Applied Analysis and Optimization
Volume 7, Number 2, 2023, 135–140
Yokohama Publishers
ISSN 2189-1664 Online Journal

A FURTHER MULTIPLICITY RESULT FOR LAGRANGIAN SYSTEMS OF RELATIVISTIC OSCILLATORS

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ABSTRACT. This is our third paper, after [4] and [5], about a joint application of the theory developed by Brezis and Mawhin in [1] with our minimax theorems ([2], [3]) to get multiple solutions of problems of the type

$$\begin{cases} (\phi(u'))' = \nabla_x F(t,u) & \text{in } [0,T] \\ u(0)=u(T)\ ,\ u'(0)=u'(T) \end{cases}$$

which are global minima of a suitable functional over a set of Lipschitzian functions. A challenging conjecture is also formulated.

1. Introduction

In what follows, L, T are two fixed positive numbers. For each $r > 0$, we set $B_r = \{x \in \mathbf{R}^n : |x| < r\}$ ($|\cdot|$ being the Euclidean norm on $\mathbf{R}^n$) and $\overline{B_r}$ is the closure of B_r. The scalar product on $\mathbf{R}^n$ is denoted by $\langle\cdot,\cdot\rangle$. We denote by $\mathcal{A}$ the family of all homeomorphisms ϕ from B_L onto $\mathbf{R}^n$ such that $\phi(0) = 0$ and $\phi = \nabla\Phi$, where the function $\Phi : \overline{B_L} \to]-\infty, 0]$ is continuous and strictly convex in $\overline{B_L}$, and of class C^1 in B_L. Notice that 0 is the unique global minimum of Φ in $\overline{B_L}$.

We denote by $\mathcal{B}$ the family of all functions $F : [0,T] \times \mathbf{R}^n \to \mathbf{R}$ which are measurable in $[0,T]$, of class C^1 in $\mathbf{R}^n$ and such that $\nabla_x F$ is measurable in $[0,T]$ and, for each $r > 0$, one has $\sup_{x\in B_r} |\nabla_x F(\cdot,x)| \in L^1([0,T])$, with $F(\cdot,0) \in L^1([0,T])$. Clearly, $\mathcal{B}$ is a linear subspace of $\mathbf{R}^{[0,T]\times\mathbf{R}^n}$. Given $\phi \in \mathcal{A}$ and $F \in \mathcal{B}$, we consider the problem $(P_{\phi,F})$

$$\begin{cases} (\phi(u'))' = \nabla_x F(t,u) & \text{in } [0,T] \\ u(0)=u(T)\ ,\ u'(0)=u'(T) \end{cases}$$

A solution of this problem is any function $u : [0,T] \to \mathbf{R}^n$ of class C^1, with $u'([0,T]) \subset B_L$, $u(0) = u(T)$, $u'(0) = u'(T)$, such that the composite function $\phi \circ u'$ is absolutely continuous in $[0,T]$ and one has $(\phi \circ u')'(t) = \nabla_x F(t,u(t))$ for a.e. $t \in [0,T]$.

Now, we set

$$K = \{u \in \mathrm{Lip}([0,T],\mathbf{R}^n) : |u'(t)| \le L \ \textit{for a.e.}\ t \in [0,T], u(0)=u(T)\}\ ,$$

2020 *Mathematics Subject Classification.* 34A34, 34C25, 49J35, 49J40.
Key words and phrases. Periodic solution, Lagrangian system of relativistic oscillators, minimax, multiplicity, global minimum, non-convex range.
The author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, PIACERI 2020-2022, Linea di intervento 2, Progetto "MAFANE".

$\mathrm{Lip}([0,T],\mathbf{R}^n)$ being the space of all Lipschitzian functions from $[0,T]$ into $\mathbf{R}^n$.

Clearly, one has

$$\sup_{[0,T]}|u| \leq LT + \inf_{[0,T]}|u| \tag{1.1}$$

for all $u \in K$. Next, consider the functional $I : K \to \mathbf{R}$ defined by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t,u(t)))dt$$

for all $u \in K$.

