must establish that (d-C;k+1) holds too. So, let $n \geq n(\delta)$ be arbitrary fixed. From the inductive hypothesis,

$d(x_n, x_{n+k}) < \varepsilon + \delta/2$; as well as
$d(x_{n+1}, x_{n+k}) < \varepsilon + \delta/2$, $d(x_{n+1}, x_{n+k+1}) < \varepsilon + \delta/2$.

On the other hand, by the asymptotic property,

$d(x_n, x_{n+1}) < \delta/4$, $d(x_{n+k}, x_{n+k+1}) < \delta/4$.

Finally, the triangular inequality gives (in a direct way)

$d(x_n, x_{n+k+1}) \leq d(x_n, x_{n+k}) + d(x_{n+k}, x_{n+k+1}) <$
$\varepsilon + \delta/2 + \delta/4 = \varepsilon + 3\delta/4 < \varepsilon + \delta$.

Putting these together, gives

$M(x_n, x_{n+k}) < \varepsilon + \delta$; whence, $P(x_n, x_{n+k}) < \varepsilon + \delta$;

if we remember that P is diametral. On the other hand, as (x_n)=full

$L_1(x_n, x_{n+k}) = \min\{\rho_n, \rho_{n+k}, d(x_n, x_{n+k}), d(x_{n+1}, x_{n+k+1})\} > 0$;
whence $P(x_n, x_{n+k}) > 0$

if we take the L_1-positive property of P into account. Combining these with the complete Meir-Keeler property (modulo $(d, \leq; P)$), one derives

$d(x_{n+1}, x_{n+k+1}) = P_0(x_n, x_{n+k}) \leq \varepsilon$.

This, along with the triangular inequality, gives

$d(x_n, x_{n+k+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k+1}) < \varepsilon + \delta/2$;

and the assertion is retainable.

Part 3. Since X is (a-o-f,d)-complete,

there exists (a unique) $z \in X$ with $x_n \xrightarrow{d} z$.

Two basic alternatives are to be discussed.

Alter 1. Suppose that T is (a-o-f,d)-continuous. Then, $u_n := Tx_n \xrightarrow{d} Tz$ as $n \to \infty$. On the other hand, $(u_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; so, $u_n \xrightarrow{d} z$ as $n \to \infty$. Combining with d=separated, yields $z = Tz$; which tells us that T is strongly Picard (modulo $(d, \leq)$).

Alter 2. Suppose that $(\leq)$ is (a-o-f,d)-selfclosed and P is orbitally singular asymptotic. By the convergence property above,

$x_n \leq z$, for all n.

Further, as $(Tx_n = x_{n+1}; n \geq 0)$ appears as full,

$H := \{n \in N; Tx_n = Tz\}$ is an asingleton;

so that, the following separation property holds:

(sepa) $\exists k = k(z) \geq 0$, such that $n \geq k$ implies
$P_0(x_n, z) > 0$; hence, $d(x_n, z) > 0$.

Denote for simplicity $(u_n = x_{n+k}; n \geq 0)$; note that (via (u_n)=full)

$(\forall n)$: $u_n \leq z$, $d(u_n, z) > 0$, $P_0(u_n, z) > 0$, $d(u_n, Tu_n) > 0$.

Suppose by contradiction that $d(z, Tz) > 0$. By the relations above,

$L_1(u_n, z) > 0$; hence, $P(u_n, z) > 0$ (as P is L_1-positive).

The contractive condition is therefore applicable to (u_n, z), for each n; and yields (by the strict nonexpansive (modulo $(d, \leq; P)$) property of T)

(s-ineq) $P_0(u_n, z) < P(u_n, z)$, $\forall n$.

Moreover, from the d-Cauchy and convergence relations, one gets (taking a metrical property of $d(.,.)$ into account)

(conv) $d(u_n, z),\ d(Tu_n, z), d(u_n, Tu_n) \to 0$;
$d(u_n, Tz),\ d(Tu_n, Tz) \to d(z, Tz)$.

By the strict inequality relation, we get, passing to (inferior) limit as $n \to \infty$ (and remembering that P is orbitally singular asymptotic)

$d(z, Tz) = \liminf_n P_0(u_n, z) \leq \liminf_n P(u_n, z) < d(z, Tz)$;

a contradiction. This tells us that the working hypothesis $d(z, Tz) > 0$ cannot be true; so that, $z = Tz$. The proof is complete. □

Note that, coincidence type versions of these facts are available, by means of related techniques in Roldán et al [32]. On the other hand, all these developments may be extended to quasi-metric structures, by following the methods in Turinici [39]. Finally, multivalued extensions of these facts are possible under the lines in Nadler [26]. We will discuss these elsewhere.

