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A FIXED POINT RESULT FOR CONTRACTIVE MAPPINGS IN GENERALIZED METRIC SPACES

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Abstract. In a recent work of ours, extending the concept of a modular space, we have introduced generalized metric spaces and proved a fixed point result for certain contractive operators of Rakotch type. These operators map a closed subset into the space and have finite orbits of arbitrary lengths. In the present paper we extend this result assuming the existence of inexact orbits of arbitrary lengths and the uniform continuity of the metric.

1. Introduction

For more than sixty years now, there has been a lot of research activity regarding the fixed point theory of contractive and of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 4, 5, 8–10, 13, 15–21, 24, 25] and references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed point problems and variational inequalities, which find important applications in mathematical analysis, optimization theory, and in engineering, medical and the natural sciences [3, 6, 7, 22–25].

In [21] we have first introduced certain generalized metric spaces by extending the concept of a modular space studied in [11, 12, 14] and then established a fixed point theorem for certain Rakotch type contractive operators which map a closed subset into the space and have finite orbits of arbitrary lengths. In the present paper we extend this result assuming the existence of inexact orbits of arbitrary lengths and the uniform continuity of the metric.

2. Modular space

Let X be a vector space. A functional $\rho : X \to [0, \infty]$ is called a *modular* [11,12,14] if the following three properties hold:

(1) $\rho(x) = 0$ if and only $x = 0$;
(2) $\rho(-x) = \rho(x)$ for all $x \in X$;

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(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for each $x, y \in X$ and each $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = 1$.

The vector space

$$X_\rho := \{x \in X : \ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

is called a *modular space*.

Assume that ρ is a modular defined on a vector space X. We say that the modular ρ satisfies a Δ_2-type condition if there exists a number $M > 0$ such that

$$\rho(2x) \leq M\rho(x), \ x \in X_\rho. \tag{2.1}$$

The authors of [12] considered a modular function space L_ρ (which is a particular case of a modular space) with a modular ρ satisfying a Δ_2-type condition. They showed that if T is a self-mapping of a closed subset K of L_ρ such that for some $c \in [0, 1)$,

$$\rho(T(x) - T(y)) \leq c\rho(x, y) \text{ for all } x, y \in K$$

and such that there exists $x_0 \in K$ satisfying

$$\sup\{\rho(2T^p(x_0)) : \ p = 1, 2, \dots\} < \infty,$$

then T has a fixed point.

Assume that ρ is a modular defined on the vector space X. For each $x, y \in X$, define

$$d(x, y) := \rho(x - y).$$

It is easy to see that for each $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$ and that $d(x, y) = d(y, x)$.

Assume that ρ satisfies the Δ_2-type condition (2.1) with a number $M > 0$. Then for each $x, y, z \in X_\rho$, we have

$$\begin{aligned} d(x, z) &= \rho(x - z) = \rho((x - y) + (y - z)) \\ &= \rho(2(2^{-1}(x - y) + 2^{-1}(y - z))) \\ &\leq M\rho(2^{-1}(x - y) + 2^{-1}(y - z)) \\ &\leq M(\rho(x - y) + \rho(y - z)) \\ &\leq Md(x, y) + Md(y, z). \end{aligned}$$

We say that a modular ρ is uniformly continuous (see Definition 5.4 of [11]) if for each $\epsilon > 0$ and each $L > 0$, there exists $\delta > 0$ such that

$$|\rho(x + y) - \rho(x)| \leq \epsilon \tag{2.2}$$

for each pair $x, y \in X_\rho$ satisfying $\rho(y) < \delta$ and $\rho(x) < L$.

Assume that the modular ρ is uniformly continuous and that $\epsilon > 0$ and $L > 0$. Then there exists a number $\delta > 0$ such that (2.2) holds for each pair $x, y \in X_\rho$ satisfying $\rho(y) \leq \delta$ and $\rho(x) \leq L$.

Assume now that the points $x, y, z \in X_\rho$ satisfy

$$d(x, y) \leq L, \ d(y, z) \leq \delta.$$

Then

$$\rho(x - y) \leq L, \ \rho(y - z) \leq \delta,$$
$$d(x - z) = \rho(x - z) = \rho((x - y) + (y - z))$$

and in view of the choice of δ,

$$|d(x,z) - d(x,y)| = |\rho(x-z) - \rho(x-y)| \le \epsilon.$$

Thus we have shown that for each $\epsilon > 0$ and each $L > 0$, there exists $\delta > 0$ such that if $x, y, z \in X_\rho$ satisfy

$$d(x,y) \le L, \ d(y,z) \le \delta,$$

then

$$|d(x,z) - d(x,y)| \le \epsilon.$$

In other words, d is uniformly continuous.

