Applied Analysis and Optimization
Volume 7, Number 2, 2023, 159–169

Yokohama Publishers
ISSN 2189-1664 Online Journal

EXISTENCE THEOREMS FOR A VARIATIONAL RELATION PROBLEM OF STAMPACCHIA TYPE

MIRCEA BALAJ AND DAN FLORIN SERAC

ABSTRACT. The aim of this paper is to investigate a variational relation problem of Stampacchia type. The existence results presented here are different by the ones obtained in other works which deal with this subject. As applications, we establish existence theorems for two types of variational inequality problems.

1. Introduction

In the seminal paper [23], Luc proposed a unitary approach for various problems from optimization that have a common structure and similar proofs. For this purpose he introduced the concept of variational relation. Given three nonempty sets X, Y and Z, a variational relation ρ can be identified with a subset of the product set $X \times Y \times Z$, so that $\rho(x, y, z)$ holds if and only if the point (x, y, z) belongs to that set. The problem studied by Luc reads as follows:

(LVRP) Find $x_0 \in X$ such that
(i) $x_0 \in S(x_0)$;
(ii) $\rho(x_0, y, z)$ holds for every $y \in P(x_0)$ and $z \in Q(x_0, y)$,

where X, Y and Z are nonempty sets, $S : X \rightrightarrows X$, $P : X \rightrightarrows Y$ and $Q : X \times Y \rightrightarrows Z$ are set-valued mappings with nonempty values and $\rho(x, y, z)$ is a relation linking elements $x \in X$, $y \in Y$, $z \in Z$.

Soon after the apparition of the paper [23], have been published many papers in which is studied the existence of the solution for problem (LVRP) [11, 24], or for other types of variational relation problems [1, 6, 8]. In the meantime, the investigation area of variational relation problems has been expanded. Thus, several systems of two variational relation problems are studied in [7, 12, 18, 22], topological and stability properties of solution set in parametric variational relation problems can be found in [17, 20] and some algorithms for determining the solution set of a certain type of variational relation problems are given in [14, 21].

The present work finds its place in this group of papers. Here, we are interested by the existence of solution for the following problem:

Let X be a nonempty convex subset of a topological vector space E, Z be a topological vector space, $T : X \to Z$ be a nonempty-valued mapping and $\rho(x, y, z)$ a be a relation linking elements $x \in X$, $y \in X$ and $z \in T(X)$.

$(SVRP)$ Find $x_0 \in X$ and $z_0 \in T(x_0)$ such that $\rho(x_0, y, z_0)$ holds for all $y \in X$.

2020 *Mathematics Subject Classification.* 47H05, 90C33.
Key words and phrases. Variational relation problem, T-KKM relation, fixed point, variational inequality.

The notation (SVRP) is motivated by the fact that the problem above includes, as a special case, the Stampacchia variational inequality problem. To justify this statement it suffices to consider that Z is E^* (the dual of T) and the relation ρ is defined by

$$\rho(x, y, z) \text{ holds } \iff \langle z, y - x\rangle \geq 0.$$

Problem (SVRP) is not new. Existence theorems for this problem can be found in [6, 10]. On the other hand, it can be regarded as a particular case of other types of variational relations problems studied in [1, 7, 8]. However, the results obtained in this paper can not be derived from the ones established in the mentioned papers.

The paper is organized as follows. In Section 2 we review some concepts concerning set-valued mappings and ternary relations. Section 3 is devoted to the existence of solution for problem (SVRP). The proofs of the main results combine an intersection theorem and a fixed point theorem. In the last section, as applications of the main results, we obtain existence theorems for the solutions of two types of variational inequality problems.

In what follows, all topological (vector) spaces are assumed to be Hausdorff. For a subset A of a topological vector space, the standard notation $\mathrm{co}A$, denotes the convex hull of A.

2. Preliminaries

In this section we recall some definitions concerning set-valued mappings and variational relations and fix the used terminology.

Let X and Y be nonempty sets. To a set-valued mapping $T : X \rightrightarrows Y$ we associate other two mappings, $T^-, T^* : Y \rightrightarrows X$ defined by

$$T^-(y) = \{x \in X : y \in T(x)\}, \;\; T^*(y) = X \setminus T^-(y) = \{x \in X : y \notin T(x)\},$$

called the lower inverse, respectively, the dual of T. The values of T^- and T^* are called the fibers, respectively, the cofibers of T.