In [1], Brezis and Mawhin proved the following result:

Theorem 1.1 ([1, Theorem 5.2]). *Any global minimum of I in K is a solution of the problem $(P_{\phi,F})$.*

On the other hand, very recently, in [6], we established the following:

Theorem 1.2 ([6, Theorem 2.2]). *Let X be a topological space, let E be a real normed space, let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $S \subseteq E^*$ be a convex set weakly-star dense in E^*. Assume that $\psi(X)$ is not convex and that $I + \eta \circ \psi$ is lower semicontinuous and inf-compact for all $\eta \in S$.*

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta} \circ \psi$ has at least two global minima in X

The aim of this paper is to establish a new multiplicity result for the solutions of the problem $(P_{\phi,F})$ as a joint application of Theorems 1.1 and 1.2.

Notice that [4] and [5] are the only previous papers on multiple solutions for the problem $(P_{\phi,F})$ which are global minima of I in K.

2. The result

Here is our result:

Theorem 2.1. *Let $\phi \in \mathcal{A}$, $F, G \in \mathcal{B}$ and $H \in C^1(\mathbf{R}^n)$. Assume that:*

(a_1) *there exists $q > 0$ such that*

$$\lim_{|x|\to+\infty} \frac{\inf_{t\in[0,T]} F(t,x)}{|x|^q} = +\infty$$

and

$$\limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,T]} |G(t,x)| + |H(x)|}{|x|^q} < +\infty \ ;$$

(a_2) *there are $\gamma \in \{\inf_{\mathbf{R}^n} H, \sup_{\mathbf{R}^n} H\}$, with $H^{-1}(\gamma)$ at most countable, and $v, w \in H^{-1}(\gamma)$ such that $\int_0^T G(t,v)dt \neq \int_0^T G(t,w)dt$.*

Then, for each $\alpha \in L^\infty([0,T])$ having a constant sign and with $\mathrm{meas}(\alpha^{-1}(0)) = 0$, *there exists $(\tilde{\lambda}, \tilde{\mu}) \in \mathbf{R}^2$ such that the problem*

$$\begin{cases} (\phi(u'))' = \nabla_x \left(F(t,u) + \tilde{\lambda} G(t,u) + \tilde{\mu}\alpha(t)H(u) \right) & \text{in } [0,T] \\ u(0) = u(T) \ , \ u'(0) = u'(T) \end{cases}$$

has at least two solutions which are global minima in K of the functional

$$u \to \int_0^T (\Phi(u'(t)) + F(t,u(t)) + \tilde{\lambda} G(t,u(t)) + \tilde{\mu}\alpha(t)H(u(t)))dt \ .$$

Proof. Fix $\alpha \in L^\infty([0,T])$ having a constant sign and with $\text{meas}(\alpha^{-1}(0)) = 0$. Let $C^0([0,T],\mathbf{R}^n)$ be the space of all continuous functions from $[0,T]$ into $\mathbf{R}^n$, with the norm $\sup_{[0,T]}|u|$. We are going to apply Theorem 1.2 taking $X = K$, regarded as a subset of $C^0([0,T],\mathbf{R}^n)$ with the relative topology, $E = \mathbf{R}^2$ and $I : K \to \mathbf{R}$, $\psi : K \to \mathbf{R}^2$ defined by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t,u(t)))dt \ ,$$

$$\psi(u) = \left(\int_0^T G(t,u(t))dt, \int_0^T \alpha(t)H(u(t))dt\right)$$

for all $u \in K$. Fix $(\lambda,\mu) \in \mathbf{R}^2$. By Lemma 4.1 of [1], the function $I(\cdot)+\langle\psi(\cdot),(\lambda,\mu)\rangle$ is lower semicontinuous in K. Let us show that it is inf-compact too. First, observe that if $P \in \mathcal{B}$ then, for each $r > 0$, there is $M \in L^1([0,T])$ such that

$$\sup_{x\in B_r} |P(t,x)| \le M(t) \tag{2.1}$$

for all $t \in [0,T]$. Indeed, by the mean value theorem, we have

$$P(t,x) - P(t,0) = \langle \nabla_x P(t,\xi), x\rangle$$

for some ξ in the segment joining 0 and x. Consequently, for all $t \in [0,T]$ and $x \in B_r$, by the Cauchy-Schwarz inequality, we clearly have