3. Generalized metric space

Assume that X is a nonempty set, $d : X \times X \to [0, \infty]$, $M > 0$, and that for each $x, y, z \in X$,

$$d(x,y) = 0 \text{ if and only if } x = y, \tag{3.1}$$

$$d(x,y) = d(y,x) \tag{3.2}$$

and

$$d(x,z) \le Md(x,y) + Md(y,z). \tag{3.3}$$

We call the pair (X, d) a generalized metric space. For each point $x \in X$ and each number $r > 0$, set

$$B_d(x,r) := \{y \in X : \ d(x,y) \le r\}.$$

Clearly, a generalized metric space is both a generalization of the concept of a modular space and a generalization of the concept of a metric space. By investigating generalized metric spaces we are able to unify the study of these two important classes of spaces. For specific examples of modular spaces, see [11, 14].

We equip the space X with the uniformity determined by the base

$$\mathcal{U}(\epsilon) := \{(x,y) \in X \times X : \ d(x,y) \le \epsilon\}, \ \epsilon > 0. \tag{3.4}$$

This uniform space is metrizable (by a metric $\tilde{d}$). We also equip the space X with the topology induced by this uniformity and assume that the uniform space X is complete.

Consider a sequence $\{x_n\}_{n=1}^\infty \subset X$ and a point $x \in X$. Clearly, $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence if and only if for each $\epsilon > 0$, there exists a natural number $n(\epsilon)$ such that $d(x_n, x_m) \le \epsilon$ for every pair of integers $n, m \ge n(\epsilon)$.

A set $E \subset X$ is said to be bounded if

$$\sup\{d(x,y) : \ x, y \in E\} < \infty.$$

Assume that $\phi : [0, \infty) \to [0, 1]$ is a decreasing function such that

$$\phi(t) < 1 \text{ for all } t > 0.$$

In [21] we proved the following fixed point result for a Rakotch type contractive operator which maps a closed subset of the space into the space.

Theorem 3.1. *Let K be a nonempty closed subset of X and let $T : K \to X$ satisfy*

$$d(T(x), T(y)) \le \phi(d(x,y))d(x,y)$$

for each $x, y \in K$ satisfying $d(x,y) < \infty$. Assume that for each integer $n \ge 1$, there exists a point $x_n \in K$ such that

$$T^n(x_n) \text{ exists and belongs to } K$$

and that the set

$$E := \{T^i(x_n) : \ n = 1, 2, \dots \text{ and } i \in \{0, \dots, n\}\}$$

is bounded. Then there exists a point $x_* \in K$ such that $T(x_*) = x_*$. Moreover, this fixed point is unique if $d(x,y) < \infty$ for each pair $x, y \in K$.

We say that the generalized metric d is uniformly continuous on bounded sets if for each nonempty bounded set $D \subset X$ and each $\epsilon > 0$, there exists a number $\delta > 0$ such that for each $x, y \in D$ and each $z \in X$ satisfying $d(y,z) \le \delta$, the inequality

$$|d(x,y) - d(x,z)| \le \epsilon$$

holds.

From now on we assume that the generalized metric d is uniformly continuous on bounded sets.

Proposition 3.2. *Assume that $K \subset X$ is a nonempty and bounded set and that $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $x_1, x_2 \in K$ and each $y_1, y_2 \in X$ which satisfy*

$$d(x_i, y_i) \le \delta, \ i = 1, 2,$$

the inequality

$$|d(x_1, x_2) - d(y_1, y_2)| \le \epsilon$$

holds.