If X and Y are topological spaces, a set-valued mapping $T : X \rightrightarrows Y$ is said to be:

(i) upper semicontinuous if for every open subset U of Y, the set
$$T^+(U) = \{x \in X : T(x) \subseteq U\}$$
is open;

(ii) lower semicontinuous if for every open subset U of Y, the set
$$T^-(U) = \{x \in X : T(x) \cap U \neq \emptyset\}$$
is open;

(iii) closed if its graph (that is, the set $\mathrm{Gr}\ \mathrm{T} = \{(x, y) \in X \times Y : y \in T(x)\}$) is a closed subset of $X \times Y$;

(iv) compact if its range, $T(X)$ is contained in a compact subset of Y.

Known results concerning above notions are collected in the next lemma.

Lemma 2.1. *Let X and Y be topological spaces and $T : X \rightrightarrows Y$.*

(i) *If Y is compact, then T is closed if and only if it is upper semicontinuous and closed-valued.*

(ii) *If* $K \subseteq X$ *is compact and* T *is compact-valued and upper semicontinuous, then* $T(K)$ *is compact.*

(iii) T *is lower semicontinuous if and only if for any net* $\{x_t\}$ *in* X *converging to* $x \in X$ *and each* $y \in S(x)$*, there exists a subnet* $\{x_{t_i}\}$ *of the net* $\{x_t\}$ *and a net* $\{y_i\}$ *in* Y *converging to* y *with* $y_i \in S(x_{t_i})$*, for each index* i*.*

(iv) *If* T *is compact-valued, then it is upper semicontinuous if and only if for every net* $\{(x_t, y_t)\}$ *in* $Gr\,T$*, with* $\{x_t\}$ *converging to some* $x \in X$*, there exists a subnet* $\{y_{t_i}\}$ *of* $\{y_t\}$ *converging to some* $y \in T(x)$*.*

Let X, Y, Z be nonempty subsets of three topological vector spaces and $\rho(x, y, z)$ a relation between elements $x \in X$, $y \in Y$ and $z \in Z$. We say that the relation ρ is:

(i) closed (convex, respectively) if the set $\{(x, y, z) \in X \times Y \times Z : \rho(x, y, z) \text{ holds}\}$ is closed (convex, respectively) in $X \times Y \times Z$;

(ii) closed (convex, respectively) in two of the three variables, say y and z, if for each $x \in X$ the set $\{(y, z) \in Y \times Z : \rho(x, y, z) \text{ holds}\}$ is closed (convex, respectively) in $Y \times Z$;

(iii) closed (convex, respectively) in the variable z, if for each $(x, y) \in X \times Y$ the set $\{z \in Z : \rho(x, y, z) \text{ holds}\}$ is closed (convex, respectively) in Z.

Given a convex-valued set-valued mapping $T : X \rightrightarrows Z$, we will say that ρ is T-convex in the variable z if for each $(x, y) \in X \times Y$, $\{z \in T(x) : \rho(x, y, z) \text{ holds}\}$ is a convex set. Clearly, if ρ iz convex in the variable z it is T-convex in z, but the converse is not true. Indeed, let us consider the set-valued mapping $T : \mathbb{R} \rightrightarrows \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the relation ρ defined by

$$T(x) = [-|x|, |x|],\ f(x, y, z) = z^4 - 6x^2z^2 + y,$$

$$\rho(x, y, z) \text{ holds } \iff f(x, y, z) \geq 0.$$

Since $\frac{\partial^2 f(x,y,z)}{\partial z^2} = 12(z^2 - x^2) \leq 0$ for all $z \in T(x)$, $f(x, y, \cdot)$ is a concave function on $T(x)$, hence the relation is T-convex in z. On the other hand one can easily check that $\rho(1, 1, 0)$ and $\rho(1, 1, 3)$ hold but $\rho(1, 1, 1)$ does not hold, hence ρ is not convex in the variable z.

The complementary relation of ρ is denoted by ρ^c, that is, $\rho^c(x, y, z)$ holds if and only if $\rho(x, y, z)$ does not hold.

In the literature can be found various KKM concepts for relations, which are nothing else than adaptations of some known KKM notions for set-valued mappings. The concept we need in the paper is given in the following definition.

Definition 2.2. Let X be nonempty convex set in a vector space, Z be a nonempty set, $T : X \rightrightarrows Z$ be a nonempty-valued mapping and $\rho(x, y, z)$ be a relation linking elements $x \in X$, $y \in X$ and $z \in T(X)$. The relation ρ is said to be KKM with respect to T (T-KKM, in short) if, for every finite subset A of X and for every $x \in \text{co}\, A$, there exist $y \in A$ and $z \in T(x)$ such that $\rho(x, y, z)$ holds.