5. Du-Rassias results

Let $(X, d, \leq)$ be a quasi-ordered metric space; and T be a selfmap of X; supposed to be semi-progressive and increasing. As precise, we have to determine whether $\mathrm{Fix}(T)$ is nonempty; and, if this holds, to establish whether T is fix-asingleton (that is: fix-singleton). The basic directions and sufficient conditions under which this problem is to be solved were already listed. In addition, the contractive Meir-Keeler setting of our problem as well as a lot of specific regularity conditions upon our data have been settled. As a by-product of these, we stated the main result of this exposition, Theorem 4.1. It is our aim in the sequel to get some particular cases of this result, with a practical finality.

To begin with, remember that a basic lot of maps was introduced, as (for $x, y \in X$)

$A_1(x,y) = d(x,y)$, $A_2(x,y) = \max\{d(x,Tx), d(y,Ty)\}$,
$L_0(x,y) = \min\{d(x,y), d(Tx,Ty)\}$,
$L_1(x,y) = \min\{d(x,y), d(Tx,Ty), d(x,Tx), d(y,Ty)\}$,
$M(x,y) = \mathrm{diam}\{x, Tx, y, Ty\}$.

Then, let us complete this system with an extra lot of maps as (for $x, y \in X$)

$B_1(x,y) = (1/2)[d(x,Tx) + d(y,Ty)]$,
$B_2(x,y) = (1/2)[d(x,Ty) + d(y,Tx)]$,
$B_3(x,y) = (1/2)[d(x,Tx) + d(y,Tx)]$,
$B_4(x,y) = (1/2)[d(y,Tx) + d(y,Ty)]$,
$C_1(x,y) = (1/3)[d(x,Tx) + d(y,Ty) + d(y,Tx)]$,
$C_2(x,y) = (1/3)[d(x,Tx) + d(x,Ty) + d(y,Tx)]$,
$C_3(x,y) = (1/3)[d(y,Ty) + d(x,Ty) + d(y,Tx)]$,
$E_1(x,y) = (1/4)[d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)]$,
$E_2(x,y) = (1/5[d(x,y) + d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)]$.

Finally, define the families of maps

$\mathcal{Q} = \{A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2\}$,
$P = \max(\mathcal{Q}) = \max\{A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2\}$.

The following fixed point statement is available.

Theorem 5.1. *Suppose that T is Meir-Keeler $(d, \leq; P)$-contractive, (or, equivalently: asymptotic Meir-Keeler $(d, \leq; P)$-contractive), where the mapping $P : X \times X \to R_+$ is as before. In addition, let X be (a-o-f,d)-complete. Then,*

(51-a) *T is a Picard operator (modulo $(d, \leq)$)*

(51-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (51-a), T is (a-o-f,d)-continuous*

(51-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (51-a), $(\leq)$ is (a-o-f,d)-selfclosed*

(51-d) *T has the fix-asingleton property.*

Proof. We show that all conditions in our main result are necessarily fulfilled by our data.

Part 1. For each $x \in X$, we have

$A_1(x, Tx) = d(x, Tx) \le A_2(x, Tx)$,
$B_1(x, Tx) = (1/2)[d(x, Tx) + d(Tx, T^2x)] \le A_2(x, Tx)$,
$B_2(x, Tx) = (1/2)d(x, T^2x) \le B_1(x, Tx) \le A_2(x, Tx)$,
$B_3(x, Tx) = (1/2)d(x, Tx) \le A_2(x, Tx)$,
$B_4(x, Tx) = (1/2)d(Tx, T^2x) \le A_2(x, Tx)$,
$C_1(x, Tx) = (1/3)[d(x, Tx) + d(Tx, T^2x)] \le A_2(x, Tx)$,
$C_2(x, Tx) = (1/3)[d(x, Tx) + d(x, T^2x)] \le$
$(1/3)[2d(x, Tx) + d(Tx, T^2x)] \le A_2(x, Tx)$,
$C_3(x, Tx) = (1/3)[d(Tx, T^2x) + d(x, T^2x)] \le$
$(1/3)[2d(Tx, T^2x) + d(x, Tx)] \le A_2(x, Tx)$,
$E_1(x, Tx) = (1/4)[d(x, Tx) + d(Tx, T^2x) + d(x, T^2x)] \le$
$(1/2)[d(x, Tx) + d(Tx, T^2x)] = B_1(x, Tx) \le A_2(x, Tx)$,
$E_2(x, Tx) = (1/5[2d(x, Tx) + d(Tx, T^2x) + d(x, T^2x)] \le$
$(1/5[3d(x, Tx) + 2d(Tx, T^2x)] \le A_2(x, Tx)$.

