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STOCHASTIC SUMMABILITY AND ITS APPLICATIONS TO PROBABILITY THEORY

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In honor of Professor Adrian Petrusel on the occasion of his 60-th birthday.

Abstract. In this paper, we study the summability properties of double sequences of real constants which map sequences of random variables to sequences of random variables that are defined on the same probability sample spaces. We show that a regular method of summability is still regular on sequences of random variables with almost everywhere convergence, almost sure convergence, and with L_p-convergence ($p \geq 1$). It is not necessarily regular on sequences of random variables with convergence in probability. We extend these results to random variables with values in *extended real numbers* (they include infinite values, see definitions 2.2 and 2.4). For this, we introduce a novel construction for the extension that allows us to multiply sequences of extended real numbers with infinite real matrices. Using of this construction is much easier and more intuitively clear than employing of the full apparatus of Nonstandard Analysis. We also introduce *column-finite regular method of summability* that allows us to deal effectively with random variables with huge, even infinite, values.

1. Introduction

Let $A = (a_{ij}),\ i = 1, 2, ...,\ j = 0, 1, 2, ...,$ be a double sequence of real constants, that is,

$$A = \begin{pmatrix} a_{10} & a_{11} & a_{12} & \ldots\ldots \\ a_{20} & a_{21} & a_{22} & \ldots\ldots \\ a_{30} & a_{31} & a_{32} & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{pmatrix}. \tag{1.1}$$

Let S denote the set of sequences of real constants. In traditional summability theory, A is, under certain assumptions, considered to be a mapping from a subset of S to S (see [1], [4], [6], [9], [14-18], [24]).

Let S_1, S_2 be non-empty subsets of S. A is said to be summable from S_1 to S_2, whenever for any $x \in S_1, Ax \in S_2$. It has been studied by many authors for S_1 to be some special space contained in S. For example, $S_1 = \ell_p, S_b$, etc, where S_b denotes the set of bounded sequences of real numbers (see [3], [10-13], [23], [27]).

In recent years, some authors have extended the concept of summability to statistical summability, which studies the convergence of sequences mapped by A in a certain proportion (see [2], [5], [7-8], [19-22], [25-26]).

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In probability theory and stochastic processes theory, the convergence of a sequence of random variables has been an important topic, e.g. in central limit theorems. It leads us to consider the summability of the sequence of random variables by a given double sequence of real numbers A.

Let $x = [X_1, X_2, X_3, ...] = \{X_n\}_{n=1}^{\infty}$ be a sequence of such real valued random variables defined on the same probability space Ω. We will consider the following types of convergence of this sequence of random variables:

(1) $X_n \to X_\infty$, a.e. in Ω;
(2) $X_n \to X_\infty$, a.s.;
(3) $X_n \to X_\infty$, in pr.;
(4) $X_n \to X_\infty$, in $L_p(\Omega)$, for $p \geq 1$.

In this paper, we are interested in finding the conditions on A, under which Ax is convergent a.e., almost surely, in pr., or in $L_p(\Omega)$, respectively, for any given x satisfying one of the above convergence conditions.

2. Some known results and a need for a new approach

The following results are straightforward consequences from the traditional summability theory ([1], [6], [24]), where random variables have all values finite.

Proposition 2.1. *If A is a method of summability, that is satisfies the following three conditions*[1]: [1]

(1) $\underset{1 \leq i < \infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;
(2) $\lim\limits_{i\to\infty} a_{ij}$ *exists for* $j = 1, 2, \ldots$;
(3) $\lim\limits_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}$ *exists.*

Then $X_n \to X_\infty$, a.e. in Ω implies $(Ax)_n \to \tilde{X}_\infty$, a.e. in Ω for some random variable $\tilde{X}_\infty$ that may be different from X_∞.

Extended real numbers are often defined as:

Definition 2.2.

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}. \tag{2.1}$$

Example 2.3. Let $A = \begin{pmatrix} 1 & -1 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$, $X_1(\omega) = \infty$, and $X_2(\omega) = \infty$, for some $\omega \in \Omega$. Then $Ax(\omega) = \begin{pmatrix} \infty - \infty \\ \cdots \end{pmatrix}$ and the first element is undefined in $\mathbb{R}^*$.

To have the matrix multiplication Ax defined, where the vector $x = \{X_n\}$ can have infinite entries, we will extend real numbers in a different way than $\mathbb{R}^*$.

Let V be a 2-dimensional vector space over $\mathbb{R}$ with basis $(1, z)$, where $z \notin \mathbb{R}$. The objects in V are linear polynomials in z. We order V by the following lexicographic ordering:

$$a_1 + b_1 z \prec a_2 + b_2 z \text{ iff } b_1 < b_2 \text{ or } b_1 = b_2 \text{ and } a_1 < a_2.$$

[1]The notation "lub" is commonly used in this setting with the meaning "least upper bound".

This ordering is now consistent with the vector addition and multiplication by a scalar[2]: [2] Let $u, v, w \in V$ and $a \in \mathbb{R}$, then:

$$u \prec v \Rightarrow u + w \prec v + w$$
$$u \prec v \Rightarrow a \cdot u \prec a \cdot v, \quad \text{whenever} \quad 0 < a$$

Definition 2.4. In the above construction, we make the following notation:

$$\begin{aligned} \infty &:= z \\ \mathbb{R}^{**} &:= V \\ \mathbb{R}^{**+}_{\infty} &:= \{u \in \mathbb{R}^{**} : \forall a \in \mathbb{R}\ (a \prec u)\} \\ \mathbb{R}^{**-}_{\infty} &:= \{u \in \mathbb{R}^{**} : \forall a \in \mathbb{R}\ (u \prec a)\} \\ \mathbb{R}^{**}_{\infty} &:= \mathbb{R}^{**+}_{\infty} \cup \mathbb{R}^{**-}_{\infty}, \end{aligned}$$

where we asume every $a \in \mathbb{R}$ to equal $a + 0 \cdot \infty$ and so be also a member of $\mathbb{R}^{**}$. We call $\mathbb{R}^{**}$ the *summable extended real numbers* and $\mathbb{R}^{**}_{\infty}$ the *summable infinite real numbers.*