$$|P(t,x)| \le r \sup_{y\in B_r} |\nabla_x P(t,y)| + |P(t,0)| \ .$$

So, to get (2.1), we can choose $M(t) := r\sup_{y\in B_r}|\nabla_x P(t,y)| + |P(t,0)|$ which is in $L^1([0,T])$ since $P \in \mathcal{B}$. Now, by (a_1), we can fix $c_1, \delta > 0$ so that

$$|G(t,x)| + |H(x)| \le c_1|x|^q \tag{2.2}$$

for all $(t,x) \in [0,T] \times (\mathbf{R}^n \setminus B_\delta)$. Then, set

$$c_2 := c_1 \max\left\{|\lambda|, |\mu|\|\alpha\|_{L^\infty([0,T])}\right\}$$

and, by (a_1) again, fix $c_3 > c_2$ and $\delta_1 > \delta$ so that

$$F(t,x) \ge c_3|x|^q \tag{2.3}$$

for all $(t,x) \in [0,T] \times (\mathbf{R}^n \setminus B_{\delta_1})$. On the other hand, for what remarked above, there is $M \in L^1([0,T])$ such that

$$\sup_{x\in B_{\delta_1}} (|F(t,x)| + |\lambda G(t,x)| + |\mu\alpha(t)H(x)|) \le M(t) \tag{2.4}$$

for all $t \in [0,T]$. Therefore, from (2.2), (2.3) and (2.4), we infer that

$$F(t,x) \ge c_3|x|^q - M(t) \tag{2.5}$$

and

$$|\lambda G(t,x)| + |\mu\alpha(t)H(x)| \le c_2|x|^q + M(t) \tag{2.6}$$

for all $(t, x) \in [0, T] \times \mathbf{R}^n$. Set

$$b := T\Phi(0) - 2\int_0^T M(t)dt \ .$$

For each $u \in K$, with $\sup_{[0,T]} |u| \geq LT$, taking (1.1), (2.5) and (2.6) into account, we have

$$I(u)+\langle\psi(u),(\lambda,\mu)\rangle \geq T\Phi(0)+\int_0^T F(t,u(t))dt-\int_0^T |\lambda G(t,u(t))|dt-\int_0^T |\mu\alpha(t)H(u(t))|dt$$

$$\geq T\Phi(0) + c_3\int_0^T |u(t)|^q dt - \int_0^T M(t)dt - c_2\int_0^T |u(t)|^q dt - \int_0^T M(t)dt$$

$$\geq (c_3 - c_2)T \inf_{[0,T]} |u|^q - b \geq (c_3 - c_2)T\left(\sup_{[0,T]} |u| - LT\right)^q + b \ .$$

Consequently

$$\sup_{[0,T]} |u| \leq \left(\frac{I(u) + \langle\psi(u),(\lambda,\mu)\rangle - b}{(c_3 - c_2)T}\right)^{\frac{1}{q}} + LT \ . \tag{2.7}$$

Fix $\rho \in \mathbf{R}$. By (2.7), the set

$$C_\rho := \{u \in K : I(u) + \langle\psi(u),(\lambda,\mu)\rangle \leq \rho\}$$

turns out to be bounded. Moreover, the functions belonging to C_ρ are equicontinuous since they lie in K. As a consequence, by the Ascoli-Arzelà theorem, C_ρ is relatively compact in $C^0([0,T],\mathbf{R}^n)$. By lower semicontinuity, C_ρ is closed in K. But K is closed in $C^0([0,T],\mathbf{R}^n)$ and hence C_ρ is compact. The inf-compactness of $I(\cdot) + \langle\psi(\cdot),(\lambda,\mu)\rangle$ is so shown. Now, we are going to prove that the set $\psi(K)$ is not convex. By (a_2), the set $\left\{\int_0^T G(t,x)dt : x \in H^{-1}(\gamma)\right\}$ is at most countable since $H^{-1}(\gamma)$ is so. Hence, since $\int_0^T G(t,v)dt \neq \int_0^T G(t,w)dt$, we can fix $\lambda \in]0,1[$ so that

$$\int_0^T G(t,x)dt \neq \int_0^T G(t,w)dt + \lambda\left(\int_0^T G(t,v)dt - \int_0^T G(t,w)dt\right) \tag{2.8}$$