Putting these together, yields

each $Q \in \mathcal{Q}$ is orbitally bounded; hence, so is $P = \max(\mathcal{Q})$.

Part 2. For each $x, y \in X$, we have

$A_i(x, y) \le M(x, y)$, $i \in \{1, 2\}$; $B_j(x, y) \le M(x, y)$, $j \in \{1, 2, 3, 4\}$,
$C_k(x, y) \le M(x, y)$, $k \in \{1, 2, 3\}$; $E_h(x, y) \le M(x, y)$, $h \in \{1, 2\}$.

This, by definition, yields

each $Q \in \mathcal{Q}$ is diametral; hence, so is $P = \max(\mathcal{Q})$.

Part 3. Let the (a-o-f) sequence $(x_n; n \ge 0)$ in X and the point $z \in X$ be such that

$x_n \xrightarrow{d} z$ (hence, $Tx_n \xrightarrow{d} z$), ($x_n \le z$, $\forall n$), and
($L_1(x_n, z) > 0$, $\forall n$) (hence, $d(z, Tz) > 0$).

We have, by definition (and a metrical property of d)

$\lim_n A_1(x_n, z) = 0$,
$\lim_n B_1(x_n, z) = (1/2)d(z, Tz)$, $\lim_n B_2(x_n, z) = (1/2)d(z, Tz)$,
$\lim_n B_3(x_n, z) = 0$, $\lim_n B_4(x_n, z) = (1/2)d(z, Tz)$
$\lim_n C_1(x_n, z) = (1/3)d(z, Tz)$, $\lim_n C_2(x_n, z) = (1/3)d(z, Tz)$
$\lim_n C_3(x_n, z) = (2/3)d(z, Tz)$,
$\lim_n E_1(x_n, z) = (1/2)d(z, Tz)$, $\lim_n E_2(x_n, z) = (2/5)d(z, Tz)$.

This yields

each $Q \in \mathcal{Q}$ is strongly orbitally singular asymptotic;
hence, so is $P = \max(\mathcal{Q})$.

Part 4. Finally, again by definition,

P is A_1-positive: $A_1(x, y) > 0$ implies $P(x, y) > 0$;
so that: P is both L_0-positive and L_1-positive.

Part 5. As a consequence of this, the main result is indeed applicable here; wherefrom, all is clear. □

As a useful particular case of this, the following statement, extending the one in Du and Rassias [12] is holding.

Theorem 5.2. *Let the selfmap $T : X \to X$ be such that*

(mk-orig) T is original Meir-Keeler $(d, \leq; P)$-contractive:
$\forall \varepsilon > 0$, $\exists \delta > 0$: ($x \leq y$, $\varepsilon \leq P(x, y) < \varepsilon + \delta$) implies $P_0(x, y) < \varepsilon$,

where the mapping $P : X \times X \to R_+$ is as before. In addition, let X be (a-o-f,d)-complete. Then,

(52-a) *T is a Picard operator (modulo $(d, \leq)$)*

(52-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (52-a), T is (a-o-f,d)-continuous*

(52-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (52-a), $(\leq)$ is (a-o-f,d)-selfclosed*

(52-d) *T has the fix-asingleton property.*

Proof. We claim that the following inclusion is holding

(mk-orig-mk) T is original Meir-Keeler $(d, \leq; P)$-contractive implies
T is Meir-Keeler $(d, \leq; P)$-contractive.

And then, by the preceding statement, all is clear. There are two steps to be passed.

Step 1. Let $x, y \in X$ be such that $x \leq y$, $P(x, y) > 0$. Put $\varepsilon = P(x, y)$; and let $\delta > 0$ be the associated by (mk-orig) number. Then,

$\varepsilon = P(x, y) < \varepsilon + \delta$; whence, $P_0(x, y) < \varepsilon = P(x, y)$;

which tells us that T is strictly nonexpansive (modulo $(d, \leq; P)$).