Remark 2.5. In this new notation, $\mathbb{R}^{**} = \mathbb{R} \cup \mathbb{R}^{**}_{\infty}$, $\mathbb{R}^{*} \subset \mathbb{R}^{**}$, and every $u \in \mathbb{R}^{**}$ can be written in a unique way as $u = a + b \cdot \infty$ for some $a, b \in \mathbb{R}$. Then $\mathbb{R}^{**+}_{\infty} = \{a + b \cdot \infty : \ a, b \in \mathbb{R} \ \text{ and } b > 0\}$ and $\mathbb{R}^{**-}_{\infty} = \{a + b \cdot \infty : \ a, b \in \mathbb{R} \ \text{ and } b < 0\}$. This allows us to explore the summability of random variables $x = \{X_n\}$ with infinite values in $\mathbb{R}^{**}$ because $\{(Ax)_n\}$ will also be a sequence of random variables with values in $\mathbb{R}^{**}$ as it is summarized in the following Proposition.

Proposition 2.6. *For every* $x = \{X_n\}$, $X_n : \ \Omega \to \mathbb{R}^{**}$, $n = 1, 2, ...$ *we have* $Ax = \{(Ax)_n\}$, $(Ax)_n : \ \Omega \to \mathbb{R}^{**}$, $n = 1, 2, ...$

Remark 2.7. We emphasize here that there is no multiplication of $\mathbb{R}^{**}$ elements defined in spite of calling them "real numbers". That is why we added the adjective "summable". We do not need such a multiplication, so we work only with the group $(\mathbb{R}^{**}, +)$ and scalar multiplication by finite real numbers.

Definition 2.8. For any $u = a + b \cdot \infty \in \mathbb{R}^{**}$ define the *absolute value* of u as $|u| := |a| + |b| \cdot \infty$.

Example 2.9. Let $u = -\pi - 3 \cdot \infty$ then $u \in \mathbb{R}^{**+}_{\infty}$, $|u| = \pi + 3 \cdot \infty \in \mathbb{R}^{**+}_{\infty}$, and $10^{100} + \infty \prec |u|$.

Remark 2.10. There is a cannonical projection from the set of random variables $X : \Omega \to \mathbb{R}^{**}$ to the set of random variables $X : \Omega \to \mathbb{R}^{*}$, where each $X(\omega)$ is mapped by the identity on $\mathbb{R}$, positive infinite values $\mathbb{R}^{**+}_{\infty}$ (see the Definition 2.4) are mapped to $+\infty$ and negative infinite values $\mathbb{R}^{**-}_{\infty}$ to $-\infty$. Considering of this projection allows us to extend the statements and proofs of all needed[3] claims and theorems for random variables $X : \Omega \to \mathbb{R}^{*}$ to random variables $X : \Omega \to \mathbb{R}^{**}$. More precisely, random variables $X : \Omega \to \mathbb{R}^{**}$ form a model of the theory of random variables $X : \Omega \to \mathbb{R}^{*}$ under the condition (3.1), albeit with more structure.

From now on, consider all random variables to be $X : \Omega \to \mathbb{R}^{**}$.

[2]See more explanation in remarks that follow the definitions below.

[3]Classical results for $X : \Omega \to \mathbb{R}^{*}$, such as Lemma 2.12, do not use any group structure.

Lemma 2.11. (a) $\mathbb{R}^{**}_{\infty} = \bigcap_{K\in\mathbb{N}} \{u \in \mathbb{R}^{**} :\ |u| \succ K\}$,

(b) $P(X \in \mathbb{R}^{**}_{\infty}) = \lim_{K\to\infty} P(|X| \succ K)$.

Proof. From definitions 2.4 and 2.8 we have $\mathbb{R}^{**}_{\infty} = \{u \in \mathbb{R}^{**} :\ \forall a \in \mathbb{R}\ (a \prec |u|)\}$, which is equivalent to (a). So $P(X \in \mathbb{R}^{**}_{\infty})$ can be expressed as:

$$P\left(\bigcap_{K\in\mathbb{N}} \{\omega :\ X(\omega) \in \mathbb{R}^{**} \text{ and } |X(\omega)| \succ K\}\right) = P\left(\bigcap_{K\in\mathbb{N}} \{\omega :\ |X(\omega)| \succ K\}\right).$$

Since $(\{\omega : |X(\omega)| \succ K\})_{K=1}^{\infty}$ is a decreasing sequence w.r.t. inclusion, we have

$$P\left(\bigcap_{K\in\mathbb{N}} \{\omega : |X(\omega)| \succ K\}\right) = \lim_{K\to\infty} P\left(\{\omega : |X(\omega)| \succ K\}\right) = \lim_{K\to\infty} P(|X| \succ K). \qquad \square$$

The following claim is well known and we include it here as a lemma without proof.

Lemma 2.12. (a) *If $\Omega' \subseteq \Omega$ and $P(\Omega') = 1$, then $X_n \to X_\infty$, in $L^p(\Omega)$, is equivalent to $X_n \to X_\infty$, in $L^p(\Omega')$.*

(b) *If countably many random variables X_n are finite a.e. in Ω, then there is $\Omega' \subseteq \Omega$ with $P(\Omega') = 1$, such that all X_n are finite in Ω'.*

The following proposition shows that a sequence $x = \{X_n\}_{n=1}^{\infty}$ of random variables converging in pr. to X_∞ does not imply that Ax converges in pr. to X_∞.

Proposition 2.13. *Suppose that A is a regular method of summability, that is:*

(1) $\underset{1\le i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;

(2) $\lim_{i\to\infty} a_{ij} = 0$, *for* $j = 1, 2, \ldots$;

(3) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then $X_n \to X_\infty$ in pr. is not sufficient for $(Ax)_n \to X_\infty$ in pr.

This is a well known result in probability theory. To show it, for the convenience of the reader, we provide an example below. This example uses only finite values.