Remark 2.3. Note that the meaning of the term "T-KKM relation" varies in the literature. Thus, in [7], a relation ρ is called T-KKM if for any A and x as above, and for each $z \in T(x)$, $\rho(x, y, z)$ holds for some $y \in A$. It should also be mentioned that the concept of T-KKM mapping used by us is a special case of those of (α, T)-KKM mapping introduced in [6, Definition 3.1].

3. Main results

The following two intersection results play an important role in our work.

Lemma 3.1 ([2, Lemma 3.1])**.** *Let X be a nonempty convex set and Y be a nonempty compact convex set, each in a topological vector space. If $S : X \rightrightarrows Y$ is a closed mapping with nonempty convex values and convex cofibers, then $\bigcap_{x\in X} S(x) \neq \emptyset$.*

Lemma 3.2 ([2, Lemma 3.2])**.** *Let X be a nonempty convex set and Y be a nonempty compact convex set, each in a topological vector space. If $S : X \rightrightarrows Y$ is a set-valued mapping with open graph, nonempty convex values and convex cofibers, then $\bigcap_{x\in X} S(x) \neq \emptyset$.*

The proof of our first existence theorem relies on Lemma 3.1.

Theorem 3.3. *Let X be a nonempty compact and convex subset of a topological vector space and Z be a topological vector space. Let $T : X \rightrightarrows Z$ be a set-valued mapping with nonempty, compact and convex values and $\rho(x, y, z)$ be a relation linking elements $x \in X$, $y \in X$ and $z \in T(X)$. Assume that:*

(i) *ρ is closed in the variables y, z and T-convex in the variable z.*
(ii) *ρ^c is convex in the variable y;*
(iii) *ρ is T-KKM;*
(iv) *for every $y \in X$, the set $\{x \in X : \exists\, z \in T(x)$ such that $\rho(x, y, z)$ holds$\}$ is closed.*

Then, there exists an $x_0 \in X$ and $z_0 \in T(x_0)$ such that $\rho(x_0, y, z_0)$ holds for all $y \in X$.

Proof. We divide the proof into two steps.

Step 1 We claim that

$$\exists x_0 \in X : \forall y \in X, \exists\, z \in T(x_0) \text{ such that } \rho(x_0, y, z) \text{ holds.} \tag{3.1}$$

Consider the set-valued mappings $G, H : X \rightrightarrows X$ defined by

$$G(x) = \{y \in X : \forall z \in T(x)\ \rho(x, y, z) \text{ does not hold}\} \text{ and } H(x) = \operatorname{co} G(x).$$

Assume that the statement (3.1) would be false. Then the set-valued mapping H has nonempty convex values. From (iv), G has open fibers and then, in view of [26, Lemma 5.1], H has open fibers too. From the Fan-Browder fixed point theorem (see [13] and [15]), the set-valued mapping H has a fixed point x_0. Consequently, there exists a nonempty finite set $A \subseteq G(x_0)$ such that $x_0 \in \operatorname{co} A$. From $A \subseteq G(x_0)$ we infer that $\rho(x_0, y, z)$ does not hold for all $y \in A$ and $z \in T(x_0)$; but clearly, this contradicts assumption (iii).

Step 2 Taking step 1 into account, the set-valued mapping $S : X \rightrightarrows T(x_0)$ defined by

$$S(y) = \{z \in T(x_0) : \rho(x_0, y, z) \text{ holds}\},$$

has nonempty values. From (i) it follows easily that the mapping S is closed and convex-valued, and from (ii) we infer that the cofibers of S are convex sets.

By Lemma 3.1, there exists a point $z_0 \in \bigcap_{y\in X} S(y)$. Consequently, $\rho(x_0, y, z_0)$ holds for all $y \in X$. □

Because assumption (iv) of the previous theorem is more difficult to verify, the following proposition is useful for concrete relations ρ.