Step 2. Let $\varepsilon > 0$ be given; and $\delta > 0$ be the associated by (mk-orig) number. Then, by the underlying condition,

$$(x \leq y,\ \varepsilon < P(x, y) < \varepsilon + \delta) \Longrightarrow (x \leq y,\ \varepsilon \leq P(x, y) < \varepsilon + \delta)$$
$$\Longrightarrow P_0(x, y) < \varepsilon \Longrightarrow P_0(x, y) \leq \varepsilon;$$

whence, T has the Meir-Keeler property (modulo $(d, \leq; P)$). Putting these together, we are done. □

Note, finally, that a variant of Theorem 5.1 with respect to the mappings used in Samet et al [34] and Du et al [11] is also possible, by the same technique. Further aspects will be discussed elsewhere.

6. Particular aspects

Let $(X, d, \leq)$ be a quasi-ordered metric space; and $T : X \to X$ be a selfmap of X; supposed to be semi-progressive and increasing. Roughly speaking, the particular statement above has been obtained by an appropriate choice of the mapping P appearing in the Meir-Keeler contractive condition. In the following, some other particular cases of our main result are stated, by working upon the contractive condition itself. As a completion, certain connections between these and some other developments in the area will be also discussed.

(A) Remember that $\varphi \in \mathcal{F}(re)(R^0_+, R)$ is called *Meir-Keeler admissible*, if it satisfies

(mk-adm) $\forall \varepsilon > 0$, $\exists \delta > 0$, $(\forall t)$: $\varepsilon < t < \varepsilon + \delta \Longrightarrow \varphi(t) \leq \varepsilon$.

Given the mapping $P : X \times X \to R_+$ and the function $\varphi \in \mathcal{F}(R^0_+, R)$, let us say that T is $(d, \leq; P; \varphi)$-*contractive*, provided

$P_0(x, y) \leq \varphi(P(x, y))$, $\forall x, y \in X$, $x \leq y$, $P(x, y) > 0$.

As precise, any such contraction is necessarily Meir-Keeler $(d, \leq; P)$-contractive, whenever $\varphi \in \mathcal{F}(re)(R^0_+, R)$ is Meir-Keeler admissible. As a direct consequence of this, we have (by means of our main result)

Theorem 6.1. *Suppose that the selfmap T is $(d, \leq; P; \varphi)$-contractive, where the function $\varphi \in \mathcal{F}(re)(R^0_+, R)$ and the mapping $P : X \times X \to R_+$ are such that the combined requirement holds*

(phi-P) φ is Meir-Keeler admissible (or, equivalently: Matkowski admissible), and P is orbitally bounded, diametral.

In addition, let X be (a-o-f,d)-complete. Then,

(61-a) *T is a Picard operator (modulo $(d, \leq)$) when, in addition, P is L_1-positive*

(61-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (61-a), T is (a-o-f,d)-continuous*

(61-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (61-a), $(\leq)$ is (a-o-f,d)-selfclosed and P is orbitally singular asymptotic*

(61-d) *T has the fix-$(\leq)$-asingleton property when (in addition) the mapping P is L_0-positive.*

Some particular cases of this result are described as follows. Let the system of maps over $\mathcal{F}(X \times X, R_+)$

$\mathcal{Q} = \{A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2\}$

be the already introduced one.

I) The regularity condition (phi-P) holds under

(phi-P-1) φ is Boyd-Wong admissible and $P = A_1$.

In this case, the corresponding version of Theorem 6.1 includes directly the related statement in Agarwal et al [1], proved by a different method. But, as shown in that paper, this result includes (under $(\leq) = X \times X$) the well known contribution due to Boyd and Wong [6] or Matkowski [21]; hence, so does Theorem 6.1.

II) The same regularity condition (phi-P) holds (see above) under

(phi-P-2) φ is Geraghty admissible and $P = A_1$.

Then, the corresponding version of Theorem 6.1 includes the related statement in Amini-Harandi and Emami [2]. But (cf. a previous remark) (phi-P-2) is a particular case of (phi-P-1) This tells us that the result due to Amini-Harandi and Emami [2] is nothing but a particular case of the one in Agarwal et al [1].

III) Finally, the same regularity condition (phi-P) holds under

(phi-P-3) φ is Geraghty admissible and $P \in \mathcal{Q}$.

In particular, when $P = B_1$, the corresponding version of Theorem 6.1 includes the related statement in Choudhury and Kundu [7] proved by a distinct argument.