Example 2.14. We take interval $[0, 1)$ as the sample space. If $n = 2^m + i$, for some given $m = 1, 2, 3, \ldots$, and for some $0 \le i < 2^m$, then we define X_n as follows:

$$X_n(\omega) = \begin{cases} 4^{m+i}, & \text{if } \omega \in \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right) \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

It is clear that $X_n \to 0$ in pr. Taking the Cesaro Summability method A, for $n \ge 16$, noting $m > 3$ and so $4^{m-1} > 2^{m+1}$, we have:

$$\begin{aligned} \left(\frac{\sum_{j=1}^{n} X_j}{n} > 1\right) &= \left(\sum_{j=1}^{n} X_j > n\right) \\ &\supseteq \left(\sum_{j=1}^{n} X_j > 2^{m+1}\right) \end{aligned}$$

$$\supseteq \left(\sum_{j=2^{m-1}}^{2^m-1} X_j > 2^{m+1} \right)$$
$$= \left(\sum_{i=0}^{2^{m-1}-1} X_{2^{m-1}+i} > 2^{m+1} \right)$$
$$= \bigcup_{i=0}^{2^{m-1}-1} \left[\frac{i}{2^{m-1}}, \frac{i+1}{2^{m-1}} \right) = [0,1).$$

It implies $P\left(\frac{\sum_{j=1}^n X_j}{n} > 1\right) = 1$ and shows that

$$\lim_{n\to\infty} P\left(\frac{\sum_{j=1}^n X_j}{n} > 1\right) = 1 > 0.$$

Hence $\frac{\sum_{j=1}^n X_j}{n}$ does not converge to 0 in pr..

The above example demonstrates that even with Cesaro Summability method A, $X_n \to X_\infty$ in pr. does not imply that $(Ax)_n \to X_\infty$ in pr.. Hence to assure $(Ax)_n \to X_\infty$ in pr. for any given regular summability method A, a stronger condition on the sequence (X_n) than $X_n \to X_\infty$ in pr. is needed.

We now review the definition of *almost sure covergence.*

Definition 2.15. If for any $\lambda > 0$
$\lim_{n\to\infty} P\left(\sup_{n\le m<\infty} |X_m - X_\infty| \succ \lambda\right) = 0$, then we denote $X_n \xrightarrow{\text{a.s.}} X_\infty$.

We have easily the following corollary, which is well-known.

Corollary 2.16. *$X_n \xrightarrow{\text{a.s.}} X_\infty$ implies $X_n \to X_\infty$ in pr.*

The following example shows that we still have problems with convergence.

Example 2.17. Let set S be such that $P(S) > 0$, and let $X_1(\omega) = \infty$ for $\omega \in S$ and zero otherwise, and $X_n(\omega) = 0$ everywhere for $n = 2, 3, ..., \infty$. Then $X_n \to X_\infty$ even in the strongest sense one can think of but, for A that has all elements in the first column nonzero, we still don't need have $(Ax)_n \to X_\infty$ even in weakest sense.

We will resolve this problem in two different ways. First, we add a condition about finitenes almost everywhere, see 3.1 in the next section. And then we will introduce "column-finite regular method of summability" in the last section.

All standard definitions of almost sure convergence are equivalent to convergence almost everywhere and the following theorem shows that it is so also in our case.

Theorem 2.18. *$X_n \xrightarrow{\text{a.s.}} X_\infty$ is equivalent to $X_n \to X_\infty$ a.e. in Ω.*

Proof. We will prove the nontrivial direction (⇒). Consider the following subsets of Ω:

$$U_{k,n} = \{\omega : \sup_{n \le m < \infty} |X_m(\omega) - X_\infty(\omega)| \succ \frac{1}{k}\},$$
$$U_k = \bigcap_n U_{k,n}, \quad n, k \in \mathbb{Z}^+$$

Then

$$U_{k,n} \supseteq U_{k,n+1},\ U_{k,n} \subseteq U_{k+1,n},\ U_k \subseteq U_{k+1},$$

In this notation, $X_n \xrightarrow{\text{a.s.}} X_\infty$ means $\lim_{n\to\infty} P(U_{k,n}) = P(U_k) = 0$ for every $k > 0$. Suppose, by contradiction, that there is $U \subseteq \Omega$, such that $P(U) > 0$ and $(\forall \omega \in U)\ (X_n(\omega) \not\to X_\infty(\omega))$. Then $U \subseteq \bigcup_k U_k$ and $0 < P(U) \le P(\bigcup_k U_k) = 0$, a contradiction. □

3. Some new stochastic summability results for random variables finite a.e.

All random variables considered in this section are real random variables defined on a probability space (Ω, P, F) that are finite almost everywhere (finite a.e.), that is, every random variable X, satisfies

$$P(X \in \mathbb{R}) = 1. \tag{3.1}$$

The following extension of the Integral and Expected value will be sufficient for us in this section.

Definition 3.1. Let X be a random variable that is finite a.e. in Ω. Then

$$E(X) = \int_\Omega X(\omega)P(d\omega) := \int_{\{\omega: X(\omega)\in\mathbb{R}\}} X(\omega)P(d\omega).$$

Theorem 3.2. *Let X_n, X_∞ be finite a.e., that is $P(X_n \in \mathbb{R}) = 1, P(X_\infty \in \mathbb{R}) = 1$, and A define a regular method of summability, that is*

(1) $\underset{1\le i<\infty}{lub} \sum_{j=1}^\infty |a_{ij}| = M < \infty$;
(2) $\lim_{i\to\infty} a_{ij} = 0$ *for* $j = 1, 2, \ldots$;
(3) $\lim_{i\to\infty} \sum_{j=1}^\infty a_{ij} = 1$.

Then $X_n \to X_\infty$ a.e. in Ω implies $(Ax)_n \to X_\infty$ a.e. in Ω.