Proposition 3.4. *Condition (iv) of Theorem 3.3 is fulfilled whenever T is upper semicontinuous and compact-valued and ρ is closed in the variables x and z.*

Proof. We have to show that under the given assumptions, for an arbitrary $y \in X$, the set

$$M_y = \{x \in X : \exists\, z \in T(x) \text{ such that } \rho(x, y, z) \text{ holds}\}$$

is closed. Let $\{x_t\}$ be a net in M_y converging to some $x \in X$. Then, for each index t, there exists $z_t \in T(x_t)$ such that $\rho(x_t, y, z_t)$ holds. Since T is upper semicontinuous and compact-valued, there exist $z \in T(x)$ and a subnet $\{z_{t_i}\}$ of $\{z_t\}$ converging to z. Since the relation ρ is closed in the variables x and z, it follows that $\rho(x, y, z)$ holds, hence $x \in M_y$. □

It is worth mentioning that condition (iv) of Theorem 3.3 can be satisfied even if T is not upper semicontinuous. For instance, let X be any compact interval in $\mathbb{R}$ and $Z = \mathbb{R}$. Let the relation ρ be defined by

$$\rho(x, y, z) \text{ holds} \iff (x + y)z = 0,$$

and the set-valued mapping $T : X \longrightarrow \mathbb{R}$ be defined by

$$T(x) = \begin{cases} [-1, 1] & \text{if } x \in X \setminus \{x_0\} \\ \{0\} & \text{if } x = x_0, \end{cases}$$

where x_0 is an arbitrarily fixed point in X. One can see that T is not upper semicontinuous, but for every $y \in X$, $\{x \in X : \exists z \in T(x) \text{ such that } (x + y)z = 0\} = X$.

Combining Theorem 3.3 and Proposition 3.4 we get the following theorem:

Theorem 3.5. *Let X be a nonempty compact and convex subset of a topological vector space and Z be a topological vector space. Let $T : X \rightrightarrows Z$ be an upper semicontinuous set-valued mapping with nonempty, compact and convex values and ρ a be a relation between the elements of the sets X, X and $T(X)$. Assume that:*

(i) *ρ is T-convex in the variable z and closed.*
(ii) *ρ^c is convex in the variable y;*
(iii) *ρ is T-KKM;*

Then, there exists an $x_0 \in X$ and $z_0 \in T(x_0)$ such that $\rho(x_0, y, z_0)$ holds for all $y \in X$.

We will see in the next theorem that, if the set X is not compact, the conclusion of Theorem 3.5 remains valid if we add certain coercivity assumptions.

Theorem 3.6. *Let X be a nonempty convex set and Z be a nonempty convex set, each in a topological vector space. Let $T : X \rightrightarrows Z$ be an upper semicontinuous set-valued mapping with nonempty, compact and convex values and $\rho(x, y, z)$ be a relation linking elements $x \in X$, $y \in X$ and $z \in T(X)$. Assume that:*

(i) *ρ is T-convex in the variable z and closed;*
(ii) *ρ^c is convex in the variable y;*
(iii) *ρ is T-KKM;*

Moreover, assume that X contains a compact convex set C_0 and a compact set K_0 such that one of the following assumptions holds:

(a) *for each $x \in X \setminus K_0$ and any $z \in T(x)$, there exists $y \in C_0$ such that $\rho(x, y, z)$ does not holds;*

(b) *$\rho(x, y, z)$ holds for all $x \in X \setminus K_0$, $z \in T(x)$ and $y \in X \setminus C_0$.*

Then, there exists an $x_0 \in X$ and $z_0 \in T(x_0)$ such that $\rho(x_0, y, z_0)$ holds for all $y \in X$.

Proof. Assume first that (a) holds. In this case the proof follows the same lines as the proof of [6, Theorem 4.3]. Consider the family of sets

$$\mathcal{C} = \{C : C_0 \subseteq C \subseteq X, C \text{ is compact and convex}\}.$$

From Theorem 3.5, for every $C \in \mathcal{C}$, there exist $x_C \in C$ and $z_C \in T(x_C)$ such that $\rho(x_C, y, z_C)$ holds for all $y \in C$. As $C_0 \subseteq C$, from (a) we infer that $x_C \in K_0$.

If $C', C'' \in \mathcal{C}$, then the set co $\big(C' \cup C''\big)$ belongs also to $\mathcal{C}$, because the convex hull of the union of a finite family of compact convex sets is compact (see [4, Lemma 5.2.9]). Consequently $\mathcal{C}$ is a directed set relative to the order relation $\subseteq$. Since the set K_0 is compact, we may assume that the net $\{x_C\}_{C \in \mathcal{C}}$ converges to some $x_0 \in K_0$. As T is upper semicontinuous and compact-valued we may also assume that the net $\{z_C\}_{C \in \mathcal{C}}$ converges to some $z_0 \in T(x_0)$.