(B) Let (ψ, φ) be a pair of functions over $\mathcal{F}(R^0_+, R)$, with

(norm) (ψ, φ) is normal:

ψ is increasing and φ is strictly positive ($\varphi(t) > 0$, $\forall t \in R^0_+$)

The following extra condition will be considered here:

(r-s-pos) φ is right sequentially positive:
for each sequence $(t_n; n \geq 0)$ in R^0_+ and each element $\varepsilon > 0$
with $t_n \to \varepsilon+$, the relation $\lim_n \varphi(t_n) = 0$ is impossible.

Given the mapping $P : X \times X \to R_+$ and the couple (ψ, φ) of functions over $\mathcal{F}(R^0_+, R)$, let us say that T is $(d, \leq; P; (\psi, \varphi))$-*contractive*, provided

$$\psi(P_0(x,y)) \leq \psi(P(x,y)) - \varphi(P(x,y)),$$
$$\forall x, y \in X,\ x \leq y,\ P_0(x,y) > 0,\ P(x,y) > 0.$$

By a previous result, any such contraction is Meir-Keeler $(d, \leq; P)$-contractive, whenever (ψ, φ) is normal and φ is right sequentially positive. As a direct consequence of this, we have (by means of our main result)

Theorem 6.2. *Suppose that T is $(d, \leq; P; (\psi, \varphi))$-contractive, where the mapping $P : X \times X \to R_+$ and the couple (ψ, φ) of functions over $\mathcal{F}(R^0_+, R)$ are such that the combined requirement holds*

(psi-phi-P) (ψ, φ) is normal, φ is right sequentially positive,
and P is orbitally bounded, diametral.

In addition, let X be (a-o-f,d)-complete. Then,

(62-a) *T is a Picard operator (modulo $(d, \leq)$) when, in addition, P is L_1-positive*

(62-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (62-a), T is (a-o-f,d)-continuous*

(62-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (62-a), $(\leq)$ is (a-o-f,d)-selfclosed and P is orbitally singular asymptotic*

(62-d) *T has the fix-$(\leq)$-asingleton property when (in addition) the mapping P is L_0-positive.*

The obtained result extends the one in Dutta and Choudhury [13]. In fact, it also includes a related statement in Găvruţa et al [14]; we do not give details.

In the following, a basic particular case of this last result is discussed.

Let $F : R^0_+ \to R$ and $\varphi : R^0_+ \to R$ be a couple of functions with

(nc-1) (F, φ) is normal: F is increasing and φ is strictly positive
(nc-2) φ is right sequentially positive (see above).

Note that this couple is just the one appearing in Theorem 6.2. As a consequence, the quoted result is applicable to contractions like

$$F(P_0(x,y)) \leq F(d(x,y)) - \varphi(d(x,y)),$$
$$x, y \in X,\ x \leq y,\ P_0(x,y) > 0,\ d(x,y) > 0;$$

referred to as: *Wardowski type contractions.* This tells us that Theorem 6.2 includes the basic fixed point result in Wardowski [44], that, in turn, extends an older statement by the same author [43]. In fact, some other statements in the area, described in the survey paper by Karapinar et al [18] may be obtained via these techniques; we do not give details. Note, finally, that our main result includes, partially, the ones described in Vujaković et al [42]. For a complete inclusion of all these, the implicit methods in Turinici [38] may be used; this will be discussed elsewhere.

Acknowledgement. This paper is dedicated to Professor Adrian Petruşel, on the occasion of his 60th birthday.

References

[1] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008), 109–116.

[2] A. Amini-Harandi and H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlin. Anal. **72** (2010), 2238–2242.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[4] R. Batra and S. Vashistha, *Fixed points of an F-contraction on metric spaces with a graph*, Int. J. Computer Math. **91** (2014), 2483–2490.

[5] P. Bernays, *A system of axiomatic set theory: Part III. Infinity and enumerability analysis*, J. Symbolic Logic **7** (1942), 65–89.

[6] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

[7] B. S. Choudhury and A. Kundu, *A Kannan-like contraction in partially ordered spaces*, Demonstr. Math. **46** (2013), 327–334.

[8] L. B. Ćirić, *A new fixed-point theorem for contractive mappings*, Publ. Inst. Math. **30** (1981), 25–27.

[9] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, New York, 1966.

[10] P. Collaco and J. C. E Silva, *A complete comparison of 25 contractive definitions*, Nonlin. Anal. **30** (1997), 441–476.

[11] W.-S. Du, E. Karapinar and Z. He, *Some simultaneous generalizations of well-known fixed point theorems and their applications to fixed point theory*, Mathematics, 2018, 6, 117.