Proof. From Lemma 2.12(b) and the assumptions, we have the existence of a sequence of sets S_n and sets S_∞, S :

$$P(|X_n| \in \mathbb{R}) = 1 \quad\Rightarrow\quad \exists S_n \subseteq \Omega : P(S_n) = 1 \text{ and } \omega \in S_n \Rightarrow |X_n(\omega)| \in \mathbb{R}$$
$$P(|X_\infty| \in \mathbb{R}) = 1 \quad\Rightarrow\quad \exists S_\infty \subseteq \Omega : P(S_\infty) = 1 \text{ and } \omega \in S_\infty \Rightarrow |X_\infty(\omega)| \in \mathbb{R}$$
$$P(X_n \to X_\infty) = 1 \quad\Rightarrow\quad \exists S \subseteq \Omega : P(S) = 1 \text{ and } \omega \in S \Rightarrow X_n(\omega) \to X_\infty(\omega)$$

Consider $T_n = \bigcap_{i=1}^n S_i \cap S \cap S_\infty$. The sequence $(T_n)_{i=1}^\infty$ is decreasing w.r.t. inclusion, so $P(T) = P(\bigcap T_n) = \lim_{n\to\infty} P(T_n) = 1$. Now, for every $\omega \in T$ we have $|X_n(\omega)| \in \mathbb{R}$ and $|X_\infty(\omega)| \in \mathbb{R}$. So the Silverman-Toeplitz theorem applies on T_n for each n and consequently $(Ax)_n \to X_\infty$ a.e. in Ω. □

Theorem 3.3. *Suppose that A is a regular method of summability and its norm $|A| = M$. That means, by Silverman-Toeplitz Theorem:*

(1) $\underset{1\le i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;

(2) $\underset{i\to\infty}{lim}\, a_{ij} = 0$, *for* $j = 1, 2, \dots$;

(3) $\underset{i\to\infty}{lim} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then $X_n \xrightarrow{\text{a.s.}} X_\infty$ *implies* $(Ax)_n \xrightarrow{\text{a.s.}} X_\infty$, *whenever*

X_n, X_∞ *are finite a.e., that is,* $P(X_n \in \mathbb{R}) = 1$, *and* $P(X_\infty \in \mathbb{R}) = 1$.

Remark 3.4. This theorem is, of course, an immediate consequence of the a.s and a.e. equivalence (Theorem 2.18). Nevertheless, we give here this less abstract proof in the classical Mathematical Analysis style. The proof of an analogical theorem in the next section based on this proof will get simpler and the abstract proof for covergence everywhere will get more complicated. Considering that the constraction of the extended real numbers is new, we hope that the reader will benefit from these additional proofs for almost sure convergence.

Proof of Theorem 3.3. By Lemma 2.11 and from finiteness a.e., we have conditions

(*) $\lim_{K\to\infty} P(|X_n| \succ K) = 0$, $\lim_{K\to\infty} P(|X_\infty| \succ K) = 0$.

For any given $\varepsilon, \delta > 0$, we have to show that there exists $N > 1$, such that for all $n > N$, the following inequality holds

$$P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta\right) < \varepsilon.$$

For the given $\varepsilon, \delta > 0$, $X_n \xrightarrow{\text{a.s.}} X_\infty$ implies that there exists $N_2 > 1$, such that

$$P\left(\sup_{n\le k<\infty} |X_k - X_\infty| \succ \frac{\delta}{3M}\right) < \frac{\varepsilon}{2}, \text{ for all } n \ge N_2. \tag{3.2}$$

From conditions (*), for the already known ε and N_2 there exists $K > 1$, such that

$$P\left(\max_{\substack{1\le k<N_2\\ or\, k=\infty}} |X_k| \succ K\right) < \frac{\varepsilon}{2} \tag{3.3}$$

For this fixed $K > 1$, we have

$$\begin{aligned} & P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta\right) \\ = \;& P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \max_{\substack{1\le k<N_2\\ or\, k=\infty}} |X_k| \le K\right) \\ & + P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \max_{\substack{1\le k<N_2\\ or\, k=\infty}} |X_k| \succ K\right) \end{aligned} \tag{3.4}$$

$$< \quad P\left(\sup_{n\le i<\infty}|(Ax)_i - X_\infty| \succ \delta : \max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right) + \frac{\varepsilon}{2}. \tag{3.5}$$

For the fixed N_2, there exists $N_3 \geq N_2$, from condition (2) in this proposition, such that

$$|a_{nj}| \leq \frac{\delta}{6KN_2}, \text{ for all } n \geq N_3, \text{ and all } 1 \leq j < N_2.$$

From condition (3) in this proposition, there exists $N \geq N_3$, such that

$$\left|\sum_{j=1}^{\infty} a_{nj} - 1\right| \leq \frac{\delta}{3K}, \text{ for all } n \geq N, \text{ and so } \sup_{n\le i<\infty}\left|\sum_{j=1}^{\infty} a_{ij} - 1\right| \leq \frac{\delta}{3K}.$$