Proof. Set

$$K_0 = \{\xi \in X : \ \text{ there is } z \in K \text{ such that } d(z, \xi) \le 1\}.$$

Clearly, K_0 is bounded. In view of the uniform continuity condition imposed on d, there exists $\delta \in (0, 2^{-1})$ such that for each $x, y_1 \in K_0$ and each $y_2 \in X$ satisfying $d(y_1, y_2) \le \delta$, we have

$$|d(x, y_1) - d(x, y_2)| \le \epsilon/4.$$

Assume that $x_1, x_2 \in K$ and $y_1, y_2 \in X$ satisfy

$$d(x_i, y_i) \le \delta, \ i = 1, 2.$$

It is not difficult to see that $y_1, y_2 \in K_0$. When combined with the choice of δ, this implies that

$$|d(x_1, x_2) - d(x_1, y_2)| \le \epsilon/4$$

and

$$|d(x_1, y_2) - d(y_1, y_2)| \le \epsilon/4.$$

These inequalities imply that

$$|d(x_1, x_2) - d(y_1, y_2)| \le \epsilon/2.$$

This completes the proof of Proposition 3.2. □

4. The main result

We use the notations and definitions introduced in Section 3 and assume that all the assumptions made there hold.

Assume that $\phi : [0, \infty) \to [0, 1]$ is a decreasing function such that

$$\phi(t) < 1 \text{ for all } t > 0. \tag{4.1}$$

Now we proceed to present and prove a fixed point result for a Rakotch type contractive operator which extends Theorem 3.1 to the case where only inexact orbits of arbitrary lengths exist.

Theorem 4.1. *Let K be a nonempty closed subset of X and let $T : K \to X$ satisfy*

$$d(T(x), T(y)) \le \phi(d(x, y))d(x, y) \tag{4.2}$$

for each $x, y \in K$ satisfying $d(x, y) < \infty$. Assume that for each integer $n \ge 1$, there exists an inexact orbit

$$\{x_i^{(n)} : \ i = 0, \dots, n\} \subset K, \tag{4.3}$$

satisfying for each $i \in \{0, \dots, n-1\}$,

$$d(T(x_i^{(n)}), x_{i+1}^{(n)}) \le 1/n \tag{4.4}$$

and that the set

$$E := \{x_i^{(n)} : \ n = 1, 2, \dots \ \text{and } i \in \{0, \dots, n\}\}$$

is bounded. Then there exists a point $x_* \in K$ such that $T(x_*) = x_*$.

Proof. We let $T^0(x) = x$, $x \in K$, and set

$$E_0 := \{z \in X : \ \text{there exists } \xi \in E \text{ for which } d(z, \xi) \le 2\}. \tag{4.5}$$

Clearly, E_0 is a bounded set. Set

$$M_0 := \sup\{d(y, z) : \ y, z \in E_0\}. \tag{4.6}$$

Let $\epsilon > 0$. Fix a positive number

$$\Delta < 4^{-1}\epsilon(1 - \phi(\epsilon/2)). \tag{4.7}$$

Proposition 3.2 implies that there exists $\delta \in (0, \epsilon)$ such that for each

$$\xi_1, \xi_2, \eta_1, \eta_2 \in E_0$$

which satisfy

$$d(\xi_i, \eta_i) \le \delta, \ i = 1, 2,$$

the inequality

$$|d(\xi_1, \xi_2) - d(\eta_1, \eta_2)| \le \Delta \tag{4.8}$$

holds. Choose a natural number

$$p(\epsilon) > \max\{8\delta^{-1} + 1, \ 2M_0\epsilon^{-1}(1 - \phi(\epsilon))^{-1} + 2\}. \tag{4.9}$$

Let $n_i, p_i, \ i = 1, 2$, be integers such that

$$p(\epsilon) \le p_i \le n_i, \ i = 1, 2. \tag{4.10}$$

We claim that there exists an integer $j \in \{0, \dots, p(\epsilon)\}$ such that

$$d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}) \le \epsilon.$$

Suppose to the contrary that this is not true. Then for all $j = 0, \dots, p(\epsilon) - 1$,

$$d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}) > \epsilon. \tag{4.11}$$

Let $j \in \{0, \dots, p(\epsilon) - 1\}$. In view of (4.2) and (4.11), we have

$$\begin{aligned}
&d(T(x^{(n_1)}_{p_1-p(\epsilon)+j}), T(x^{(n_2)}_{p_2-p(\epsilon)+j}))\\
&\le \phi(d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}))d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j})\\
&\le \phi(\epsilon)d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}).
\end{aligned} \tag{4.12}$$

Inequalities (4.9)–(4.11) imply that for $i = 1, 2$, we have

$$d(x^{(n_i)}_{p_i-p(\epsilon)+j+1}, T(x^{(n_i)}_{p_i-p(\epsilon)+j})) \le n_i^{-1} \le p(\epsilon)^{-1} < 8^{-1}\delta. \tag{4.13}$$