for all $x \in H^{-1}(\gamma)$. Since K contains the constant functions, the points

$$\left(\int_0^T G(t,v)dt, \gamma\int_0^T \alpha(t)dt\right)$$

and

$$\left(\int_0^T G(t,w)dt, \gamma\int_0^T \alpha(t)dt\right)$$

belong to $\psi(K)$. So, to show that $\psi(K)$ is not convex, it is enough to check that the point

$$\left(\int_0^T G(t,w)dt + \lambda\left(\int_0^T G(t,v)dt - \int_0^T G(t,w)dt\right), \gamma\int_0^T \alpha(t)dt\right)$$

does not belong to $\psi(K)$. Arguing by contradiction, suppose that there exists $u \in K$ such that

$$\int_0^T G(t, u(t))dt = \int_0^T G(t, w)dt + \lambda \left(\int_0^T G(t, v)dt - \int_0^T G(t, w)dt \right) \ , \tag{2.9}$$

$$\int_0^T \alpha(t)H(u(t))dt = \gamma \int_0^T \alpha(t)dt \ . \tag{2.10}$$

Since the functions α and $H \circ u - \gamma$ do not change sign, (2.10) implies that $\alpha(t)(H(u(t)) - \gamma) = 0$ a.e. in $[0, T]$. Consequently, since $\text{meas}(\alpha^{-1}(0)) = 0$, we have $H(u(t)) = \gamma$ a.e. in $[0, T]$ and hence $H(u(t)) = \gamma$ for all $t \in [0, T]$ since $H \circ u$ is continuous. In other words, the connected set $u([0, T])$ is contained in $H^{-1}(\gamma)$ which is at most countable. This implies that the function u must be constant and so (2.9) contradicts (2.8). Therefore, I and ψ satisfy the assumptions of Theorem 1.2 and hence there exists $(\tilde{\lambda}, \tilde{\mu}) \in \mathbf{R}^2$ such that the function $I(\cdot) + \langle \psi(\cdot), (\tilde{\lambda}, \tilde{\mu}) \rangle$ has at least two global minima in K. Thanks to Theorem 1.1, they are solutions of the problem we are dealing with, and the proof is complete. □

Remark 2.2. Of course, (a_2) is the leading assumption of Theorem 2.1. The request that $H^{-1}(\gamma)$ must be at most countable cannot be removed. Indeed, if we remove such a request, we could take $H = 0$, $G(t, x) = \langle x, \omega \rangle$, with $\omega \in \mathbf{R}^n \setminus \{0\}$ and $F(t, x) = \frac{1}{p}|x|^p$, with $p > 1$. Now, observe that, by Proposition 3.2 of [1], for all $\lambda \in \mathbf{R}$, the problem

$$(\phi(u'))' = |u|^{p-2}u + \lambda\omega \ in \ [0, T]$$

$$u(0) = u(T) \ , \ u'(0) = u'(T)$$

has a unique solution. To the contrary, the question of whether the condition $\int_0^T G(t, v)dt \neq \int_0^T G(t, w)dt$ (keeping $v \neq w$) can be dropped remains open at present. We feel, however, that it cannot be removed. In this connection, we propose the following

Conjecture 2.3. There exist $\phi \in \mathcal{A}$, $F \in \mathcal{B}$, $H \in C^1(\mathbf{R}^n)$, $\alpha \in L^\infty([0, T])$, with $\alpha \geq 0$ and $\text{meas}(\alpha^{-1}(0)) = 0$, and $q > 0$ for which the following assertions hold:

(b_1)

$$\lim_{|x| \to +\infty} \frac{\inf_{t \in [0,T]} F(t, x)}{|x|^q} = +\infty$$

and

$$\limsup_{|x| \to +\infty} \frac{|H(x)|}{|x|^q} < +\infty \ ;$$

(b_2) the function H has exactly two global minima ;

(b_3) for each $\mu \in \mathbf{R}$, the functional

$$u \to \int_0^T (\Phi(u'(t)) + F(t, u(t)) + \mu\alpha(t)H(u(t)))dt$$

has a unique global minimum in K.

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Manuscript received June 26 2022
revised September 19 2022

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