Now, for all $n \geq N$, we have

$$\begin{aligned}
&\left(\sup_{n\le i<\infty}|(Ax)_i - X_\infty| \succ \delta : \max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right)\\
=\;&\left(\sup_{n\le i<\infty}\left|\sum_{j=1}^{\infty}a_{ij}X_j - \sum_{j=1}^{\infty}a_{ij}X_\infty + \left(\sum_{j=1}^{\infty}a_{ij}-1\right)X_\infty\right| \succ \delta : \max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right)\\
=\;&\left(\sup_{n\le i<\infty}\left|\sum_{j=1}^{\infty}a_{ij}(X_j - X_\infty) + \left(\sum_{j=1}^{\infty}a_{ij}-1\right)X_\infty\right| \succ \delta : \max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right)\\
=\;&\left(\sup_{n\le i<\infty}\left|\sum_{j=1}^{N_2-1}a_{ij}(X_j - X_\infty) + \sum_{j=N_2}^{\infty}a_{ij}(X_j - X_\infty) + \left(\sum_{j=1}^{\infty}a_{ij}-1\right)X_\infty\right| \succ \delta :\right.\\
&\left.\max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right)\\
\subseteq\;&\left[\left(\sup_{n\le i<\infty}\left|\sum_{j=1}^{N_2-1}a_{ij}(X_j - X_\infty)\right| \succ \frac{\delta}{3}\right)\bigcup\left(\sup_{n\le i<\infty}\left|\sum_{j=N_2}^{\infty}a_{ij}(X_j - X_\infty)\right| \succ \frac{\delta}{3}\right)\right.\\
&\left.\bigcup\left(\sup_{n\le i<\infty}\left|\left(\sum_{j=1}^{\infty}a_{ij}-1\right)X_\infty\right| \succ \frac{\delta}{3}\right)\right]\bigcap\left(\max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right)\\
\subseteq\;&\left[\left(\sup_{n\le i<\infty}\sum_{j=1}^{N_2-1}|a_{ij}|(|X_j|+|X_\infty|) \succ \frac{\delta}{3}\right)\bigcup\left(\sup_{n\le i<\infty}\sum_{j=N_2}^{\infty}|a_{ij}|\,|X_j - X_\infty| \succ \frac{\delta}{3}\right)\right.\\
&\left.\bigcup\left(\sup_{n\le i<\infty}\left|\left(\sum_{j=1}^{\infty}a_{ij}-1\right)\right||X_\infty| \succ \frac{\delta}{3}\right)\right]\cap\ \Omega\\
\subseteq\;&\left[\left(\sup_{n\le i<\infty}\sum_{j=1}^{N_2-1}\frac{\delta}{6KN_2}\left(\max_{1\le k<N_2}|X_k|+|X_\infty|\right) \succ \frac{\delta}{3}\right)\right.
\end{aligned}$$

$$\bigcup\left(\sup_{n\le i<\infty}\sum_{j=N_2}^{\infty}|a_{ij}|\sup_{N_2\le k<\infty}|X_k-X_\infty|\succ\frac{\delta}{3}\right)\bigcup\left(\frac{\delta}{3K}|X_\infty|\succ\frac{\delta}{3}\right)\Bigg]$$

$$\subseteq\quad\left[\left(\left(\max_{1\le k<N_2}|X_k|+|X_\infty|\right)\succ 2K\right)\bigcup\left(\sup_{N_2\le k<\infty}|X_k-X_\infty|\succ\frac{\delta}{3M}\right)\right.$$
$$\left.\bigcup\left(|X_\infty|\succ K\right)\right]$$

$$\subseteq\quad\varnothing\cup\left(\sup_{N_2\le k<\infty}|X_k-X_\infty|\succ\frac{\delta}{3M}\right)\cup\varnothing.$$

Applying probability to these sets and by (3.2) we have:

$$P\left(\sup_{n\le i<\infty}|(Ax)_i-X_\infty|\succ\delta:\max_{\substack{1\le k<N_2\\ or\ k=\infty}}|X_k|\le K\right)$$
$$\le P\left(\sup_{N_2\le k<\infty}|X_k-X_\infty|\succ\frac{\delta}{3M}\right)<\frac{\varepsilon}{2}$$

Now going back to (3.5) we can finish our proof:

$$P\left(\sup_{n\le i<\infty}|(Ax)_i-X_\infty|\succ\delta\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

□

If we consider the Cesaro Summability method A, then we have the following corollary of Theorem 3.3.

Corollary 3.5. *For $n=1,2,...$, let random variables X_n,X_∞ satisfy*

(a) X_n, X_∞ *are finite a.e.,*

(b) $X_n\xrightarrow{\text{a.s.}}X_\infty$.

Then $\dfrac{\sum_{i=1}^{n}X_i}{n}\xrightarrow{\text{a.s.}}X_\infty$.

Theorem 3.6. *Let $p\ge 1$. Suppose that A is a regular method of summability, that is it satisfies the following conditions:*

(1) $\underset{1\le i<\infty}{lub}\sum_{j=1}^{\infty}|a_{ij}|=M<\infty$;

(2) $\lim_{i\to\infty}a_{ij}=0$, *for* $j=1,2,\dots$;

(3) $\lim_{i\to\infty}\sum_{j=1}^{\infty}a_{ij}=1$.

Then $X_n\to X_\infty$, in $L^p(\Omega)$, implies $(Ax)_n\to X_\infty$, in $L^p(\Omega)$, whenever

X_n, X_∞ are finite a.e., that is, $P(X_n\in\mathbb{R})=1,P(X_\infty\in\mathbb{R})=1$.

Proof. For any given $\varepsilon > 0$, the condition $X_n \to X_\infty$, in $L^p(\Omega)$ (refer to Definition 3.1 and Lemma 2.12(a)) implies that there exists $N_1 > 1$, such that for all $n \geq N_1$, the following inequality holds

$$(E|X_n - X_\infty|^p)^{\frac{1}{p}} < \frac{\varepsilon}{3M}. \tag{3.6}$$

The condition $X_n, X_\infty \in L^p(\Omega)$ implies that there exists $K > 1$, such that $||X_n||_p \leq K$ (again refer to Definition 3.1) for all $n = 1, 2, ..., \infty$. For the fixed N_1, Condition (2) implies that there exists $N_2 \geq N_1$, such that

$$|a_{nj}| \leq \frac{\varepsilon}{6KN_1}, \text{ for all } n \geq N_2, \text{ and all } 1 \leq j \leq N_1. \tag{3.7}$$

From the condition (3) in this theorem, there exists $N \geq N_2$, such that

$$\left| \sum_{j=1}^{\infty} a_{nj} - 1 \right| \leq \frac{\varepsilon}{3K}, \text{ for all } n \geq N. \tag{3.8}$$