[12] W.-S. Du and T. M. Rassias, *Simultaneous generalizations of known fixed point theorems for a Meir-Keeler type condition with applications*, Int. J. Nonlin. Anal. Appl. **11** (2020), 55–66.

[13] P. N. Dutta and B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Th. Appl. **2008** (2008): Article ID 406368.

[14] L. Găvruţa, P. Găvruţa, and F. Khojasteh, *Two classes of Meir-Keeler contractions*, Arxiv, 1405-5034-v1, 20 May 2014.

[15] M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc.,**40** (1973), 604–608.

[16] J. Jachymski, *Common fixed point theorems for some families of mappings*, Indian J. Pure Appl. Math. **25** (1994), 925–937.

[17] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl. **194** (1995), 293–303.

[18] E. Karapinar, A. Fulga and R. P. Agarwal, *A survey: F-contractions with related fixed point results*, J. Fixed Point Th. Appl. **22** (2020), Article number 69.

[19] M. A. Krasnoselskii and V. Ya. Stetsenko, *Towards a theory of equations with concave operators* (Russian), Sib. Math. Zh. **10** (1969), 565–572.

[20] S. Leader, *Fixed points for general contractions in metric spaces*, Math. Japonica **24** (1979), 17–24.

[21] J. Matkowski, *Integrable Solutions of Functional Equations*, Dissert. Math., vol. 127, Polish Sci. Publ., Warsaw, 1975.

[22] J. Matkowski, *Fixed point theorems for contractive mappings in metric spaces*, Časopis Pest. Mat. **105** (1980), 341–344.

[23] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

[24] G. H. Moore, *Zermelo's Axiom of Choice: its Origin, Development and Influence*, Springer, New York, 1982.

[25] Y. Moskhovakis, *Notes on Set Theory*, Springer, New York, 2006.

[26] S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[27] A. Petruşel and I. A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc. **134** (2006), 411–418.

[28] M. Piticari, *Successive approximations method and fixed point principle* (Romanian), Diss. Thesis, A. I. Cuza Univ., Iaşi, 1982.

[29] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. **5** (1972), 26–42.

[30] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.

[31] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlin. Anal. **47** (2001), 2683–2693.

[32] A.-F. Roldán, E. Karapinar, C. Roldán, and J. Martinez-Moreno, *Coincidence point theorems on metric spaces via simulation method*, Fixed Point Th. Appl. **2015** (2015): 98.

[33] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[34] B. Samet, C. Vetro, and H. Yazidi, *A fixed point theorem for a Meir-Keeler type contraction through rational expression*, J. Nonlin. Sci. Appl. **6** (2013), 162–169.

[35] E. Schechter, *Handbook of Analysis and its Foundation*, Academic Press, New York, 1997.

[36] A. Tarski, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, Fund. Math. **35** (1948), 79–104.

[37] M. Turinici, *Fixed points of implicit contraction mappings*, An. Şt. Univ. "Al. I. Cuza" Iaşi (Sect I-a, Mat) **22** (1976), 177–180.

[38] M. Turinici, *Implicit contractive maps in ordered metric spaces*, in: Topics in Mathematical Analysis and Applications, T. M. Rassias and L. Tóth (eds.), Springer Intl. Publ., Switzerland, 2014, pp. 715–746.

[39] M. Turinici, *Contractive maps in locally transitive relational metric spaces*, The Sci. World J. **2014** (2014): Article ID 169358.

[40] M. Turinici, *Contraction maps in pseudometric structures*, in: Essays in Mathematics and Applications), T. M. Rassias and P. M. Pardalos (eds.), Springer Intl. Publ., Switzerland, 2016, pp. 513–562.

[41] M. Turinici, *Reports in Metrical Fixed Point Theory*, Pim Editorial House, Iaşi, 2020.

[42] J. Vujaković, S. Mitrović, M. Pavlović and S. Radenović, *On recent results concerning F-contraction in generalized metric spaces*, Mathematics, 2020, 8, 767.

[43] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Th. Appl. **2012** (2012): 94.

[44] D. Wardowski, *Solving existence problems via F-contractions*, Proc. Amer. Math. Soc. **146** (2018), 1585–1598.

[45] E. S. Wolk, *On the principle of dependent choices and some forms of Zorn's lemma*, Canad. Math. Bull. **26** (1983), 365–367.

Manuscript received February 22 2022
revised January 5 2023

Mihai Turinici
A. Myller Math Seminar; A I Cuza University; 700506 Iaşi, Romania
E-mail address: mturi@uaic.ro