By (4.5), (4.12), (4.13) and the choice of δ (see (4.8)),

$$\begin{aligned}
d(x^{(n_1)}_{p_1-p(\epsilon)+j+1}, x^{(n_2)}_{p_2-p(\epsilon)+j+1}) &\le \Delta + d(T(x^{(n_1)}_{p_1-p(\epsilon)+j}), T(x^{(n_2)}_{p_2-p(\epsilon)+j}))\\
&\le \Delta + \phi(\epsilon)d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}).
\end{aligned} \tag{4.14}$$

It follows from (4.7), (4.11) and (4.14) that

$$\begin{aligned}
&d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}) - d(x^{(n_1)}_{p_1-p(\epsilon)+j+1}, x^{(n_2)}_{p_2-p(\epsilon)+j+1})\\
&\ge (1 - \phi(\epsilon))d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}) - \Delta\\
&\ge \epsilon(1 - \phi(\epsilon)) - \Delta \ge 2^{-1}\epsilon(1 - \phi(\epsilon))
\end{aligned} \tag{4.15}$$

for all $j = 0, \dots, p(\epsilon) - 1$. By (4.5), (4.6), (4.10) and (4.15),

$$\begin{aligned}
M_0 &\ge d(x^{(n_1)}_{p_1-p(\epsilon)}, x^{(n_2)}_{p_2-p(\epsilon)})\\
&\ge d(x^{(n_1)}_{p_1-p(\epsilon)}, x^{(n_2)}_{p_2-p(\epsilon)}) - d(x^{(n_1)}_{p_1}, x^{(n_2)}_{p_2})\\
&= \sum_{j=0}^{p(\epsilon)-1} (d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}) - d(x^{(n_1)}_{p_1-p(\epsilon)+j+1}, x^{(n_2)}_{p_2-p(\epsilon)+j+1})\\
&\ge 2^{-1}\epsilon(1 - \phi(\epsilon))p(\epsilon)
\end{aligned}$$

and

$$p(\epsilon) \le 2M_0\epsilon^{-1}(1 - \phi(\epsilon))^{-1}.$$

This, however, contradicts (4.9). The contradiction we have reached shows that there does exist $j \in \{0, \dots, p(\epsilon)\}$ such that

$$d(x^{(n_1)}_{p_1-p(\epsilon)+j}, x^{(n_2)}_{p_2-p(\epsilon)+j}) \le \epsilon, \tag{4.16}$$

as claimed.

Next. we show that for all integers

$$i \in \{j, \dots, n(\epsilon)\},$$

we have

$$d(x^{(n_1)}_{p_1-p(\epsilon)+i}, x^{(n_2)}_{p_2-p(\epsilon)+i}) \le \epsilon. \tag{4.17}$$

Suppose to the contrary that this is not true. Then there exists $k \in \{j, \dots, p(\epsilon)\}$ such that

$$d(x^{(n_1)}_{p_1-p(\epsilon)+k}, x^{(n_2)}_{p_2-p(\epsilon)+k}) > \epsilon. \tag{4.18}$$

In view of (4.16) and (4.18),

$$k > j.$$

We may assume without any loss of generality that (4.17) holds for all integers $i = j, \dots, k-1$ and, in particular,

$$d(x^{(n_1)}_{p_1-p(\epsilon)+k-1}, x^{(n_2)}_{p_2-p(\epsilon)+k-1}) \le \epsilon. \tag{4.19}$$

There are two cases:

$$d(x^{(n_1)}_{p_1-p(\epsilon)+k-1}, x^{(n_2)}_{p_2-p(\epsilon)+k-1}) \le \epsilon/2, \tag{4.20}$$

$$d(x^{(n_1)}_{p_1-p(\epsilon)+k-1}, x^{(n_2)}_{p_2-p(\epsilon)+k-1}) > \epsilon/2. \tag{4.21}$$

Assume that (4.20) holds. In view of (4.2) and (4.20),

$$\begin{aligned} &d(T(x^{(n_1)}_{p_1-p(\epsilon)+k-1}), T(x^{(n_2)}_{p_2-p(\epsilon)+k-1})) \\ &\le d(x^{(n_1)}_{p_1-p(\epsilon)+k-1}, x^{(n_2)}_{p_2-p(\epsilon)+k-1}) \le \epsilon/2. \end{aligned} \tag{4.22}$$