Now, for all $n \geq N$, from (3.6), (3.7), and (3.8), we have

$$\begin{aligned}
&(E|(Ax)_n - X_\infty|^p)^{\frac{1}{p}} = \\
&= \left(E \left| \sum_{j=1}^{\infty} a_{nj} X_j - \sum_{j=1}^{\infty} a_{nj} X_\infty + \left(\sum_{j=1}^{\infty} a_{nj} - 1 \right) X_\infty \right|^p \right)^{\frac{1}{p}} \\
&\leq \left(E \left| \sum_{j=1}^{\infty} a_{nj}(X_j - X_\infty) \right|^p \right)^{\frac{1}{p}} + \left(E \left| \left(\sum_{j=1}^{\infty} a_{nj} - 1 \right) X_\infty \right|^p \right)^{\frac{1}{p}} \\
&\leq \left(E \left| \sum_{j=1}^{N_1} a_{nj}(X_j - X_\infty) \right|^p \right)^{\frac{1}{p}} + \left(E \left| \sum_{j=N_1+1}^{\infty} a_{nj}(X_j - X_\infty) \right|^p \right)^{\frac{1}{p}} \\
&\quad + \left(E \left| \left(\sum_{j=1}^{\infty} a_{nj} - 1 \right) X_\infty \right|^p \right)^{\frac{1}{p}} \\
&\leq \sum_{j=1}^{N_1} |a_{nj}| \left(E \left| (X_j - X_\infty) \right|^p \right)^{\frac{1}{p}} + \sum_{j=N_1+1}^{\infty} |a_{nj}| \left(E \left| (X_j - X_\infty) \right|^p \right)^{\frac{1}{p}} \\
&\quad + \left| \left(\sum_{j=1}^{\infty} a_{nj} - 1 \right) \right| \left(E \left| X_\infty \right|^p \right)^{\frac{1}{p}} \\
&\leq 2K \sum_{j=1}^{N_1} |a_{nj}| + \sum_{j=N_1+1}^{\infty} |a_{nj}| \frac{\varepsilon}{3M} + \frac{\varepsilon}{3K} K \\
&< \varepsilon.
\end{aligned}$$

□

Similarly to the proof of Theorem 3.6, by applying the completeness property of $L^p(\Omega)$, we can prove the following corollary.

Corollary 3.7. *Let $p \geq 1$. If A is a regular method of summability, that is it satisfies the following conditions:*

(1) $\underset{1\leq i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;

(2) $\underset{i\to\infty}{lim}\, a_{ij} = 0$, *for* $j = 1, 2, \ldots$;

(3) $\underset{i\to\infty}{lim} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then x converges in $L^p(\Omega)$ implies that Ax converges in $L^p(\Omega)$, whenever

$$X_n,\ X_\infty \text{ are finite a.e., that is, } P(X_n \in \mathbb{R}) = 1, \text{ and } P(X_\infty \in \mathbb{R}) = 1.$$

Remark 3.8. By following the steps in the proof of Theorem 3.6, one can show that both Theorem 3.6 and Corollary 3.7 hold for $p = \infty$.

4. Stochastic summability results, where only X_∞ is finite a.e.

All random variables considered in this section are real random variables defined on a probability space (Ω, P, F) not necessarily finite almost everywhere. However, we will still assume that random variable X_∞ is finite a.e., that is

$$P(X_\infty \in \mathbb{R}) = 1. \tag{4.1}$$

Changing this would require to introduce a new definition of pointwise convergence on $\mathbb{R}^{**}_\infty$ and that is beyond the scope of this article[4].[4]

Lemma 4.1. *Given X_n, let there be a sequence of sets $\{S_n\}$ that satisfies $S_n \subseteq S_{n+1}$, $\lim_{n\to\infty} P(S_n) = 1$, where $S_n \subseteq \{\omega : X_n(\omega) \in \mathbb{R}\}$, let X_∞ be finite a.e., that is $P(X_\infty \in \mathbb{R}) = 1$, and let A define a column-finite regular method of summability, that is*

(1) $\underset{1\leq i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;

(2) *there is finitely many* $a_{ij} \neq 0$ *for each* $j = 1, 2, \ldots$

(3) $\underset{i\to\infty}{lim} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then $X_n \to X_\infty$ a.e. in Ω implies $(Ax)_n \to X_\infty$ a.e. in Ω.

Proof. From Lemma 2.12(b) and the assumptions, we have the existence of a sequence of sets S_n as above and sets S_∞, S :

$$P(X_\infty \in \mathbb{R}) = 1 \quad \Rightarrow \quad \exists S_\infty \subseteq \Omega : \ P(S_\infty) = 1 \text{ and } X_\infty(\omega) \in \mathbb{R} \text{ for } \omega \in S_\infty$$

$$P(X_n \to X_\infty) = 1 \quad \Rightarrow \quad \exists S \subseteq \Omega : \ P(S) = 1 \text{ and } X_n(\omega) \to X_\infty(\omega) \text{ for } \omega \in S$$

Consider $T_n = S_n \cap S \cap S_\infty$. The sequence $(T_n)_{i=1}^{\infty}$ is increasing w.r.t. inclusion, so $P(T) = P(\bigcup T_n) = \lim_{n\to\infty} P(T_n) = 1$. Now, fix $\omega \in T$. There is N, such that for all $n > N$ we have $\omega \in T_n$, so $X_n(\omega) \in \mathbb{R}$, $X_\infty(\omega) \in \mathbb{R}$, and $X_n(\omega) \to X_\infty(\omega)$. However, due to the matrix A condition (2), there is also K, such that $A_{ij} = 0$ for all $i > K$ and $j \leq N$. Now, for this ω we have a sequence of finite numbers

[4]We currently have essentially no convergence on $\mathbb{R}^{**}_\infty$, for example, $\lim_{n\to\infty} |a_n - a_\infty| = \lim_{n\to\infty} |(1 + \frac{1}{n})\infty - \infty| \neq 0$, so $a_n \not\to a_\infty$, that is the point-wise convergence in classic sense fails here.