Take an arbitrary $y \in X$ and denote by

$$C_y := \text{ co } \big(C_0 \cup \{y\}\big).$$

Clearly, $C_y \in \mathcal{C}$ and for every $C \in \mathcal{C}$ satisfying $C_y \subseteq C$, $\rho(x_C, y, z_C)$ holds. Because the relation ρ is closed, it follows that $\rho(x_0, y, z_0)$ holds. Thus, the proof is complete in case (a).

Let us now deal with the case when (b) occurs. If assumption (a) is fulfilled, we have nothing to do. Hence, let us assume that there exists $x_0 \in X \setminus K_0$ and $z_0 \in T(x_0)$ such that $\rho(x_0, y, z_0)$ holds for all $y \in C_0$. Since, by (b), $\rho(x_0, y, z_0)$ holds also for all $y \in X \setminus C_0$, we reach the desired conclusion. □

Remark 3.7. When condition (a) is satisfied, Theorem 3.6 can be considered as a version of [6, Theorem 4.3]. However, although they are close, the assumptions of the two results are different. For instance, the condition that the relation ρ be T-KKM is replaced in [6, Theorem 4.3] with a stronger condition (marked below with (P)):

(P) *for every finite subset A of X and for every $x \in$ co A, there exists $y \in A$ such that $\rho(x, y, z)$ holds for all $z \in T(x)$.*

In what follows we provide an example in which Theorem 3.6 is applicable but the mentioned result from [6] does not work.

Example 3.8. Let $X = [0, \infty[$, $T(x) = [x, x + 1]$ for all $x \in [0, \infty[$ and relation ρ defined by

$$\rho(x, y, z) \text{ holds } \iff f(x, y, z) \geq 0,$$

where $f(x, y, z) = \frac{x}{x+1} - \frac{z}{y+1}$, $x, y, z \geq 0$.

The relation ρ is T-convex in the variable z and ρ^c is convex in the variable y, because the function f is affine in the third variable and increasing in the second variable.

Let A be a nonempty finite subset of $[0,\infty[$ and $y = \max A$. Then, for every $x \in$ co A we have $x \leq y$ and

$$f(x,y,x) = \frac{x}{x+1} - \frac{x}{y+1} \geq \frac{x}{x+1} - \frac{x}{x+1} = 0,$$

hence relation ρ is T-KKM.

We prove further that the coercivity condition (a) from Theorem 3.6 is fulfilled if we take $C_0 = K_0 = [0,1]$. Indeed, if $x \in]1,\infty[$, then $\frac{1}{x} \in [0,1]$ and for every $z \in [x, x+1]$ we have

$$f\big(x, \frac{1}{x}, z\big) = \frac{x}{x+1} - \frac{z}{\frac{1}{x}+1} = \frac{x(1-z)}{x+1} < 0,$$

hence $\rho\big(x, \frac{1}{x}, z\big)$ does not hold.

From Theorem 3.6, there exists $(x_0, z_0) \in$ Gr T such that $f(x_0, y, z_0) \geq 0$ for every $y \geq 0$. One can easily see that $(0,0)$ is the unique pair (x_0, z_0) with this property.

In order to see that [6, Theorem 4.3] can not be applied, it suffices to show that condition (P) is not satisfied. For this, take $A = \{1\}$, $z = 2 \in T(1)$ and observe that $f(1,1,2) < 0$.

The next theorem, established in the setting of finite dimensional spaces, can be regarded as an open version of Theorem 3.5.

Theorem 3.9. *Let X be a nonempty compact and convex subset of a finite dimensional topological vector space and Z be a topological vector space. Let $T : X \rightrightarrows Z$ be a lower semicontinuous and convex-valued set-valued mapping and $\rho(x,y,z)$ a relation between elements $x, y \in X$ and $z \in T(X)$. Assume that the following conditions are satisfied:*

(i) *ρ is convex in the variable z and open;*
(ii) *ρ^c is convex in the variable y;*
(iii) *ρ is T-KKM;*

Then, there exists an $x_0 \in X$ and $z_0 \in T(x_0)$ such that $\rho(x_0, y, z_0)$ holds for all $y \in X$.

Proof. The proof is similar to that of Theorem 3.3, the set-valued mappings G, H and S being defined as there.

Step 1 We need first to prove that the statement (3.1) from the aforementioned proof is fulfilled. By way of contradiction, suppose that for each $x \in X$, there exists $y \in X$ such that $\rho(x,y,z)$ does not hold, for all $z \in T(x)$. Then, the set-valued mapping G has nonempty values.