By (4.4), (4.5), (4.9), (4.10) and the choice of δ (see (4.8)),

$$|d(x^{(n_1)}_{p_1-p(\epsilon)+k}, x^{(n_2)}_{p_2-p(\epsilon)+k}) - d(T(x^{(n_1)}_{p_1-p(\epsilon)+k-1}), T(x^{(n_2)}_{p_2-p(\epsilon)+k-1}))| \le \Delta. \tag{4.23}$$

Inequalities (4.7), (4.22) and (4.23) imply that

$$d(x^{(n_1)}_{p_1-p(\epsilon)+k}, x^{(n_2)}_{p_2-p(\epsilon)+k}) \le \Delta + \epsilon/2 < \epsilon.$$

This contradicts (4.18). The contradiction we have reached proves that (4.21) holds. By (4.2), (4.19) and (4.21),

$$\begin{aligned} &d(T(x^{(n_1)}_{p_1-p(\epsilon)+k-1}), T(x^{(n_2)}_{p_2-p(\epsilon)+k-1})) \\ &\le \phi(d(x^{(n_1)}_{p_1-p(\epsilon)+k-1}, x^{(n_2)}_{p_2-p(\epsilon)+k-1}))d(x^{(n_1)}_{p_1-p(\epsilon)+k-1}, x^{(n_2)}_{p_2-p(\epsilon)+k-1}) \\ &\le \phi(\epsilon/2)\epsilon. \end{aligned} \tag{4.24}$$

Clearly, (4.23) holds in this case too. It follows from (4.1), (4.23) and (4.24) that

$$d(x^{(n_1)}_{p_1-p(\epsilon)+k}, x^{(n_2)}_{p_2-p(\epsilon)+k}) \le \phi(\epsilon/2)\epsilon + \Delta \le \epsilon.$$

This, however, contradicts (4.18). The contradiction we have reached yields that (4.17) holds for all $i \in \{j, \dots, p(\epsilon)\}$ and

$$d(x^{(n_1)}_{p_1}, x^{(n_2)}_{p_2}) \le \epsilon.$$

For each integer $p \ge 1$, denote by E_p the closure of the set

$$\{x^{(n)}_k : n \ge p \text{ is an integer and } k \in \{p, \dots, n\}\}.$$

We have shown that the diameters of E_p (with respect to the metric $\tilde{d}$) tend to zero as $p \to \infty$. Therefore

$$\cap_{p=1}^{\infty} E_p = \{x_*\}, \tag{4.25}$$

where $x_* \in K$.

Next, we show that $T(x_*) = x_*$. To this end, let $\epsilon > 0$. In view of (3.4), there exists an integer $p \geq 1$ such that

$$E_p \times E_p \subset \mathcal{U}(\epsilon),\ p^{-1} < \epsilon. \tag{4.26}$$

By (4.25) and (4.26),

$$d(x_*, z) \leq \epsilon \text{ for all } z \in E_p. \tag{4.27}$$

It follows from (4.27) and the definition of E_p that

$$d(x_*, x_p^{(p+1)}) \leq \epsilon,\ d(x_*, x_{p+1}^{(p+1)}) \leq \epsilon. \tag{4.28}$$

In view of (4.2) and (4.28), we have

$$d(T(x_*), T(x_p^{(p+1)})) \leq \epsilon.$$

When combined with (4.26) and (4.28), this implies that

$$\begin{aligned} d(T(x_*), x_{p+1}^{(p+1)}) &\leq Md(T(x_*), T(x_p^{(p+1)})) + Md(T(x_p^{(p+1)}), x_{p+1}^{(p+1)}) \\ &\leq M\epsilon + Mp^{-1} < 2M\epsilon. \end{aligned}$$

When combined with (4.28), this implies that

$$\begin{aligned} d(x_*, T(x_*)) &\leq Md(x_*, x_p^{(p+1)}) + Md(x_{p+1}^{(p+1}, T(x_*)) \\ &\leq M\epsilon + 2M^2\epsilon. \end{aligned}$$

Since ϵ is an arbitrary positive number, we conclude that $T(x_*) = x_*$, as asserted.

This completes the proof of Theorem 4.1. □

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