$\{X_n(\omega)\}_{n=N+1}^{\infty}$ and a matrix A' obtained from A by deleting first K rows and first N columns, such that the sequence $\{(A'x)_k(\omega)\}_{k=1}^{\infty}$ is same as $\{(Ax)_n(\omega)\}_{n=K+1}^{\infty}$. So the Silverman-Toeplitz theorem applies to A' and $\{X_n(\omega)\}_{n=N+1}^{\infty}$, and we have $(A'x)_k(\omega) \to X_\infty(\omega)$, which is equivalent to $(Ax)_n(\omega) \to X_\infty(\omega)$. This holds for all $\omega \in T$, consequently $(Ax)_n \to X_\infty$ a.e. in Ω. □

Example 4.2. In this example, the random variables are the same kind as X_1 in Example 2.17, but the convergence problem is resolved. Suppose n is a positive integer, $P(V_n) = \frac{1}{n} > 0$, $V_{n+1} \subset V_n$, $X_\infty(\omega) = 0$ for all ω, and

$$X_n(\omega) = \begin{cases} \infty, & \text{if } \omega \in V_n \\ 0, & \text{if } \omega \notin V_n. \end{cases} \tag{4.2}$$

Then $P\left(\bigcap V_n\right) = 0$ and for every $\omega \notin \bigcap V_n$ there is N, such that $\omega \notin V_N$ and for all $n \geq N$ we have $X_n(\omega) = 0$. Consequently $X_n \to X_\infty$ a.e. in Ω and also $\lim_{n\to\infty} P(X_n \in \mathbb{R}) = 1$. Considering $S_n = V_n^{complement} = \Omega - V_n$, we can apply the previous Lemma to a column-finite regular Summability method A, such as *diagonal Cesaro Summability method* $\frac{\sum_{i=n}^{2n-1} X_n}{n}$, since S_n satisfy the lemma's conditions, and get $(Ax)_n \to 0 = X_\infty$ a.e. in Ω.

Lemma 4.3. *Let $X_n \to X_\infty$ a.e. in Ω and X_∞ be finite a.e.. Then there exists a sequence of sets S_n that satisfy all their conditions required in Lemma 4.1.*

Proof. From the convergence a.e. and finiteness a.e. conditions, there is a set S', $P(S') = 1$, where $X_n(\omega) \to X_\infty(\omega) \in \mathbb{R}$. But then for any $\omega \in S'$ also exists N, such that for all integers $n > N$ we have $X_n(\omega) \in \mathbb{R}$. Let N_ω be least of such integers. So $S' \subseteq \bigcup S_n$, where

$$S_n = \{\omega : \ N_\omega < n\}.$$

Moreover $S_n \subseteq \{\omega : X_n(\omega) \in \mathbb{R}\}$, $S_n \subseteq S_{n+1}$, and $\lim_{n\to\infty} P(S_n)$ exists. However, $\lim_{n\to\infty} P(S_n) = P(\bigcup S_n) \geq P(S') = 1$. Consequently $\lim_{n\to\infty} P(S_n) = 1$ and all conditions hold now. □

As a corollary of the two lemmas above we have the following theorem.

Theorem 4.4. *Let X_∞ be finite a.e., that is $P(X_\infty \in \mathbb{R}) = 1$, and A define a column-finite regular method of summability, that is*

(1) $\underset{1\leq i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;
(2) *there is finitely many $a_{ij} \neq 0$ for each $j = 1, 2, \ldots$*
(3) $\underset{i\to\infty}{lim} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then $X_n \to X_\infty$ a.e. in Ω implies $(Ax)_n \to X_\infty$ a.e. in Ω.

And we have a theorem with the same conditions also for almost sure convergence.

Theorem 4.5. *Suppose that A is a column-finite regular method of summability and its norm $|A| = M$. That means:*

(1) $\underset{1\leq i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;
(2) *there is finitely many $a_{ij} \neq 0$ for each $j = 1, 2, \ldots$;*
(3) $\underset{i\to\infty}{lim} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then $X_n \xrightarrow{\text{a.s.}} X_\infty$ *implies* $(Ax)_n \xrightarrow{\text{a.s.}} X_\infty$ *, whenever*

$$X_\infty \textit{ is finite a.e., that is, } P(X_\infty \in \mathbb{R}) = 1.$$

Proof. This proof is based on the proof of Theorem 3.3. Please see Remark 3.4. By Lemma 2.11 and from finiteness a.e., we have a condition

(*) $\lim_{K\to\infty} P(|X_\infty| \succ K) = 0$

For any given $\varepsilon, \delta > 0$, we have to show that there exists $N > 1$, such that for all $n > N$, the following inequality holds

$$P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta\right) < \varepsilon.$$

For the given $\varepsilon, \delta > 0$, $X_n \xrightarrow{\text{a.s.}} X_\infty$ implies that there exists $N_2 > 1$, such that

$$P\left(\sup_{n\le k<\infty} |X_k - X_\infty| \succ \frac{\delta}{3M}\right) < \frac{\varepsilon}{2}, \text{ for all } n \ge N_2. \tag{4.3}$$

From the condition (*), for the already known ε there exists $K > 1$, such that

$$P\left(|X_\infty| \succ K\right) < \frac{\varepsilon}{2}.$$

For this fixed $K > 1$, we have

$$\begin{aligned} & P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta\right) \\ = \ & P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \ |X_\infty| \le K\right) \\ & + P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \ |X_\infty| \succ K\right) \\ < \ & P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \ |X_\infty| \le K\right) + \frac{\varepsilon}{2}. \qquad (4.4) \end{aligned}$$

For the fixed N_2, there exists $N_3 \ge N_2$, from condition (2) in this proposition, such that

$$|a_{nj}| = 0, \text{ for all } n \ge N_3, \text{ and all } 1 \le j < N_2.$$

From condition (3) in this proposition, there exists $N \ge N_3$, such that

$$\left|\sum_{j=1}^{\infty} a_{nj} - 1\right| \le \frac{\delta}{3K}, \text{ for all } n \ge N, \text{ and so } \sup_{n\le i<\infty} \left|\sum_{j=1}^{\infty} a_{ij} - 1\right| \le \frac{\delta}{3K}.$$