Denote by $\big(\text{Gr } G\big)^c$ the complement in $X \times X$ of the graph of G, that is

$$\big(\text{Gr } G\big)^c = \{(x,y) \in X \times X : \exists z \in T(x) \text{ such that } \rho(x,y,z) \text{ holds}\}.$$

Let $(x,y) \in \big(\text{Gr}G\big)^c$ and $z \in T(x)$ such that $\rho(x,y,z)$ holds. Since relation ρ is open, there exist the open neighborhoods U_x of x, U_y of y and U_z of z such that $\rho(x', y', z')$

holds for all $(x',y',z') \in U_x \times U_y \times U_z$. Because T is lower semicontinuous, there exists a neighborhood V_x of x such that for all $x' \in V_x$, $T(x') \cap U_z \neq \emptyset$. Then, for every $(x',y') \in (U_x \cap V_x) \times U_y$, there exists $z' \in T(x') \cap U_z$ for which $\rho(x',y',z')$ holds. Consequently, $(U_x \cap V_x) \times U_y$ is a neighborhood of (x,y) contained in $(\text{Gr } G)^c$, hence the set $(\text{Gr } G)^c$ is open in $X \times X$, Therefore, G is a closed mapping and, in view of Lemma 2.1 (i), it is upper semicontinuous with compact values. From [4, Theorem 17.35], we infer that H is upper semicontinuous and clearly it has nonempty compact and convex values. The Kakutani fixed point theorem ([19, Theorem 1]) guarantees that the set-valued mapping H has a fixed point x_0. Consequently, there exists a nonempty finite set $A \subseteq G(x_0)$ such that $x_0 \in \text{ co } A$. Then, for all $y \in A$ and $z \in T(x_0)$, $\rho(x_0,y,z)$ does not holds; but this contradicts the fact that the relation ρ is T-KKM.

Step 2 In view of Step 1, S has nonempty values. Moreover, from (ii), it follows easily that the values and the cofibers of S are convex sets. Since the relation ρ is open, the graph of S is open in $X \times T(x_0)$. By Lemma 3.2, there exists a point $z_0 \in \bigcap_{y\in X} S(y)$. Hence, $z_0 \in T(x_0)$ and $\rho(x_0,y,z_0)$ holds for all $y \in X$. □

Remark 3.10. Theorem 3.9 generalizes slightly [3, Theorem 5.1]. In the mentioned theorem is required the convexity of ρ in the variables x and z, while here it is needed for ρ to be convex only in the variable z.

When the set-valued mapping T has convex graph, we will show that the condition as relation ρ be T-KKM can be replaced by a weaker one. It can be seen that the convexity of the graph of T can be expressed by the condition

$$\lambda T(x_1) + (1-\lambda)T(x_2) \subseteq T(\lambda x_1 + (1-\lambda)x_2) \ \forall x_1, x_2 \in X, \ \forall \lambda \in [0,1]. \tag{3.2}$$

A set-valued mapping T satisfying condition (3.2) is often called convex (see, for instance, [5, Definition 2.21]). However, because in the case of a function (uni-valued mapping), the classic concept of convexity and the one given by (3.2) are different, we prefer to use the term of set-valued mapping with convex graph.

Theorem 3.11. *Let X be a nonempty convex subset of topological vector space, Z be a topological vector space, $T : X \rightrightarrows Z$ be a compact set-valued mapping with nonempty values and convex graph and $\rho(x,y,z)$ be a relation between elements $x, y \in X$ and $z \in T(x)$. Problem (SVRP) has at least a solution whenever the following conditions are satisfied:*

(i) *ρ is either closed or open;*
(ii) *ρ is convex in the variables x and z and ρ^c is convex in the variable y;*
(iii) *for each $y \in X$, there exists $(x,z) \in$ Gr T such that $\rho(x,y,z)$ holds.*

Proof. Consider the set-valued mapping $S : X \rightrightarrows$ Gr T defined by

$$S(y) = \{(x,z) \in \text{ Gr } T : \rho(x,y,z) \text{ holds } \}.$$

From (ii) and (iii) it follows that S has nonempty convex values and convex cofibers. By (i), the graph of S is either closed or open in $X \times$ Gr T. According to Lemmas 3.1 (when ρ is closed) and 3.2 (when ρ is open), there exists $(x_0,z_0) \in$ Gr T such that $(x_0,z_0) \in \bigcap_{y\in X} S(y)$. This completes the proof. □

Remark 3.12. In order to show that any T-KKM relation satisfies condition (iii) of Theorem 3.11 it suffices to take in Definition 2.2, $A = \{y\}$. On the other side, simple examples can be given to prove that the converse is not true.

4. Variational inequalities

For bringing our previous results closer to some problems encountered in the literature, in this section we investigate the existence of solutions for two types of variational inequality problems. In what follows, E is a real normed linear space, E^* is its dual and $\langle \cdot, \cdot \rangle$ is the duality pairing between E^* and E.

Given a nonempty compact convex subset X of E, a function $\varphi : X \times X \to E$ and a nonempty-valued mapping $T : X \rightrightarrows E^*$ we consider the following two problems:

$$(VIP-1) \quad \text{Find } \ (x_0, x_0^*) \in \text{Gr}\, T \text{ such that } \ \langle x_0^*, \varphi(x_0, y)\rangle \geq 0, \ \text{ for all } y \in X.$$

$$(VIP-2) \quad \text{Find } \ (x_0, x_0^*) \in \text{Gr}\, T \text{ such that } \ \langle x_0^*, \varphi(x_0, y)\rangle > 0, \ \text{ for all } y \in X.$$

Problem (VIP-1) is known in the literature under the name of variational-like inequality problem and its study has been initiated by Parida, Sahoo and Kumar [25] (when E is a finite dimensional space and T is a uni-valued mapping). It is worth noting that if $\varphi(x, y) = y - x$, (VIP-1) reduces to the Stampacchia variational inequality problem. To the best of our knowledge, problem (VIP-2) is new, although it can be considered a particular case of a vector variational-like inequality studied in [16].

Theorem 4.1. *Let X, φ and T as above. Assume that:*

(i) *the function φ is affine in the second variable and continuous;*
(ii) *T is norm-to-weak* upper semicontinuous with weak* compact convex values;*
(iii) *for each $x \in X$, there exists $x^* \in T(x)$ such that $\langle x^*, \varphi(x, x)\rangle \geq 0$.*

Then, problem (VIP-1) has a solution.

Proof. We intend to apply Theorem 3.5 when Z is replaced with E^* and the relation ρ is defined by

$$\rho(x, y, x^*) \text{ holds } \Leftrightarrow \langle x^*, \varphi(x, y)\rangle \geq 0.$$

By Lemma 2.1 (ii), $T(X)$ is a weak* compact subset of E^*, and by Alaoglu's Theorem (see [4, Theorem 6.21]), it is norm bounded. Considering $T(X)$ equipped with the weak* topology, from [9, Lemma 4.3], the duality pairing $\langle \cdot, \cdot \rangle$ restricted to $T(X) \times E$ is jointly continuous. Then, the function $(x, y, x^*) \longrightarrow \langle x^*, \varphi(x, y)\rangle$ is continuous on $X \times X \times T(X)$, hence the set $\{(x, y, x^*) \in X \times X \times T(X) : \langle x^*, \varphi(x, y)\rangle \geq 0\}$ is closed in $X \times X \times T(X)$, relative to the product (norm $\times$ norm $\times$ weak*) topology.

Let $x \in X$, $x^* \in T(X)$ and $y_1, y_2 \in X$ such that $\langle x^*, \varphi(x, y_1)\rangle < 0$, $\langle x^*, \varphi(x, y_2)\rangle < 0$. Since φ is affine in the variable y, for every $\lambda \in [0, 1]$,

$$\langle x^*, \varphi(x, \lambda y_1 + (1 - \lambda) y_2)\rangle = \lambda \langle x^*, \varphi(x, y_1)\rangle + (1 - \lambda)\langle x^*, \varphi(x, y_2)\rangle < 0,$$

hence ρ^c is convex in y.

Let $\{y_1, \ldots, y_n\}$ be a finite subset of X and $x = \sum_{i=1}^n \lambda_i y_i$, with $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$. In view of (iii), there exists $x^* \in T(x)$ such that

$$0 \leq \langle x^*, \varphi(x,x)\rangle = \sum_{i=1}^n \lambda_i \langle x^*, \varphi(x, y_i)\rangle.$$

Consequently, for some index $i \in \{1, \ldots, n\}$, $\langle x^*, \varphi(x, y_i)\rangle \geq 0$. This proves that relation ρ is T-KKM. To conclude, it remains to apply Theorem 3.5. □

Using similar arguments as in the previous proof, we can derived from Theorem 3.9 the following existence result for problem (VIP-2).