Now, for all $n \ge N$, we have

$$\begin{aligned} & \left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \ |X_\infty| \le K\right) \\ = \ & \left(\sup_{n\le i<\infty} \left|\sum_{j=1}^{\infty} a_{ij}X_j - \sum_{j=1}^{\infty} a_{ij}X_\infty + \left(\sum_{j=1}^{\infty} a_{ij} - 1\right) X_\infty\right| \succ \delta : \ |X_\infty| \le K\right) \end{aligned}$$

$$= \left(\sup_{n\le i<\infty} \left| \sum_{j=1}^{\infty} a_{ij}(X_j - X_\infty) + \left(\sum_{j=1}^{\infty} a_{ij} - 1 \right) X_\infty \right| \succ \delta : \ |X_\infty| \le K \right)$$

$$= \left(\sup_{n\le i<\infty} \left| \sum_{j=1}^{N_2-1} a_{ij}(X_j - X_\infty) + \sum_{j=N_2}^{\infty} a_{ij}(X_j - X_\infty) + \left(\sum_{j=1}^{\infty} a_{ij} - 1 \right) X_\infty \right| \succ \delta : \right.$$
$$\left. |X_\infty| \le K \right)$$

$$\subseteq \left[\left(\sup_{n\le i<\infty} \left| \sum_{j=1}^{N_2-1} a_{ij}(X_j - X_\infty) \right| \succ \frac{\delta}{3} \right) \bigcup \left(\sup_{n\le i<\infty} \left| \sum_{j=N_2}^{\infty} a_{ij}(X_j - X_\infty) \right| \succ \frac{\delta}{3} \right) \right.$$
$$\left. \bigcup \left(\sup_{n\le i<\infty} \left| \left(\sum_{j=1}^{\infty} a_{ij} - 1 \right) X_\infty \right| \succ \frac{\delta}{3} \right) \right] \bigcap \ (|X_\infty| \le K)$$

$$\subseteq \left[\left(\sup_{n\le i<\infty} \sum_{j=1}^{N_2-1} |a_{ij}|(|X_j| + |X_\infty|) \succ \frac{\delta}{3} \right) \bigcup \left(\sup_{n\le i<\infty} \sum_{j=N_2}^{\infty} |a_{ij}| \, |X_j - X_\infty| \succ \frac{\delta}{3} \right) \right.$$
$$\left. \bigcup \left(\sup_{n\le i<\infty} \left| \left(\sum_{j=1}^{\infty} a_{ij} - 1 \right) \right| |X_\infty| \succ \frac{\delta}{3} \right) \right] \cap \ \Omega$$

$$\subseteq \left[\left(\sup_{n\le i<\infty} \sum_{j=1}^{N_2-1} 0 \cdot \left(\max_{1\le k<N_2} |X_k| + |X_\infty| \right) \succ \frac{\delta}{3} \right) \right.$$
$$\left. \bigcup \left(\sup_{n\le i<\infty} \sum_{j=N_2}^{\infty} |a_{ij}| \sup_{N_2\le k<\infty} |X_k - X_\infty| \succ \frac{\delta}{3} \right) \bigcup \left(\frac{\delta}{3K} |X_\infty| \succ \frac{\delta}{3} \right) \right]$$

$$\subseteq \left[\varnothing \bigcup \left(\sup_{N_2\le k<\infty} |X_k - X_\infty| \succ \frac{\delta}{3M} \right) \bigcup \left(|X_\infty| \succ K \right) \right]$$

$$= \varnothing \cup \left(\sup_{N_2\le k<\infty} |X_k - X_\infty| \succ \frac{\delta}{3M} \right) \cup \varnothing.$$

Applying probability to these sets and by (4.3) we have:

$$P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta : \ |X_\infty| \le K \right)$$
$$\le P\left(\sup_{N_2\le k<\infty} |X_k - X_\infty| \succ \frac{\delta}{3M} \right) < \frac{\varepsilon}{2}$$

Now going back to (4.4) we finish our proof:

$$P\left(\sup_{n\le i<\infty} |(Ax)_i - X_\infty| \succ \delta \right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

If we consider the diagonal Cesaro Summability method A, then we have the following corollary of Theorem 4.5.

Corollary 4.6. *For $n = 1, 2, ...,$ let random variables X_n, X_∞ satisfy*

(a) X_∞ *is finite a.e.,*
(b) $X_n \xrightarrow{\text{a.s.}} X_\infty$.

Then $\dfrac{\sum_{i=n}^{2n-1} X_i}{n} \xrightarrow{\text{a.s.}} X_\infty$.

However, diagonal Cesaro Summability method fails for convergence in pr. giving us the following Proposition.

Proposition 4.7. *Suppose that A is a column-finite regular method of summability, that is:*

(1) $\underset{1\le i<\infty}{lub} \sum_{j=1}^{\infty} |a_{ij}| = M < \infty$;
(2) *there is finitely many* $a_{ij} \neq 0$ *for each* $j = 1, 2, \ldots$;
(3) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1$.

Then $X_n \to X_\infty$ in pr. is not sufficient for $(Ax)_n \to X_\infty$ in pr.

This follows from a well known example 2.14 after applying it to diagonal Cesaro Summability method. For the convenience of the reader, we provide it in the example below. This example uses only finite values.

Example 4.8. Taking the diagonal Cesaro Summability method A in Example 2.14, and again for $n \geq 16$, we have:

$$\begin{aligned}
\left(\frac{\sum_{j=n}^{2n-1} X_j}{n} > 1\right) &= \left(\sum_{j=n}^{2n-1} X_j > n\right) \\
&\supseteq \left(\sum_{j=n}^{2n-1} X_j > 2^{m+1}\right) \\
&= \left(\sum_{k=i}^{2^m-1} X_{2^m+k} + \sum_{k=0}^{2i-1} X_{2^{m+1}+k} > 2^{m+1}\right) \\
&= \bigcup_{k=i}^{2^m-1} \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right) \cup \bigcup_{k=0}^{2i-1} \left[\frac{k}{2^{m+1}}, \frac{k+1}{2^{m+1}}\right) \\
&= \left[\frac{i}{2^m}, 1\right) \cup \left[0, \frac{i}{2^m}\right) = [0, 1).
\end{aligned}$$

This implies $P\left(\frac{\sum_{j=n}^{2n-1} X_j}{n} > 1\right) = 1$ and shows that

$$\lim_{n\to\infty} P\left(\frac{\sum_{j=n}^{2n-1} X_j}{n} > 1\right) = 1 > 0.$$

Hence $\frac{\sum_{j=n}^{2n-1} X_j}{n}$ does not converge to 0 in pr..

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