Theorem 4.2. *Assume that the normed space E is finite dimensional (in which case the dual coincides with E) and that the following conditions are satisfied:*

(i) *the function φ is affine in the second variable and continuous;*
(ii) *T is lower semicontinuous with compact convex values;*
(iii) *for each $x \in X$, there exists $x^* \in T(x)$ such that $\langle x^*, \varphi(x,x)\rangle > 0$.*

Then, problem (VIP-2) has a solution.

References

[1] R. P. Agarwal, M. Balaj and D. O'Regan, *A unifying approach to variational relation problems*, J. Optim. Theory Appl. **155** (2012), 417–429.
[2] R. P. Agarwal, M. Balaj and D. O'Regan, *Intersection theorems with applications in optimization*, J. Optim. Theory Appl. **179** (2018), 761–777.
[3] R. P. Agarwal, M. Balaj and D. O'Regan, *Intersection theorems for weak KKM set-valued mappings in the finite dimensional setting*, Topology Appl. **262** (2019), 64–79.
[4] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis. A hitchhiker's guide*, Springer, Berlin, 2006.
[5] Q. H. Ansari, E. Köbis and J. C. Yao, *Vector variational inequalities and vector optimization. Theory and applications. Vector Optimization*, Springer, Cham, 2018.
[6] M. Balaj, *Three types of variational relation problems*, Taiwanese J. Math. **17** (2013), 47–61.
[7] M. Balaj, *Solutions existence for two types of mixed variational relation problems*, Appl. Anal. **95** (2016), 426–443.
[8] M. Balaj, *New existence criteria of the solutions for a variational relation problem*, Optimization **66** (2017), 675–689.
[9] M. Balaj, *Stampacchia variational inequality with weak convex mappings*, Optimization **67** (2018), 1571–1577
[10] M. Balaj, *Intersection theorems for generalized weak KKM set-valued mappings with applications in optimization*, Math. Nachr. **294** (2021), 1262–1276.
[11] M. Balaj and L. J. Lin, *Luc's variational relation problem under some lower semicontinuity assumption*, J. Nonlinear Convex Anal. **17** (2016), 39–54.
[12] M. Balaj and D.T. Luc, *On mixed variational relation problems*, Comput. Math. Appl. **60** (2010), 2712–2722.
[13] F. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.
[14] A. Dhara and D. T. Luc, *A solution method for linear variational relation problems*, J. Global Optim. **59** (2014), 729–756.
[15] K. Fan, *A generalization of Tychonoff 's fixed point theorem*, Math. Ann. **142** (1961), 305–310.
[16] A. P. Farajzadeh, A. Amini-Harandi and K. R. Kazmi, *Existence of solutions to generalized vector variational-like inequalities* J. Optim. Theory Appl. **146** (2010), 95–104.
[17] N. V. Hung, *Continuity of solutions for parametric generalized quasi-variational relation problems*, Fixed Point Theory Appl. 2012;102:10 p.

[18] D. Inoan, *Factorization of quasi-variational relations systems*, Acta Math. Vietnam. **39** (2014), 359–365.
[19] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–459.
[20] P. Q. Khanh and D. T. Luc, *Stability of solutions in parametric variational relation problems*, Set-Valued Anal. **16** 1015–1035.
[21] A. Latif and D. T. Luc, *Variational relation problems: Existence of solutions and fixed points of contraction mappings*, Fixed Point Theory Appl. **2013**, 2013:315, 10 pp
[22] L. J. Lin and S. Y. Wang, *Simultaneuous variational relation problems and related applications*, Comput. Math. Appl. **58** (2009), 1711–1721.
[23] D. T. Luc, *An abstract problem in variational analysis*, J. Optim. Theory Appl. **138** (2008), 65–76.
[24] D. T. Luc, E. Sarabi and A. Soubeyran, *Existence of solutions in variational relation problems without convexity*, J. Math. Anal. Appl. **364** (2010), 544–555
[25] J. Parida, M. Sahoo and A. Kumar, *A variationallike inequality problem*, Bull. Austral. Math. Soc. **39** (1989), 225–231.
[26] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological space*, J. Math. Econom. **12** (1983), 233–245.

Manuscript received May 16 2022
revised November 16 2022

M. Balaj
Department of Mathematics, University of Oradea, Romania
E-mail address: mbalaj@uoradea.ro

D. F. Serac
Department of Mathematics, University of Oradea, Romania
E-mail address: dany16121986@yahoo.com